A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 2

Elementary Matrix Algebra

Although usually studied together in a linear algebra course, **matrix algebra** can be studied separately from **linear algebraic equations** and **abstract linear algebra (Vector Space Theory)**. They are studied together since matrices are useful in the representation and solution of **linear algebraic equations** and yield important examples of the abstract algebraic structure called a **vector space**. Matrix algebra begins with the **definition of a matrix**. It is assumed that you have some previous exposure to matrices as an array of **scalars**. Scalars are elements of another abstract algebraic structure called a **field**. However, unless otherwise stated, the scalar entries in matrices can be assumed to be real or complex numbers (Halmos 1958,p.1). The definition of a matrix is followed closely by the definition of **basic algebraic operations** (computational procedures) which involve the scalar entries in the matrices (e.g., real or complex numbers) as well as possibly additional scalars. These include **unary**, **binary** or other types of operations.

Some basic operations which you may have previously seen are:

1. **Transpose** of a matrix.
2. **Complex Conjugate** of a matrix.
3. **Adjoint** of a matrix. (Not the same as the **Classical Adjoint** of a matrix.)
4. **Addition** of two matrices of a given size.
5. **Componentwise multiplication** of two matrices of a given size (not what is usually called multiplication of matrices).
6. Multiplication of a matrix (or a vector) by a scalar (**scalar multiplication**).
7. **Dot** (or **scalar**) **product** of two column vectors (or row vectors) to obtain a scalar. Also called an **inner product**. (Column and row vectors are one dimensional matrices.)
8. **Multiplication** (perhaps more correctly called composition) of two matrices with the right sizes (dimensions).

A **unary operation** is an operation which maps a matrix of a given size to another matrix of the same size. Unary operations are functions or mappings, but algebraists and geometers often think in terms of operating on an object to transform it into another object rather than in terms of mapping one object to another (already existing) object. Here the objects being mapped are not numbers, they are matrices. That is, you are given one matrix and asked to compute another. For square matrices, the first three operations, **transpose**, **complex conjugate**, and **adjoint**, are unary operations. Even if the matrices are not square, the operations are still **functions**.

A **binary operation** is an operation which maps two matrices of a given size to a third matrix of the same size. That is, instead of being given one matrix and being asked to compute another, you are given two matrices and asked to compute a third matrix. Algebraist think of combining or transforming two elements into a new element. **Addition** and **componentwise multiplication** (not what is usually called matrix multiplication), and **multiplication of square matrices** are examples of binary operations. Since on paper we write one element, the symbol for the binary operation, and then the second element, the operation may, but does not have to, depend on the order in which the elements are written. (As a mapping, the **domain** of a binary operation is a **cross product** of sets with elements that are **ordered pairs**.) A binary operation that does not depend on order is said to be **commutative**. Addition and componentwise multiplication are commutative; multiplication of square matrices is not.

This may be the first you have become aware that there are binary operations in mathematics that are not commutative. However, it is not the first time you have encountered a non-commutative binary operation. Instead of thinking of subtraction as the inverse operation of addition (which is effected by adding the additive inverse element), we may think of subtraction as a binary operation which is not commutative ($a - b$ is not usually equal to $b - a$). However, there is another fundamental binary operation on $\mathscr{F}(\mathbf{R},\mathbf{R})$ that is not commutative. If $f(x) = x^2$ and $g(x) = x + 2$, it is true that $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)\ \forall x\in\mathbf{R}$ so that $f+g = g+f$. Similarly, $(fg)(x) = f(x)\,g(x) = g(x)\,f(x) = (gf)(x)\forall x\in\mathbf{R}$ so that $fg = gf$. However, $(f\circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2$, but $(g\circ f)(x) = g(f(x)) = g(x^2) = x^2 + 2$ so that $f\circ g \neq g\circ f$. Hence the **composition of functions** is not commutative.

