A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 3

Vector Spaces and Subspaces

Abstract linear algebra begins with the definition of a **vector space** (or **linear space**) as an abstract algebraic structure. We may view the eight properties in the definition as the fundamental axioms for **vector space theory**. The definition requires knowledge of another abstract algebraic structure, a **field** where we can always add (and subtract) as well as multiply (and divide except for zero), but nothing essential is lost if you always think of the field (of **scalars**) as being the real or complex numbers (Halmos 1958,p.1).

DEFINITION #1. A nonempty set of objects (**vectors**), V, together with an **algebraic field** (of **scalars**) **K**, and two algebraic operations (**vector addition** and **scalar multiplication**) which satisfy the algebraic properties listed below (**Laws of Vector Algebra**) comprise a **vector space**. (Following standard convention, although technically incorrect, we will usually refer to the **set of vectors** V as the vector space). The set of scalars **K** are usually either the real numbers **R** or the complex numbers **C** in which case we refer to V as a **real or complex vector space**. Let $\vec{x}$, $\vec{y}$, $\vec{z} \in V$ be any vectors and $\alpha, \beta \in \mathbf{K}$ be any scalars. Then the following must hold:

VS1) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ — Associativity of vector addition

VS2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ — Commutativity of vector addition

VS3) There exists a vector $\vec{0}$, such that for every $\vec{x} \in V$, $\vec{x} + \vec{0} = \vec{x}$ — Existence of a right additive identity vector for vector addition

VS4) For each $\vec{x} \in V$, there exist a vector, denoted by $-\vec{x}$, such that $\vec{x} + (-\vec{x}) = \vec{0}$. — Existence of a right additive inverse vector for each vector in V

VS5) $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ — An associativity property for scalar multiplication

VS6) $(\alpha+\beta)\vec{x} = (\alpha\vec{x}) + (\beta\vec{x})$ — A distributive property for scalar multiplication and vector addition

VS7) $\alpha(\vec{x} + \vec{y}) = (\alpha\vec{x}) + (\alpha\vec{y})$ — Another distributive property for scalar multiplication and vector addition

VS8) $1\vec{x} = \vec{x}$ — A scaling property for scalar multiplication

These eight properties are an essential part of the definition of a vector space. In abstract algebra terminology, the first four properties establish a vector space with vector addition as an **Abelian (or commutative) group** (another abstract algebraic structure). The last four properties give rules on how vector addition and scalar multiplication must behave together.

Although technically not correct, we often refer to the set V of vectors as the vector space. Thus the **R**, $\mathbf{R}^2$,$\mathbf{R}^3$, $\mathbf{R}^n$, and $\mathbf{R}^{m\times n}$ are referred to as real vector spaces and **C**, $\mathbf{C}^n$, and $\mathbf{C}^{m\times n}$ as complex vector spaces. Also **Q**, $\mathbf{Q}^n$ and $\mathbf{Q}^{m\times n}$, and indeed, **K**, $\mathbf{K}^n$, and $\mathbf{K}^{m\times n}$ for any field are referred to as vector spaces. However, the definition of a vector space requires that its structure be given. Thus to rigorously represent a vector space, we use a 5-tuple consisting of 1) the set of vectors, 2) the set of scalars, 3) vector addition, 4) scalar multiplication, and 5) the zero vector (which is the only vector required to be in all vector spaces). Thus V = (V, **K**, +, , $\vec{0}$). (Since we use juxtaposition to indicate scalar multiplication, we have no symbol for multiplication of a vector by a scalar and hence just leave a space.)

We consider some properties that hold in all vector spaces (i.e., they are vector space properties like the original eight, but unlike the original eight, these follow from the original eight using mathematical logic). Specific examples of vector spaces are given in the next handout. Once they have been shown to be vector spaces, it is not logically necessary to show directly that the properties in Theorems #1, #2 and #3 and Corollary #4 hold in these spaces. The properties hold since we can show that they are vector space properties (i.e., they hold in an abstract vector space). However, you may wish to check out these properties in specific vector spaces (i.e., provide a direct proof) to improve your understanding of the concepts. We start with an easy property for which we provide proof.

