DEFINITION #1. Let $\underset{m\times n}{A}$ be an m×n matrix and α a scalar ($\alpha \in \mathbf{K}$ where **K** is a field):

$$\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [a_{ij}] \in \mathbf{K}^{mxn}, \qquad \alpha\in \mathbf{K} \text{ (e.g., } \mathbf{K} = \mathbf{R} \text{ or } \mathbf{K} = \mathbf{C}) \qquad (1)$$

then we define the product of α and A (called **multiplication by a scalar**, but not scalar product) by

$$\alpha \underset{mxn}{A} = \begin{bmatrix} \alpha a_{1,1} & \alpha a_{1,2} & \cdot & \cdot & \cdot & \alpha a_{1,n} \\ \alpha a_{2,1} & \alpha a_{2,2} & \cdot & \cdot & \cdot & \alpha a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \alpha a_{m,1} & \alpha a_{m,1} & \cdot & \cdot & \cdot & \alpha a_{m,n} \end{bmatrix} = [\,\alpha\, a_{ij}\,] \in \mathbf{K}^{mxn} \qquad (1)$$

that is, $\alpha A \equiv C = [c_{ij}]$ where $c_{ij} = \alpha a_{ij}$ for i = 1,..., m and j = 1,..., m. That is, we multiply each component in A by α. Again defining an arbitrary component c_{ij} of αA takes less space but is less graphic, that is, does not give a good visualization of the operation. However, you should learn to provide that visualization, (i.e., look at $c_{ij} = \alpha a_{ij}$ and visualize each component being multiplied by α, for example, the nine element graphic above or the four corner graphic.) Sometimes we place the scalar on the right hand side (RHS): $\alpha A = \alpha[a_{ij}] = [a_{ij}]\alpha = A\alpha$.

PROPERTIES. The following theorems can be proved.

THEOREM #1. Let A be an m×n matrix and 1 be the multiplicative identity in the associated scalar field **K** (e.g., $1 \in \mathbf{R}$ or $1 \in \mathbf{C}$), then 1A = A.

THEOREM #2. Let A and B be an m×n matrix and α and β be scalars in **K**(e.g., $\alpha,\beta \in \mathbf{R}$ or $\alpha,\beta \in \mathbf{C}$), then

a. $\alpha(\beta A) = (\alpha\beta)A$ (Note the difference in the meaning of the two sets of parentheses.)
b. $(\alpha + \beta)A = (\alpha A) + (\beta A)$ (Note the difference in the meaning of the two plus signs.)
c. $\alpha(A+B) = (\alpha A) + (\beta B)$ (What about parentheses and plus signs here?)

INVERSE OPERATION. If α is a nonzero scalar, the division of a matrix A by the scalar α is defined to be multiplication by the multiplicative inverse of the scalar. Thus A/α = (1/α) A.

THEOREM #3. Let A, B ∈ $\mathbf{C}^{m\times n}$. Then the following hold:
1) $(\alpha A)^T = \alpha A^T$
2) $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$
3) $(\alpha A)^* = \overline{\alpha}\, A^*$

EXERCISES on Multiplication by a Scalar

EXERCISE #1. Multiply A by the scalar α if

a) α = 2 and $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$ b) α = 2+i and A = [1,2] c) α = 1 and $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$

d) α =1 /2 and $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$

EXERCISE #2. Compute $(\alpha A)^T$ (that is, first multiply α by A and take the transpose) and then αA^T (that is, multiply α by A transpose) and show that you get the same answer if

a) α = 2 and $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$ b) α = 2+i and A = [1,2] c) α = 1 and $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$

d) α = 1/2 and $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$

EXERCISE #3. Compute $\overline{\alpha A}$ (that is, multiply α by A and then compute the complex conjugate) and then compute $\overline{\alpha}\overline{A}$ (that is, compute $\overline{\alpha}$ and then $\overline{A}$ and then multiply $\overline{\alpha}$ by $\overline{A}$) and then show that you get the same answer if:

a) α = 2 and $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$ b) α = 2+i and A = [1,2] c) α = 1 and $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$

d) α = ½ and $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$

DEFINITION #1. Let $\vec{x} = [x_1,x_2,...,x_n]$ and $\vec{y} = [y_1,y_2,...,y_n]$ be row vectors in $\mathbf{R}^{1\times n}$ or $\mathbf{C}^{1\times n}$ (we could just as well use column vectors). Then define the **scalar product** of $\vec{x}$ and $\vec{y}$ by

$$\vec{x} \circ \vec{y} = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots + x_n\bar{y}_n = \sum_{i=1}^{n} x_i\bar{y}_i$$

In $\mathbf{R}^2$ and $\mathbf{R}^3$, this is usually called the **dot product**. (In an abstract setting, this operation is usually called an **inner product**.)