When multiplying a matrix by a scalar (i.e., **scalar multiplication**), you are given a matrix and a scalar, and asked to compute another matrix. When computing a **dot** (or **scalar**) **product**, you are given two column (or row) vectors and asked to compute a scalar.

EXERCISES on Introduction and Basic Operations

EXERCISE #1. True or False.

_____ 1. Computing the transpose of a matrix is a binary operation.
_____ 2. Multiplying two square matrices is a binary operation.
_____ 3. Computing the dot product of two column vectors is a binary operation.

Halmos, P. R.1958, *Finite Dimensional Vector Spaces* (Second Edition) Van Nostrand Reinhold Company, New York.

We begin with an informal definition of a matrix as an array of scalars. Thus an **m×n matrix** is an array of elements from an algebraic field **K** (e.g.,real or complex numbers) which has m rows and n columns. We use the notation $\mathbf{K}^{m\times n}$ for the set of all matrices over the field **K**. Thus, $\mathbf{R}^{3x4}$ is the set of all 3x4 real matrices and $\mathbf{C}^{2x3}$ is the set of all 2x3 matrices of complex numbers. We represent an arbitrary matrix over **K** by using nine elements:

$$\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [\, a_{ij} \,] \in \mathbf{K}^{mxn} \tag{1}$$

Recall that an **algebraic field** is an abstract algebra structure. **R** and **C** as well as **Q** are examples of algebraic fields. Nothing essential is lost if you think of the elements in the matrices as real or complex numbers in which case we refer to them as **real or complex matrices**. Parentheses as well as square brackets may be used to set off the array. However, the same notation should be used within a given discussion (e.g., a homework problem). That is, you must be consistent in your notation. Also, since vertical lines are used to denote the **determinant** of a matrix (it is assumed that you have had some exposure to determinants) they are not acceptable to denote matrices. (Points will be taken off if your notation is not consistent or if vertical lines are used for a matrix.) Since $\mathbf{K}^{mxn}$ for the set of all matrices over the field **K**, $\mathbf{R}^{mxn}$ is the set of all real matrices and $\mathbf{C}^{mxn}$ is the set of all complex matrices. In a similar manner to the way we embed **R** in **C** and think of the reals as a subset of the complex numbers, we embed $\mathbf{R}^{mxn}$ in $\mathbf{C}^{mxn}$ and, unless told otherwise, assume $\mathbf{R}^{mxn} \subseteq \mathbf{C}^{mxn}$. In fact, we may think of $\mathbf{C}^{mxn}$ as $\mathbf{R}^{mxn} + i\,\mathbf{R}^{mxn}$ so that every complex matrix has a real and imaginary part. Practically, we simply expand the set where we are allowed to choose the numbers $a_{i\,j}$, i = 1, ..., m and j = 1, ..., n.

The elements of the algebraic field, $a_{i\,j}$, i = 1, ..., m and j = 1, ..., n, (e.g., real or complex numbers) are called the **entries** in (or **components** of) A. To indicate that $a_{i\,j}$ is the element in A is the i^{th} row and j^{th} column we have also used the shorthand notation $A = [a_{ij}]$. We might also use $A = (a_{ij})$. The first subscript of $a_{i\,j}$ indicates the row and the second subscript indicates the column. This is particularly useful if there is a formula for $a_{i\,j}$. We say that the **size** or **dimension** of the matrix is m×n and use the notation $\underset{mxn}{A}$ to indicate that A is a matrix of size m×n. The entries in A will also be called **scalars**. For clarity, it is sometimes useful to separate the subscripts with a comma instead of a space. For example we might use $a_{12,2}$ and $a_{i+3,i+j}$ instead of $a_{12\;2}$ and $a_{i+3\;i+j}$.

EXAMPLES. $\underset{3x3}{A} = \begin{bmatrix} 1 & 3/2 & 4 \\ 7 & 2 & 6 \\ 1 & 8 & 0 \end{bmatrix}$ and $B = (1/3, \quad 2/5, \quad 3)$ are examples of rational matrices.