<u>THEOREM #1.</u> Let V = (V,**K**, +, , $\vec{0}$) be a vector space. Then for all $\vec{x} \in V$, $\vec{0} + \vec{x} = \vec{x}$.

Proof. Let $\vec{x}$ be any vector in V and let $\vec{0}$ be the right additive identity for V (or V). Then

<u>STATEMENT</u>	<u>REASON</u>
$\vec{0} + \vec{x} = \vec{x} + \vec{0}$	VS2. Vector addition is commutative
$= \vec{x}$	VS3 $\vec{0}$ is the right additive identity element for V (or V).

Hence for all $\vec{x} \in V$, $\vec{0} + \vec{x} = \vec{x}$ (i.e., $\vec{0}$ is a **left additive identity element** as well as a right additive identity element).

Q.E.D.

As in the proofs of any identity, the replacement of $\vec{0} + \vec{x}$ by $\vec{x}$ is effected by the property of equality that says that in any equality, a quantity may be replaced by an equal quantity. (Note that the second equality is really $\vec{0} + \vec{x} = \vec{x}$ as the LHS is assumed.) This proof is in some sense identical to the proof that for all x∈**K**, 0 + x = 0 for fields. This is because the property is really a **group theory property** and vectors with vector addition (as well as scalars with scalar addition) form <u>groups</u>. We now list additional properties of a <u>vector space</u> that follow since the vectors in a vector space along with vector addition form a <u>group</u>. Since $\vec{0}$ is both a <u>left and right additive identity element</u>, we may now say that it is **an additive identity element**.

<u>THEOREM #2.</u> Let V be a vector space over the field **K**. (i.e., Let V = (V, **K**, +, , $\vec{0}$) be a vector space.) Then

1. The zero vector is unique (i.e., there is only one additive identity element in V).
2. Every vector has a unique additive inverse element.
3. $\vec{0}$ is its own additive inverse element (i.e., , $\vec{0} + \vec{0} = \vec{0}$).
4. The additive inverse of an additive inverse element is the element itself. (i.e., if - $\vec{x}$ is the additive inverse of $\vec{x}$, then $-(-\vec{x}) = \vec{x}$).

5. $-(\vec{x} = \vec{y}) = (-\vec{x}) + (-\vec{y})$. (i.e., the additive inverse of a sum is the sum of their additive inverses.)
6. Sums of vectors can be written in any order you wish.
7. If $\vec{x} + \vec{y} = \vec{x} + \vec{z}$, then $\vec{y} = \vec{z}$. (Cancellation Law for Addition)

We now give a theorem for vector spaces analogous to the one for fields that says that if the product of two numbers is zero, one of the numbers must be zero.

THEOREM #3. Let V be a vector space over the field **K**. (i.e., Let V = (V, **K**, +, , $\vec{0}$) be a vector space. The scalars **K** may be thought of as either **R** or **C** so that we have a real or complex vector space, but V may not be thought of as $\mathbf{R}^n$ or $\mathbf{C}^n$.)

1. $\forall\ \vec{x} \in V$, $\qquad 0\ \vec{x} = \vec{0}$.
2. $\forall\ \alpha \in \mathbf{K}$, $\qquad \alpha\vec{0} = \vec{0}$
3. $\forall \alpha \in \mathbf{K}$, $\ \forall\ \vec{x} \in V$, $\qquad \alpha\vec{x} = \vec{0}$ implies either $\alpha = 0$ or $\vec{x} = \vec{0}$.

COROLLARY#4 (Zero Product). Let V be a vector space over the field **K**. (**K** may be thought of is either **R** or **C**.) Let $\alpha \in \mathbf{K}$, $\vec{x} \in V$. Then $\alpha\vec{x} = \vec{0}$ if and only if $\alpha = 0$ or $\vec{x} = \vec{0}$.