PROPERTIES. The following theorem can be proved.

THEOREM #1. Let $\vec{x} = [x_1,x_2,...,x_n]$, $\vec{y} = [y_1,y_2,...,y_n]$, and $\vec{z} = [z_1,z_2,...,z_n]$ be row vectors in $\mathbf{R}^{1\times n}$ or $\mathbf{C}^{1\times n}$; we could just as well use column vectors) and α be a scalar. Then

a. $\vec{x}\cdot\vec{y} = \overline{\vec{y}\cdot\vec{x}} = \bar{\vec{y}}\cdot\bar{\vec{x}}$
b. $(\vec{x}+\vec{y})\cdot\vec{z} = \vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{z}$
c. $(\alpha\vec{x})\cdot\vec{y} = \alpha(\vec{x}\cdot\vec{y})$
d. $\vec{x}\cdot\vec{x} \geq 0$
   $\vec{x}\cdot\vec{x} = 0$ if and only if $\vec{x} = \vec{0}$

The term **inner product** is used for an operation on an abstract vector space if it has all of the properties given in Theorem #1. Hence we have that the operation of scalar product defined above is an example of an inner product and hence $\mathbf{R}^{1\times n}$ and $\mathbf{K}^{1\times n}$ (and $\mathbf{R}^{n\times 1}$ and $\mathbf{C}^{n\times 1}$ and $\mathbf{R}^n$ and $\mathbf{C}^n$) are **inner product spaces**.

EXERCISES on Dot or Scalar Product.

EXERCISE #1. Compute $\vec{x}\cdot\vec{y}$ if
a) $\vec{x}=[1,2,1]$, $\vec{y}=[2,3,1]$ b) $\vec{x}=[1,2,3,4]$, $\vec{y}=[2,4,0,1]$ c) $\vec{x}=[1,0,1,0]$, $\vec{y}=[1,2,0,1]$

EXERCISE #2. Compute $\vec{x}\cdot\vec{y}$ if
a) $\vec{x}=[1+i,2+i,0,1], \vec{y}=[1+i,2+i,0,1]$ b) $\vec{x}=[1+i,0,0,i], \vec{y}=[1+i,2+i,0,1]$
c) $\vec{x}=[1,0,1,i], \vec{y}=[1+i,2+i,0,1]$

Matrix multiplication should probably be called matrix composition, but the term matrix multiplication is standard and we will use it.

DEFINITION #1. Let $\underset{m\times p}{A} = [\,a_{ij}\,]$, $\underset{p\times n}{B} = [\,a_{ij}\,]$. Then $\underset{m\times n}{C} = AB = [\,c_{ij}\,]$ where $c_{ij} = \sum_{i=1}^{p} a_{ik} b_{kj}$.

Although the definition of the product matrix C is given without reference to the scalar product, it is useful to note that the element c_{ij} is obtained by computing the scalar product of the i^{th} row of A with the j^{th} column of B and placing this result in the i^{th} row and j^{th} column of C. Using tensor notation, we may just write $c_{ij} = \sum_{i=1}^{p} a_{ik} b_{kj}$. If we adopt the **Einstein summation convention** of summing over repeated indices, then this may be written as $c_{ij} = a_{ik}\, b_{kj}$. This assumes that the values of m, n, and p are known.

EXAMPLE.

$$\underset{A}{\begin{bmatrix} -1 & 2 & 1 \\ 0 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 3 & -4 & 5 \\ 7 & 3 & 11 \\ 4 & 1 & 12 \end{bmatrix}}$$

$c_{11} = [-1,2,1]\circ[0,1,1] = [-1,2,1][0,1,1]^T \quad = (-1)(0) + (2)(1) + (1)(1) = 3$

$c_{12} = [-1,2,1]\circ[3,-1,1] = [-1,2,1][3,-1,1]^T \quad = (-1)(3) + (2)(-1) + (1)(1) = -4$

.
.
.