$\underset{3x3}{A} = \begin{bmatrix} \pi & 3 & 4 \\ 7 & 2 & 6 \\ 1 & 8 & 0 \end{bmatrix}$, and $\underset{3x1}{B} = \begin{bmatrix} 1 \\ 4 \\ e \end{bmatrix}$ are examples of real (non-rational) matrices.

$A = (1+i, \quad 2, \quad 3+2i)$, and $\underset{3x2}{B} = \begin{bmatrix} \pi + 2i & e+i \\ 3/4 + 3i & 1.5 \\ 2e & 7\pi \end{bmatrix}$ are examples of complex (non-real) matrices.

As was done above, commas (instead of just spaces) are often used to separate the entries in $1\times n$ matrices (often called row vectors). Although technically incorrect, this practice is acceptable since it provides clarity.

COLUMN AND ROW VECTORS. An $m\times 1$ matrix (e.g., $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$) is often called a **column vector**. We often use $[1,4,3]^T$ to denote this column vector to save space. The T stands for transpose as explained later. Similarly, a $1\times n$ matrix (e.g., [1, 4, 3]) is often called a **row vector**. This stems from their use in representing physical vector quantities such as position, velocity, and force. Because of the obvious isomorphisms, we will often use $\mathbf{R}^n$ for the set of real column vector $\mathbf{R}^{n\times 1}$ as well as for the set of real row vectors $\mathbf{R}^{1\times n}$. Similarly conventions hold for complex column and row vectors and indeed for $\mathbf{K}^n$, $\mathbf{K}^{n\times 1}$, and $\mathbf{K}^{1\times n}$. That is, we use the term "vector" even when n is greater than three and when the field is not $\mathbf{R}$. Thus we may use $\mathbf{K}^n$ for a finite sequence of elements in $\mathbf{K}$ of length n, no matter how we choose to write it.

EQUALITY OF MATRICES. We say that two matrices are **equal** if they are the same; that is, if they are identical. This means that they must be of the same size and have the same entries in corresponding locations. (We do not use the bracket and parenthesis notation within the same discussion. All matrices in a particular discussion must use the same notation.)

A FORMAL DEFINITION OF A MATRIX. Since $\mathbf{R}$ and $\mathbf{C}$ are examples of an (abstract) algebraic structure called a **field** (not to be confused with a vector field considered in vector analysis), we formally define a **matrix** as follows: An $m\times n$ **matrix** A over the (algebraic) field $\mathbf{K}$ (usually $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$) is a function from the set $D = \{1,...,. m\} \times \{1, ..., n\}$ to the set $\mathbf{K}$ (i.e., $A: D \rightarrow \mathbf{K}$). D is the **cross product** of the sets $\{1,...,. m\}$ and $\{1, ..., n\}$. Its elements are **ordered pairs** (i,j) where $i\in\{1,...,. m\}$ and $j\in\{1, ..., n\}$. The value of the function A at the point $(i.j) \in D$ is denoted by a_{ij} and is in the codomain of the function. The value (i.j) in the domain that corresponds to a_{ij} is indicated by the location of a_{ij} in the array. Thus the array gives the values of the function at each point in the domain D. Thus the set of m×n matrices over the field K is the set of functions $\mathscr{F}(D,\mathbf{K})$ which we normally denote by $\mathbf{K}^{m\times n}$. The set of real m×n matrices is $\mathbf{R}^{m\times n} = \mathscr{F}(D,\mathbf{R})$ and the set of complex m×n matrices is $\mathbf{C}^{m\times n} = \mathscr{F}(D,\mathbf{C})$.

EXERCISES on Definition of a Matrix

EXERCISE #1. Write the 2×3 matrices whose entries are:
(a) $a_{ij} = i+j$, (b) $a_{ij} = (3i + j^2)/2$, (c) $a_{ij} = [\pi i + j]$.