In the abstract algebraic definition of a vector space, vectors do not have a magnitude (or length). Later, we will discuss **normed linear spaces** where this structure is added to a vector space. However, the concept of **direction**, in the sense indicated in the following theorem, is in all vector spaces.

DEFINITION #2. Let V be a vector space over a field **K**. Two nonzero vectors $\vec{x}$, $\vec{y} \in V$ are said to be **parallel** if there is a scalar $\alpha \in \mathbf{K}$ such that $\alpha\vec{x} = \vec{y}$. (The zero vector has no direction.) If V is a real vector space so that **K**=**R**, then two (non-zero) parallel vectors are in the **same direction** if $\alpha > 0$, but in **opposite directions** if $\alpha < 0$.

EXERCISES on The Definition of an Abstract Vector Space

EXERCISE #1. True or False.

For all questions, assume V is a vector space over a field **K**, $\vec{x}$, $\vec{y}$, $\vec{z} \in V$ are any vectors and $\alpha, ß \in \mathbf{K}$ are any scalars.

_____ 1. The following property is an axiom in the definition of a vector space:
VS) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

_____ 2. The following property is an axiom in the definition of a vector space:
VS) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

_____ 3. The following property is an axiom in the definition of a vector space:

VS)There exists a vector $\vec{0}$ such that for every $\vec{x} \in V$, $\vec{x} + \vec{0} = \vec{x}$

_____ 4. The following property is an axiom in the definition of a vector space:

VS) For each $\vec{x} \in V$, there exist a vector, denoted by $-\vec{x}$, such that $\vec{x} + (-\vec{x}) = \vec{0}$.

_____ 5. The following property is an axiom in the definition of a vector space:

VS)) $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$

_____ 6. The following property is an axiom in the definition of a vector space:

VS) $(\alpha+\beta)\vec{x} = (\alpha\vec{x}) + (\beta\vec{x})$

_____ 7. The following property is an axiom in the definition of a vector space:

VS) $\alpha(\vec{x} + \vec{y}) = (\alpha\vec{x}) + (\alpha\vec{y})$

_____ 8. The following property is an axiom in the definition of a vector space:

VS)) $1\vec{x} = \vec{x}$

_____ 9. In V, vector addition is associative.

_____ 10. In V, vector addition is commutative.

_____ 11.In V, for vector addition, there exist a right additive identity such that for every $\vec{x} \in V$, $\vec{x} + \vec{0} = \vec{x}$.

_____ 12. For vector addition, every vector in V has a right additive inverse, denoted by $-\vec{x}$, such that $\vec{x} + (-\vec{x}) = \vec{0}$.

_____ 13. For all $\vec{x} \in V$, we have $\vec{0} + \vec{x} = \vec{x}$.

_____ 14. In V, the zero vector is unique.

_____ 15. There is only one additive identity element in V

_____ 16. In V, every vector has a unique additive inverse element.

_____ 17. In V, $\vec{0}$ is its own additive inverse element

_____ 18. The additive inverse of an additive inverse element is the element itself

_____ 19. if $-\vec{x}$ is the additive inverse of $\vec{x}$ in V, then $-(-\vec{x}) = \vec{x}$.

_____ 20. $\forall\alpha \in \mathbf{K}$, $\forall\ \vec{x} \in V$, $\alpha\vec{x} = \vec{0}$ implies either $\alpha = 0$ or $\vec{x} = \vec{0}$.

Halmos, P. R.1958, *Finite Dimensional Vector Spaces* (Second Edition) Van Nostrand Reinhold Company, New York.

If we define a specific set of vectors, a set of scalars and two operations that satisfy the eight properties in the definition of a vector space (i.e., the Laws or Axioms of Vector Algebra) we obtain an example of a vector space. Note that this requires that the eight properties given in the definition of an (abstract) vector space be verified for the (concrete) example. We give examples of vector spaces, but the verification that they are indeed vector spaces is left to the exercises. We also give two ways of building a more complicated vector space from a given vector space.