SIZE REQUIREMENT: In order to take the dot product of the i^{th} row of A with the j^{th} column of B, they must have the same number of elements. Thus the number of columns of A must be the same as the number of rows of B.

$$\underset{n\times p}{A}\ \underset{p\times n}{B} = \underset{n\times m}{C}$$

THE SCALAR PRODUCT IN TERMS OF MATRIX MULTIPLICATION. The dot product of two row vectors in $\mathbf{R}^{1\times n}$ can be given in terms of matrix multiplication. Let $\vec{x} = [x_1, x_2, \ldots, x_n]$ and $\vec{y} = [y_1, y_2, \ldots, y_n]$ be row vectors in $\mathbf{R}^{1\times n}$. Then

$$\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{i=1}^{n} x_iy_i = \underset{1xn}{\vec{x}}\ \underset{nx1}{\vec{y}}^{T}$$

If $\vec{x}$ and $\vec{y}$ are defined as column vectors, we have

$$\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{i=1}^{n} x_iy_i = \underset{1xn}{\vec{x}}^{T}\ \underset{nx1}{\vec{y}}$$

Using tensor notation, including the Einstein summation convention, we have $\vec{x}\cdot\vec{y} = x_i\, y_i$.

<u>PROPERTIES</u>. For multiplication properties, we must first be sure that all of the operations are possible. Note that unless A and B are square, we can not compute both AB and BA. Even for square matrices, AB does not always equal BA (except for 1×1's). However, we do have:

<u>THEOREM #1.</u> Let $\underset{mxp}{A}$, $\underset{pxr}{B}$, and $\underset{rxn}{C}$ be matrices so that the multiplications $\underset{mxp}{A}\ \underset{pxr}{B}$, $\underset{pxr}{B}\ \underset{rxn}{C}$, $\underset{mxp}{A}\ (\underset{pxr}{B}\ \underset{rxn}{C})$, and $(\underset{mxp}{A}\ \underset{pxr}{B})\ \underset{rxn}{C}$ are all possible. Then

A(BC) = (AB)C. matrix multiplication is **associative**

<u>THEOREM #2.</u> Let $\underset{mxp}{A}$, $\underset{pxn}{B}$, and $\underset{pxn}{C}$ be matrices so that the multiplications $\underset{mxp}{A}\ \underset{pxn}{B}$, $\underset{mxp}{A}\ \underset{pxn}{C}$ and the additions $\underset{pxn}{B} + \underset{pxn}{C}$ and $\underset{mxp}{A}\ \underset{pxn}{B} + \underset{mxp}{A}\ \underset{pxn}{C}$ are all possible. Then

A(B + C) = AB + AC matrix multiplication on the left **distributes** over matric addition

Now let $\underset{mxp}{A}$, $\underset{mxp}{B}$, and $\underset{pxn}{C}$ be matrices so that the multiplications $\underset{mxp}{A}\ \underset{pxn}{B}\ \underset{pxn}{C}$, $\underset{mxp}{B}\ \underset{pxn}{C}$, and the additions $\underset{mxp}{A} + \underset{mxp}{B}$, and $\underset{mxp}{A}\ \underset{pxn}{C} + \underset{mxp}{B}\ \underset{pxn}{C}$ are all possible. Then

(A+B)C = AC + BC matrix multiplication on the right **distributes** over matric addition

<u>THEOREM #3.</u> Let $\underset{mxp}{A}$ and $\underset{pxn}{B}$ be matrices so that the matrix multiplications AB, (αA)B, and A(αB) are all possible. Then (αA)B = A(αB) = α(AB).

EXERCISES on Matrix Multiplication

<u>EXERCISE #1</u>. If possible, multiply A times B (i.e., find the product AB) if

a) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1+i \\ 1 & 1-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$ c) A = [1,2] , B = [1,2,3]

d) $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ 2 & 0 \\ 0 & 1-i \end{bmatrix}$ e) $A = \begin{bmatrix} 1 & 0 \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$ f) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$

g) $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$, $B = \begin{bmatrix} 2+2\sqrt{2}i \\ 0 \\ 1-2\pi i \end{bmatrix}$ h) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1+i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>EXERCISE #2</u>. . If posible, compute A(B+C) (that is, first add B to C and then add A times this sum) and then AB and BC (that is, multiply AB and then BC) and then show that you get A(B+C) = AB +AC if

a) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1+i \\ 1 & 1-i \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ 2 & 0 \\ 0 & 1-i \end{bmatrix}$ $C = \begin{bmatrix} 2 & 2+i \\ 3e & 3 \\ 3-i & i \end{bmatrix}$

If A and B are square, then we can compute both AB and BA. Unfortunately, these may not be the same.