EXERCISE #2. Write the 3×1 matrices whose entries are given by the formulas in Exercise 1.

EXERCISE # 3. Choose x and y so that A = B if:

(a) $\underset{1\times 3}{A} = [x,2,3]$ and $\underset{1\times 3}{B} = [4,y,3]$,

(b) $\underset{2\times 2}{A_2} = \begin{bmatrix} 2 & x \\ 1 & 3 \end{bmatrix}$, $\underset{2\times 2}{B} = \begin{bmatrix} 2 & 3 \\ y & 3 \end{bmatrix}$,

(c) $\underset{1\times 2}{A} = [x,2]$, $B = [y,2]$,

(d) $\underset{1\times 2}{A} = [x,2]$, $B = [3,1]$.

Let **K** be an arbitrary algebraic field. (e.g. **K** = **Q**, **R**, or **C**). Nothing essential is lost if you think of **K** as **R** or **C**. Recall that we use $\mathbf{K}^{mxn}$ to denote the set of all matrices with entries in **K** that are of size (or dimension) mxn (i.e. having m rows and n columns). We first define the **transpose function** that can be applied to any matrix $A \in \mathbf{K}^{mxn}$. It maps $\mathbf{K}^{mxn}$ to $\mathbf{K}^{nxm}$. We then define the **(complex) conjugate function** from $\mathbf{C}^{mxn}$ to $\mathbf{C}^{mxn}$. Finally, we define the **adjoint function** (not the same as the classical adjoint) from $\mathbf{C}^{mxn}$ to $\mathbf{C}^{nxm}$. We denote an arbitrary matrix A in $\mathbf{K}^{mxn}$ by using nine elements.

$$\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [\, a_{ij} \,] \in \mathbf{K}^{mxn} \qquad (1)$$

<u>TRANSPOSE</u>.

<u>DEFINITION #1</u>. If $\underset{mxn}{A} = [\, a_{i\,j} \,] \in \mathbf{K}^{mxn}$, then $\underset{nxm}{A^T} = [\, \tilde{a}_{ij} \,] \in \mathbf{K}^{nxm}$ is defined by $\tilde{a}_{ij} = a_{j\,i}$ for i=1,...,n, j=1,...,m. A^T is called the **<u>transpose</u>** <u>of</u> <u>the</u> <u>matrix</u> <u>A</u>. (The transpose of A is obtained by exchanging its rows and columns.) The **transpose function** maps (or transforms) a matrix in $\mathbf{K}^{mxn}$ to (or into) a matrix in $\mathbf{K}^{nxm}$. Thus for the <u>transpose function</u> T, we have T: $\mathbf{K}^{mxn} \to \mathbf{K}^{nxm}$.

Note that we have not just defined one function, but rather an infinite number of functions, one for each set $\mathbf{K}^{mxn}$. We have named the function T, but we use A^T instead of T(A) for the element in the codomain $\mathbf{K}^{nxm}$ to which A is mapped. Note that, by convention, the notation $\underset{mxn}{A} = [\, a_{ij} \,]$ means that a_{ij} is the entry in the i^{th} row and j^{th} column of A: that is, unless otherwise specified, the **order of the subscripts** (rather than the choice of index variable) indicates the row and column of the entry. Hence $\underset{nxm}{A^T} \neq [\, a_{ji} \,]$.

EXAMPLE. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$. If $B = \begin{bmatrix} -1 & i \\ 3+i & 2i \\ 5 & 1+i \end{bmatrix}$, the $B^T = \begin{bmatrix} -1 & 3+i & 5 \\ i & 2i & 1+i \end{bmatrix}$. If A is given by (1) above, then $A^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdot & \cdot & \cdot & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdot & \cdot & \cdot & a_{n,2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{1,m} & a_{2,m} & \cdot & \cdot & \cdot & a_{n,m} \end{bmatrix}$. Note that this makes the rows and columns hard to follow.