EXAMPLE #1. The set of **sequences** of a fixed length in a field **K**. Since we are not interested in matrix multiplication at this time, we may think of them not only as sequences, but as **column vectors** or as **row vectors**. When we consider linear operators, we wish them to be column vectors and so we use this interpretation. For example, if **K** = **R**, we let

i) $$V = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} : x_i \in R, i = 1,2,3,..,n \right\} = \mathbf{R}^{n\text{x}1} \cong \mathbf{R}^n$$

ii) The scalars are the **real numbers R**.

iii) **Vector addition** is defined by

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n + y_n \end{bmatrix}.$$ (i.e. Add componentwise as in matrix addition)

iv) **Scalar multiplication** is defined by

$$\alpha \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha x_n \end{bmatrix}.$$ (i.e. Multiply each component in x by α as in multiplication of a matrix by a scalar.)

Again, the space of **row vectors**: $V = \{(x_1,...,x_n)\colon x_i \in \mathbf{R}, i = 1,...,n\} = \mathbf{R}^{1\text{xn}} \cong \mathbf{R}^n$ is technically different from the space of column vectors. However, as far as being a vector space is concerned, the distinction is merely technical and not substantive. Unless we need to make use of the technical difference we will denote both spaces by $\mathbf{R}^n$. We refer to $\mathbf{R}^n$ as the space of n-tuples of real numbers whether they are written as row vectors or column vectors. If we wish to think of row vectors as $n \times 1$ matrices, column vectors as $1 \times n$ matrices, and consider matrix multiplication, then we must distinguish between row and column vectors. If we wish the scalars to be the complex numbers, we must consider the space $\mathbf{C}^n$ of n-tuples of complex numbers.

EXAMPLE #2. Consider the set $\mathbf{R}^{m\times n}$ of all **matrices** of real numbers with a fixed size, $m \times n$. We define vector addition as matrix addition. (To find the sum of two matrices, add them componentwise.) Scalar multiplication of a vector (i.e. a matrix) by a scalar is defined in the usual way (multiply each component in the matrix by the scalar). Note that matrix multiplication is not required for the set of matrices of a given size to have the vector space structure. The Laws of Vector Algebra can be shown to hold and hence $\mathbf{R}^{m\times n}$ is a vector space. We may also consider $\mathbf{C}^{m\times n}$, the set of all matrices of complex numbers.

EXAMPLE #3. **Function Spaces**. Let $\mathscr{F}(D,\mathbf{R}) = \{f: D \to \mathbf{R}\}$ where $D \subseteq \mathbf{R}$; that is, let $\mathscr{F}(D)$ be the collection of all real valued functions of a real variable which have a common domain D in $\mathbf{R}$. Often, $D = I = (a,b)$. The scalers will be $\mathbf{R}$. Vector addition is defined as function addition. Recall that a function f is defined by knowing (the rule that determines) the values $y = f(x)$ for each x is the domain of f. Given two functions f and g whose domains are both D, we can define a new function $h = f + g$ (h is called the sum of f and g) by the rule $h(x) = f(x) + g(x)$ for all $x \in D$. Similarly we define the function αf by the rule $(\alpha f)(x) = \alpha\ f(x)$ for all $x \in D$. This defines scalar multiplication of a "vector" (i.e. a function) by a scalar. The Laws of Vector Algebra can be shown to hold and hence $\mathscr{F}(D)$ is a vector space. We may also define $\mathscr{F}(D)$ to be $\{f: D \to \mathbf{C}\}$ where $D \subseteq \mathbf{C}$. That is, we may also let $\mathscr{F}(D)$ describe the set of all complex valued functions of a complex variable that have a common domain D in $\mathbf{C}$. Function spaces are very important when you wish to solve differential equations.