THEOREM #1. If n >1, then there exists A, B$\in\mathbf{R}^{n\times n}$ such that AB $\neq$ BA. Thus matrix multiplication is **not commutative**.

Thus AB=BA is not an identity. Can you give a **counter example** for n=2? (i.e. an example where AB $\neq$ BA. See Exercise #2.)

DEFINITION #1. For square matrices, there is a **multiplicative identity element**. We define the n×n matrix I by

$$\underset{mxn}{I} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

One's down the diagonal. Zero's everywhere else.

THEOREM #2. We have $\underset{nxn}{A}\ \underset{nxn}{I} = \underset{nxn}{I}\ \underset{nxn}{A} = \underset{nxn}{A} \qquad \forall A \in \mathbf{K}^{nxn}$

DEFINITION #2. If there exists B such that AB = I., then B is a **right (multiplicative) inverse** of A. If there exists C such that CA = I., then C is a **left (multiplicative) inverse** of A. If AB = BA = I, then B is a (**multiplicative) inverse** of A and we say that A is **invertible**. If B is the only matrix with the property that AB = BA = I, then B is the **inverse** of A. If A has a unique inverse, then we say A is **nonsingular** and denote its inverse by A^{-1}.

THEOREM #3. Th identity matrix is its own inverse.

Later (Chapter 9) we show that if A has a right and a left inverse, then it has a unique inverse. Hence we prove that A is invertible if and only if it is nonsingular. Even later, we show that if A has a right (or left) inverse, then it has a unique inverse. Thus, even though matrix multiplication is not commutative, a right inverse is always a left inverse and is indeed the inverse. Some matrices have inverses; others do not. Unfortunately, it is usually not easy to look at a matrix and determine whether or not it has a (multiplicative) inverse.

THEOREM #4. There exist A, B$\in\mathbf{R}^{n\times n}$ such that A$\neq$I is invertible and B has no inverse.

INVERSE OPERATION. If B has a right and left inverse then it is a unique inverse ((i.e., $\exists B^{-1}$ such that $B^{-1}B = BB^{-1} = I$) and we can define **Right Division** AB^{-1} and **Left Division** $B^{-1}A$ of A by B (provided B^{-1} exists). But since matrix multiplication is not commutative, we do not know that these are the same. Hence $\frac{A}{B}$ is not well defined since no indication of whether we mean left or right division is given.

EXERCISES on Matrix Algebra for Square Matrices

EXERCISE #1. True or False.

_____ 1. If A and B are square, then we can compute both AB and BA.

_____ 2. If n >1, then there exists A, $B \in \mathbf{R}^{n\times n}$ such that $AB \neq BA$.

_____ 3. Matrix multiplication is not commutative.

_____ 4. AB=BA is not an identity.

_____ 5. For square matrices, there is a multiplicative identity element, namely the n×n matrix I, given by

$$I_{nxn} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

_____ 6. $\forall A \in \mathbf{K}^{nxn}$ we have $A_{nxn} I_{nxn} = I_{nxn} A_{nxn} = A_{nxn}$

_____ 7. If there exists B such that AB = I., then B is a right (multiplicative) inverse of A.

_____ 8. If there exists C such that CA = I., then C is a left (multiplicative) inverse of A.

_____ 9. If AB = BA = I, then B is a multiplicative inverse of A and we say that A is invertible.

_____ 10. If B is the only matrix with the property that AB = BA = I, then B is the inverse of A.

_____ 11. If A has a unique inverse, then we say A is nonsingular and denote its inverse by A^{-1}.

_____ 12. The identity matrix is its own inverse.

_____ 13. If A has a right and a left inverse, then it has a unique inverse.

_____ 14. A is invertible if and only if it is nonsingular.

_____ 15. If A has a right (or left) inverse, then it has a unique inverse.

_____ 16. Even though matrix multiplication is not commutative, a right inverse is always a left inverse.

_____ 17. The inverse of a matrix is unique.

_____ 18. Some matrices have inverses; others do not.

_____ 19. It is usually not easy to look at a matrix and determine whether or not it has a (multiplicative) inverse.

_____ 20. There exist A, $B \in \mathbf{R}^{n\times n}$ such that $A \neq I$ is invertible and B has no inverse

EXERCISE #2. Let $\alpha = 2$, $A = \begin{bmatrix} 1+i & 1-i \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ i & 1+i \end{bmatrix}$. Compute the following:

$\overline{A}$ = ____________. A^T =_____________. A^* = _____________. αA = _____________.