PROPERTIES. There are lots of properties, particularly if **K** = **C** but they generally involve other operations not yet discussed. Hence we give only one at this time. We give its proof using the standard format for proving identities in the STATEMENT/REASON format. We represent matrices using nine components, the first , second, and last components in the first, second, and last rows. We refer to this as the **nine element method**. For many proofs involving matrix properties, the nine element method is sufficient to provide a convincing argument.

THEOREM #1. For all $A \in \mathbf{K}^{m\times n}$ = the set of all m×n complex valued matrices, $(A^T)^T = A$.

Proof: Let

$$\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [\,a_{ij}\,] \in \mathbf{K}^{mxn} \tag{1}$$

Then by the definition of the transpose of A we have

$$\underset{mxn}{A}^{T} = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdot & \cdot & \cdot & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdot & \cdot & \cdot & a_{m,2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{1,n} & a_{2,n} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix}_{nxm} = [\tilde{a}_{i,j}] \in \mathbf{K}^{nxm} \tag{2}$$

where $\tilde{a}_{ji,j} = a_{j,i}$. We now show $(A^T)^T = A$ using the standard procedure for proving identities in the STATEMENT/REASON format.

<u>STATEMENT</u>	<u>REASON</u>
$\left\{\underset{mxn}{A}^{T}\right\}^{T} = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,2} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix}^{T} \right\}^{T}$	Definition of A (using the nine element representation) (Notation)
$= \begin{bmatrix} a_{1,1} & a_{2,1} & \cdot & \cdot & \cdot & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdot & \cdot & \cdot & a_{m,2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{1,n} & a_{2,n} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix}^{T}$	Definition of transpose

$$= \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,2} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} \qquad \text{Definition of transpose}$$

$$= \quad A \qquad \text{Definition of A}$$

Since A was arbitrary, we have for all $A \in \mathbf{K}^{m\text{x}n}$ that $(A^T)^T = A$ as was to be proved.

Q.E.D.

Thus, if you take the transpose twice you get the original matrix back. This means that the function or mapping defined by the operation of taking the transpose is in some sense, its own inverse operation. (We call such functions or mappings **operators**.) If f: $\mathbf{K}^{m\times n} \to \mathbf{K}^{m\times n}$ is defined by $f(A) = A^T$, then $f^{-1}(B) = B^T$. However, $f \neq f^{-1}$ unless m=n. That is, taking the transpose of a square matrix is its own inverse function.

DEFINITION #2. If $A \in \mathbf{K}^{n\times n}$ (i.e. A is square) and $A = A^T$, then A is **symmetric**.

COMPLEX CONJUGATE.

DEFINITION #3. If $\underset{m\text{x}n}{A} = [\, a_{ij} \,] \in \mathbf{C}^{m\text{x}n}$, then $\underset{m\text{x}n}{\overline{A}} = [\, \overline{a}_{ij} \,] \in \mathbf{C}^{m\text{x}n}$. The **(complex) conjugate function** maps (or transforms) a matrix in $\mathbf{C}^{m\text{x}n}$ to (or into) a matrix in $\mathbf{C}^{m\text{x}n}$. Thus for the complex conjugate function C we have C: $\mathbf{C}^{m\text{x}n} \to \mathbf{C}^{m\text{x}n}$.

That is, the entry in the i^{th} row and j^{th} column is $\overline{a}_{ij}$ for 1,...,n, j=1,...,m. (i.e. the **complex conjugate** of A is obtained by taking the complex conjugate componentwise; that is, by taking the complex conjugate of each entry in the matrix.) Again we have not just defined one function, but rather an infinite number of functions, one for each set $\mathbf{K}^{m\text{x}n}$. We have named the function C, but we use $\overline{A}$ instead of C(A) for the element in the codomain $\mathbf{K}^{n\text{x}m}$ to which A is mapped. Note that unlike A^T there is no problem in defining $\overline{A}$ directly.