EXAMPLE #4. Suppose $V_{\mathbf{R}}$ is a real vector space. We construct a complex vector space $V_{\mathbf{C}}$ as follows: As a set, let $V_{\mathbf{C}} = V_{\mathbf{R}} \times V_{\mathbf{R}} = \{(\vec{x}, \vec{y}): \vec{x}, \vec{y} \in V_{\mathbf{R}}\}$. However, we will use the Eulerian notation, $\vec{z} = \vec{x} + i\,\vec{y}$ for elements in $V_{\mathbf{C}}$. We define vector addition and scalar multiplication componentwise in the obvious way:

If $\vec{z}_1 = \vec{x}_1 + i\,\vec{y}_1$, $\vec{z}_2 = \vec{x}_2 + i\,\vec{y}_2 \in V_{\mathbf{C}}$, then $\vec{z}_1 + \vec{z}_2 = (\vec{x}_1 + \vec{x}_2) + i(\vec{y}_1 + \vec{y}_2)$.

If $\gamma = \alpha + i\beta \in \mathbf{C}$ and $\vec{z} = \vec{x} + i\,\vec{y} \in V_{\mathbf{C}}$, then $\gamma\,\vec{z} = (\alpha\,\vec{x} - \beta\,\vec{y}) + i\,(\beta\,\vec{x} + \alpha\,\vec{y})$.

It is straight forward to show that with these definitions, all eight of the Laws of Vector Algebra are satisfied so that $V_{\mathbf{C}}$ is a complex vector space. We see that $V_{\mathbf{R}}$ can be embedded in $V_{\mathbf{C}}$ and hence can be considered as a subset of $V_{\mathbf{C}}$. It is not a subspace (see the next handout) since they use a different set of scalars. However, if scalars are restricted to $\mathbf{R}$ and vectors to the form $\vec{z} = \vec{x} + i\,\vec{0}$, then the vector space structure of $V_{\mathbf{R}}$ is also embedded in $V_{\mathbf{C}}$.

It is important to note that this process can be done for any vector space $V_{\mathbf{R}}$. If we start with $\mathbf{R}$, we get $\mathbf{C}$. If we start with $\mathbf{R}^n$, we get $\mathbf{C}^n$. If we start with $\mathbf{R}^{m\times n}$, we get $\mathbf{C}^{m\times n}$. If we start with real functions of a real variable, we get complex functions of a real variable. For example, the complex vector space $C^1(\mathbf{R}^2,\mathbf{C}) = C^1(\mathbf{R}^2,\mathbf{R})+iC^1(\mathbf{R}^2,\mathbf{R})$ with vectors that are complex valued functions of two real variables of the form u(x,y) +iv(x,y) with $u,v\in C^1(\mathbf{R}^2,\mathbf{R})$ will be of interest later. Since there is a one-to-one correspondence between $\mathbf{R}^2$ and $\mathbf{C}$ (in fact, they are isomorphic as real vector spaces) and have the same topology (they have the same norm and hence the same metric, see Chapter 8) we may identify $C^1(\mathbf{R}^2,\mathbf{R})$ with set of real valued functions of a complex variable $C^1(\mathbf{C},\mathbf{R})= \{u(z)= \tilde{u}(x, y) \in C^1(\mathbf{R}^2):z=x+iy\}$ and hence $C^1(\mathbf{R}^2,\mathbf{R})+iC^1(\mathbf{R}^2,\mathbf{R})$ with the complex vector space of complex valued functions of a complex variable $C^1(\mathbf{C},\mathbf{C}) = C^1(\mathbf{C},\mathbf{R})+iC^1(\mathbf{C},\mathbf{R})$ of the form w(z) = u(z) +iv(z) where $u,v\in C^1(\mathbf{R}^2,\mathbf{R})$.

EXAMPLE #5. **Time varying vectors**. Suppose V is a real vector space (which we think of as a state space). Now let $V(I) = \{x(t):I\to V\}=\mathcal{F}(I,V)$ where $I = (a,b)\subseteq\mathbf{R}$. That is, V is the set of all "vector valued" functions on the open interval I. (Thus we allow the state of our system to vary with time.) To make V(I) into a vector space, we must equip it with a set of scalars, vector addition, and scalar multiplication. The set of scalars for V(I) is the same as the scalars for V (i.e.,$\mathbf{R}$). Vector addition and scalar multiplication are simply function addition and scalar multiplication of a function. To avoid introducing to much notation, the engineering convention of using the same symbol for the function and the dependent variable will be used (i.e., instead of y=f(x), we use y=y(x)). Hence instead of $\vec{x} = \vec{f}(t)$, for a function in V(I), we use $\vec{x} = \vec{x}(t)$. The context will explain whether $\vec{x}$ is a vector in V or a function in V(I).