A+B =_____________. AB =______________.

EXERCISE #3. Let $\alpha = 3$, $A = \begin{bmatrix} i & 1-i \\ 0 & 1+i \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ i & 1+i \end{bmatrix}$. Compute the following:

$\overline{A}$ = ____________. A^T =_____________. A^* = _____________. αA = _____________.

A+B =_____________. AB =______________.

EXERCISE #4. Solve $\underset{2x2}{A}\underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

EXERCISE #5. Solve $\underset{2x2}{A}\underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

EXERCISE #6 Solve $\underset{2x2}{A}\underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Recall that if **K** is a field and $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then the following properties of matrices are true.

THEOREM #1. Let $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**. Then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have

1) $A + (B + C) = (A + B) + C$ — associativity of matrix addition
2) $A + B = B + A$ — commutativity of matrix addition
3) There exists a unique matrix **O** such that for every matrix A ε $\mathbf{K}^{m\times n}$, $A + \mathbf{O} = A$.
4) For each A ε $\mathbf{K}^{m\times n}$, there exist a unique matrix called $-A$ such that $A + (-A) = \mathbf{O}$.

By using these properties, but without resorting to looking at the components of the matrices we can prove

THEOREM #2. Let $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**. Then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have

1) $\mathbf{O} + A = A$.
2) $(-B) + (A + B) = (A + B) + (-B) = A$
3) $-(A + B) = (-A) + (-B)$. The additive inverse of a sum is the sum of the additive inverses.
4) If $A + B = A + C$, then $B = C$ This is the cancellation law (for addition).

The proofs of properties 1) to 3) in Theorem #2 are easily proved using the standard method for proving identities in STATEMENT/REASON format. However none of the reasons rely on looking at the components of the matrices and hence do not rely directly on the properties of the underlying field. Now note that property 4) in Theorem #2 is not an identity. The conclusion is an equality, but it is a **conditional equality**. Although one could write a somewhat contorted proof of the concluding equality (B = C) by starting with one side and using the **substitution axiom of equality** to achieve the other side, a better proof is achieved by simply adding the same element to both sides of the given equation. Explanations of why the element to be added exists (Property 4) in Theorem #4) and why you can add the same element to both sides of an equation and the result is the same (axiom of equality) are needed.

The properties given in Theorem #1 establish $\mathbf{K}^{m\times n}$ as a **Abelian** (or **commutative**) **group**. Since only these properties are needed to prove Theorem #2, we see that any mathematical structure having these properties (i.e., any commutative group) also has the properties given in Theorem #2. We refer to both the defining properties given in Theorem #1 and the resulting properties given in Theorem #2 as **Abelian group properties**.

Now recall that if **K** is a field and $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then the following additional properties of matrices have been proved.

THEOREM #3. Let $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**. Then for all scalars α ß $\in$ **K** and A, B $\in\mathbf{K}^{m\times n}$, we have

1) $\alpha (ß A) = (\alpha ß) A$ (Note that $(\alpha ß)$ indicates multiplication in the field **K**.)
2) $(\alpha + ß) A = \alpha A + ß A$
3) $\alpha (A + B) = \alpha A + \alpha B$
4) $1 A = A$.

The properties in Theorem #4 below are easy to prove directly for matrices. However, they can also be proved by using the properties in Theorem #1 and Theorem #3 (and the properties in Theorem #2 since these can be proved using only Theorem #1), but without resorting to looking at the components of the matrices.

THEOREM #4. Let $\mathbf{K}^{m \times n}$ is the set of all matrices over **K**. Then for all scalars $\alpha \in \mathbf{K}$ and $A \in \mathbf{K}^{m \times n}$, we have

1) $0A = \mathbf{O}$,
2) $\alpha \mathbf{O} = \mathbf{O}$,
3) If $\alpha A = \mathbf{O}$, then either $\alpha = 0$ or $A = \mathbf{O}$.

The first two properties are identities and can be proved using the standard method. Some might argue that such proofs can become contorted and other methods may be clearer. The third is neither an identity nor a conditional equality. The hypothesis is an equality, but the conclusion states that there are only two possible reasons why the suppose equality could be true, either of the possibilities given in 1) or 2) (or both) but no others.