PROPERTIES. Again, there are lots of properties but most of them involve other operations not yet discussed. We give two at this time.

THEOREM #2. For all $A \in \mathbf{C}^{m\times n}$ = the set of all m×n complex valued matrices, $\overline{\overline{A}} = A$.

Thus if you take the complex conjugate twice you get the original matrix back. As with the transpose, taking the complex conjugate of a square matrix is its own inverse function.

THEOREM #3. $(\overline{A})^T = \overline{A^T}$.

That is, the operations of computing the complex conjugate and computing the transpose in some sense commute. That is, it does not matter which operation you do first. If A is square, then these two functions on $\mathbf{C}^{n\times n}$ do indeed commute in the technical sense of the word.

ADJOINT.

DEFINITION #4. If $\underset{m\times n}{A} = [\,a_{ij}\,] \in \mathbf{C}^{m\times n}$, then $\underset{m\times n}{A}^{*} = (\overline{A})^T \in \mathbf{C}^{n\times m}$. The **adjoint function** maps (or transforms) a matrix in $\mathbf{C}^{m\times n}$ to (or into) a matrix in $\mathbf{K}^{n\times m}$. Thus for the adjoint function *A*, we have , A: $\mathbf{C}^{m\times n} \rightarrow \mathbf{C}^{n\times m}$.

That is, the entry in the i^{th} row and j^{th} column is $\overline{a}_{ji}$ for 1,...,n, j=1,...,m. (i.e. the **adjoint** of A is obtained by taking the complex conjugate componentwise and then taking the transpose.) Again we have not just defined one function, but rather an infinite number of functions, one for each set $\mathbf{C}^{m\times n}$. We have named the function *A* (using italics to prevent confusion with the matrix A, but we use A* instead of *A*(A) for the element in the codomain $\mathbf{C}^{n\times m}$ to which A is mapped. . Note that having defined A^T and $\overline{A}$, A^* can be defined in terms of these two functions without having to directly refer to the entries of A.

PROPERTIES. Again, there are lots of properties but most of them involve other operations. Hence we give only one at this time.

THEOREM #4. $A^* = \overline{A^T}$.

That is, in computing A*, it does not matter whether you take the transpose or the complex conjugate first.

DEFINITION #4. If $A^* = A$, then A is said to be a **Hermitian matrix**.

THEOREM #5. If $A\in\mathbf{R}^{n\times n}$, then A is **symmetric** if and only if A is Hermitian.

Hermitian matrices (and hence real symmetric matrices) have nice properties; but more background is necessary before these can be discussed. Later, when we think of a matrix as (defining) an operator, then a Hermitian matrix is (defines) a self adjoint operator.

EXERCISES on Some Unary Matrix Operations

EXERCISE # 1. Compute the transpose of each of the following matrices

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ b) A = [1, ½] c) $A = \begin{bmatrix} 1 & i & 1+i \\ 2 & 2+i & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}$

e) $A=[\pi,2]$ f) $A = \begin{bmatrix} i \\ 2+i \\ 3 \end{bmatrix}$ g) $A = \begin{bmatrix} 2 & 2i \\ 0 & -1 \end{bmatrix}$ h) $\begin{bmatrix} 0 & 0 \\ 1+i & 2+i \\ 1 & 0 \end{bmatrix}$ i) $A = [i,e,\pi]$

EXERCISE # 2. Compute the (complex) conjugate of each of the matrices in Exercise 1 above.

EXERCISE # 3. Compute the adjoint of each of the matrices in Exercise 1 above.

EXERCISE # 4. The notation $B = [b_{i\,j}]$ defines the entries of an arbitrary m×n matrix B. Write out this matrix as an array using a nine component representation (i.e, as is done in Equation (1). Now write out the matrix $C = [c_{ij}]$ as an array. Now write out the matrix C^T as an array using the nine component method.