1) If $\vec{x}, \vec{y} \in V(I)$, then we define $\vec{x}+\vec{y}$ pointwise as , $(\vec{x}+\vec{y})(t) = \vec{x}(t) + \vec{y}(t)$.

2) If $\vec{x} \in V(I)$ and α is a scalar, then we define $(\alpha\,\vec{x})(t) \in V(I)$ pointwise as $(\alpha\,\vec{x})(t) = \alpha\,\vec{x}(t)$.

The proof that V(I) is a vector space is left to the exercises. We use the notation V(t) instead of V(I), when, for a math model, the interval of validity is unknown and hence part of the problem. Since V is a real vector space, so is V(t). V(t) can then be embedded in a complex vector space as described above. Although we rarely think of time as being a complex variable, this is often useful mathematically to solve dynamics problems since we may wish state variables to be analytic. Thus the holomorphic function spaces are of interest.

EXAMPLE #6. (Holomorphic functions) Consider $C^1(\mathbf{R}^2,\mathbf{R})+iC^1(\mathbf{R}^2,\mathbf{R})$.

EXERCISES on Examples of Vector Spaces

<u>EXERCISE</u> <u>#1</u>. True or False.

For all questions, assume V is a vector space over a field **K**, $\vec{x}$, $\vec{y}$, $\vec{z} \in V$ are any vectors and $\alpha, \beta \in \mathbf{K}$ are any scalars.

_____ 1. $V = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} : x_i \in R, i = 1,2,3,..,n \right\} = \mathbf{R}^n$ is a vector space.

_____ 2. The scalars for the vector space $\mathbf{R}^3$ are the real numbers **R**.
_____ 3. $\mathbf{R}^{m\mathrm{x}n}$ is a vector space
_____ 4.The scalars for the vector space $\mathbf{R}^{3\mathrm{x}2}$ are the real numbers **R**.
_____ 5. $\mathbf{C}^{1\mathrm{x}2}$ is a vector space
_____ 6. $\mathbf{R}^{m\mathrm{x}n}$ is a real vector space
_____ 7. The scalars for the vector space $\mathbf{R}^{3\mathrm{x}2}$ are the real numbers **R**.
_____ 8. The scalars for the vector space $\mathbf{C}^{3\mathrm{x}2}$ are the real numbers **R**.
_____ 9. The scalars for the vector space $\mathbf{R}^{10}$ are the real numbers **R**.
_____ 10. $\mathbf{R}^{1\mathrm{x}2}$ is a real vector space.
_____ 11. $\mathbf{C}^{m\mathrm{x}n}$ is a real vector space.
_____ 12. The function space $\mathscr{F}(D,\mathbf{R}) = \{f: D \to \mathbf{R}\}$ where $D \subseteq \mathbf{R}$ is a vector space.
_____ 13. The set of all continuous functions on the open interrval I = (a,b), C(I,**R**), is a vector space
_____ 14. The set of all analytic functions on the open interrval I = (a,b), A (I,**R**), is a vector space.
_____ 15. The scalars for the vector space $\mathscr{F}(D,\mathbf{R}) = \{f: D \to \mathbf{R}\}$ where $D \subseteq \mathbf{R}$ are the real numbers **R**.
_____ 16. $\mathscr{F}(D,\mathbf{R}) = \{f: D \to \mathbf{R}\}$ where $D \subseteq \mathbf{R}$ is a real vector space.
_____ 17. The set of all real valued continuous functions on the open interrval I = (a,b), C(I,**R**), is a real vector space.