The properties given in Theorems #1 and #3 establish $\mathbf{K}^{m \times n}$ as a **vector space** (see Chapter 2-3) over **K**. Since only properties in Theorems #1 and #3 are needed to prove Theorems #2 and #4, we see that any mathematical structure having the properties given in Theorems #1 and #3 (i.e., any vector space) also has the properties given in Theorems #2 and 4. We refer to both the defining properties given in Theorems #1 and #3 and the resulting properties given in Theorems #2 and #4 as **vector space properties**. We consider additional vector spaces in the next chapter. Later, we consider more propertiess of (multiplicative) inverses of square matrices.

EXERCISES on Additional Matrix Properties

EXERCISE #1. True or False.

_____ 1.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have
$A + (B + C) = (A + B) + C$.

_____ 2.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have
the associativity of matrix addition.

_____ 3.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B $\in\mathbf{K}^{m\times n}$, we have
$A + B = B + A$.

_____ 4.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B $\in\mathbf{K}^{m\times n}$, we have
the commutativity of matrix addition.

_____ 5.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then we have that there exists a unique matrix **O**
such that for every matrix A ε $\mathbf{K}^{m\times n}$, $A + \mathbf{O} = A$.

_____ 6.If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then we have that for each A ε $\mathbf{K}^{m\times n}$, there exist
a unique matrix called $-A$ such that $A + (-A) = \mathbf{O}$.

_____ 7. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then we have $O + A = A$.

_____ 8. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then $(-B) + (A + B) = (A + B) + (-B) = A$

_____ 9. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have
$-(A + B) = (-A) + (-B)$.

_____ 10. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have that the
additive inverse of a sum is the sum of the additive inverses.

_____ 11. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all A, B, C $\in\mathbf{K}^{m\times n}$, we have that if
$A + B = A + C$, then $B = C$

_____ 12. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then $A + B = A + C$, then $B = C$ is called the
cancellation law for addition.

_____ 13. Many of the properties of matrices are identies and can be proved using the
STATEMENT/REASON format.

_____ 14. Some of the properties of matrices do not rely on looking at the components of the
matrices.

_____ 15. Some of the properties of matrices do not rely directly on the properties of the
underlying field.

_____ 16. Some of the properties of matrices are not identities.

_____ 17. Some of the properties of matrices are conditional equalities.

_____ 18. Some of the properties of matrices can be proved by starting with one side and using
the substitution axiom of equality to achieve the other side.

_____ 19. Sometimes a better proof of a matrix property can be obtained by simply adding the
same element to both sides of a given equation.

_____ 20. Sometimes a proofs of a matrix properties can be obtained by using properties of
equality.

_____ 21. A group is an abstract algebraic structure.

_____ 22. An Abelian group is an abstract algebraic structure.

_____ 23. A commutative group is an abstract algebraic structure.

______ 24. The set of all matrices over a field **K** is a group.
______ 25. The set of all matrices over a field **K** is an abelian group.
______ 26. The set of all matrices over a field **K** is a commutative group.
______ 27. The set of all matrices over a field **K** is a group.
______ 28. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all scalars α, $\beta \in$ **K** and $A \in \mathbf{K}^{m\times n}$, we have $\alpha(\beta A) = (\alpha\beta)A$
______ 29. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all scalars α, $\beta \in$ **K** and $A,B \in \mathbf{K}^{m\times n}$, we have $(\alpha + \beta) A = \alpha A + \beta A$
______ 30. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all scalars $\alpha \in$ **K** and $A,B \in \mathbf{K}^{m\times n}$, we have $\alpha(A + B) = \alpha A + \alpha B$
______ 31. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all $A \in \mathbf{K}^{m\times n}$, we have $1 A = A$.
______ 32. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all $A \in \mathbf{K}^{m\times n}$, we have $0A = O$.
______ 33. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all all scalars $\alpha \in$ **K** we have $\alpha O = O$.
______ 34. If $\mathbf{K}^{m\times n}$ is the set of all matrices over **K**, then for all scalars $\alpha \in$ **K** and $A \in \mathbf{K}^{m\times n}$, we have that if $\alpha A = O$, then either $\alpha = 0$ or $A = O$.
______ 35. The set of all matrices over **K** is a vector space.