EXERCISE # 5. Write a proof of Theorem #2. Since it is an identity, you can use the standard form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using the nine component method. Then start with the left hand side and go to the right hand side, justifying every step in a STATEMENT/REASON format.

EXERCISE # 6. Write a proof of Theorem #3. Since it is an identity, you can use the standard form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using the nine component method. Then start with the left hand side and go to the right hand side, justifying every step.

DEFINITION #1. Let A and B be $m \times n$ matrices in $\mathbf{K}^{m\times n}$ defined by

$$\underset{m\times n}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [a_{ij}], \quad \underset{m\times n}{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdot & \cdot & \cdot & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdot & \cdot & \cdot & b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ b_{m,1} & b_{m,1} & \cdot & \cdot & \cdot & b_{m,n} \end{bmatrix} = [b_{ij}] \qquad (1)$$

We define the **sum** of A and B by

$$\underset{m\times n}{A} + \underset{m\times n}{B} = \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdot & \cdot & \cdot & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \cdot & \cdot & \cdot & a_{2,n}+b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1}+b_{m,1} & a_{m,1}+b_{m,1} & \cdot & \cdot & \cdot & a_{m,n}+b_{m,n} \end{bmatrix} \in \mathbf{K}^{mxn} \qquad (2)$$

that is, $A + B \equiv C \equiv [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1,\ldots, m$ and $j = 1,\ldots, n$.

We say that we add A and B componentwise. Note that it takes less space (but is not as graphic) to define the $i,j^{\underline{th}}$ component (i.e., c_{ij}) of $A + B = C$ then to write out the sum in rows and columns. Since m and n are arbitrary, in writing out the entries, we wrote only nine entries. However, these nine entries may give a more graphic depiction of a concept that helps with the visualization of matrix computations. 3×3 matrices are often used to represent physical quantities called **tensors**. Similar to referring to elements in $\mathbf{K}^n$, $\mathbf{K}^{n\times1}$, and $\mathbf{K}^{1\times n}$ as "vectors" even when $n>3$ and $\mathbf{K}\neq\mathbf{R}$, we will refer to $A = a_{ij}$ (without parentheses or brackets) as **tensor notation** for any matrix A even when A is not 3×3. Thus $c_{ij} = a_{ij} + b_{ij}$ defines the matrix c_{ij} as the sum of the matrices a_{ij} and b_{ij} using tensor notation. (The **Einstein summation convention** is discussed later, but it is **not assumed** unless so stated .)

As a mapping, + maps an element in $\mathbf{K}^{m\times n} \times \mathbf{K}^{m\times n}$ to an element in $\mathbf{K}^{m\times n}$, $+ : \mathbf{K}^{m\times n} \times \mathbf{K}^{m\times n} \to \mathbf{K}^{m\times n}$ where $\mathbf{K}^{m\times n} \times \mathbf{K}^{m\times n} = \{(A,B):A,B\in\mathbf{K}^{m\times n}\}$ is the **cross product** of $\mathbf{K}^{m\times n}$ with itself. The elements in $\mathbf{K}^{m\times n} \times \mathbf{K}^{m\times n}$ are **ordered pairs** of elements in $\mathbf{K}^{m\times n}$. Hence the order in which we add matrices could be important. Part b of Theorem 1 below establishes that it is not, so that matrix addition is **commutative**. Later we will see that multiplication of square matrices is a binary operation that is not commutative so that the order in which we multiply matrices is important.

PROPERTIES. Since the entries in a matrix are elements of a field, the field properties can be used to prove corresponding properties of matrix addition. These properties are then given the same names as the corresponding properties for fields.

THEOREM #1. Let A, B, and C be $m \times n$ matrices. Then

a. $\underset{mxn}{A} + (\underset{mxn}{B} + \underset{mxn}{C}) = (\underset{mxn}{A} + \underset{mxn}{B}) + \underset{mxn}{C}$ (matrix addition is **associative**) (3)

b. $\underset{mxn}{A} + \underset{mxn}{B} = \underset{mxn}{B} + \underset{mxn}{A}$ (matrix addition is **commutative**) (4)

THEOREM #2. There exists an $m \times n$ matrix **O** such that for all $m \times n$ matrices
$\underset{mxn}{A} + \underset{mxn}{\mathbf{O}} = \underset{mxn}{A}$ (existence of a **right additive identity** in $\mathbf{K}^{m\times n}$).

We call the right additive identity matrix the **zero matrix** since it can be shown to be the $m \times n$ matrix whose entries are all the zero element in the field **K**. (You may think of this element as the number zero in **R** or **C**.) To distinguish the zero matrix from the number zero we have denoted it by **O**; that is

$$\underset{mxn}{\mathbf{O}} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} = [\,0\,] \in \mathbf{K}^{mxn} \qquad (1)$$

THEOREM #3. The matrix $\mathbf{O} \in \mathbf{K}^{mxn}$ is the unique **additive identity element** in $\mathbf{K}^{mxn}$.

PARTIAL PROOF. The definition of and proof that **O** is an additive identity element is left to the exercises. We prove only that it is unique. To show that **O** is the unique element in $\mathbf{K}^{mxn}$ such that for all $A \in \mathbf{K}^{mxn}$, $A + \mathbf{O} = \mathbf{O} + A = A$, we assume that there is some other matrix, say $\Theta \in \mathbf{K}^{mxn}$ that has the same property, that is, for all $A \in \mathbf{K}^{mxn}$, $A + \Theta = \Theta + A = A$, and show that $\Theta = \mathbf{O}$. A proof can be written in several ways including showing that each element in Θ is the zero element of the field so that by the definition of equality of matrices we would have $\Theta = \mathbf{O}$. However, we can write a proof without having to consider the entries in Θ directly. Such a prove is considered to be more elegant. Although the conclusion, $\Theta = \mathbf{O}$ is an equality, it is not an identity. We will write the proof in the STATEMENT/REASON format, but not by starting with the left hand side. We start with a known identity.

STATEMENT	REASON
$\forall A \in \mathbf{K}^{mxn}$, $A = \mathbf{O} + A$	It is assumed that $\mathbf{O}$ has been established as an additive identity element for $\mathbf{K}^{mxn}$.
$\Theta = \mathbf{O} + \Theta$	Let $A = \Theta$ in this known identity.
$= \mathbf{O}$	Θ is assumed to be an identity so that for all $A \in \mathbf{K}^{mxn}$, $A + \Theta = \Theta + A = A$. This includes $A = \mathbf{O}$ so that $\mathbf{O} + \Theta = \mathbf{O}$.

Since we have shown that the assumed identity Θ must in fact be $\mathbf{O}$, (i.e., $\Theta = \mathbf{O}$), we have that $\mathbf{O}$ is the unique additive identity element for $\mathbf{K}^{mxn}$.

Q.E.D.

THEOREM #4. For every matrix $\underset{mxn}{A}$, there exists a matrix, call it $\underset{mxn}{B}$ such that

$\underset{mxn}{A} + \underset{mxn}{B} = \underset{mxn}{\mathbf{O}}$ (existence of a **right additive inverse**).

It is easily shown that B is the matrix containing the negatives of the entries in A. That is, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $b_{ij} = -a_{ij}$. Hence we denote B by $-A$. (This use of the minus sign is technically different from the multiplication of a matrix by the scalar) 1. Multiplication of a matrix by a scalar is defined later. Only then can we prove that these are the same.)

COROLLARY #5. For every matrix $\underset{mxn}{A}$ the unique right additive inverse matrix, call it $-\underset{mxn}{A}$, is

$-\underset{mxn}{A} = [-a_{ij}]$ (Computation of the unique right additive inverse).

THEOREM #6. For every matrix $\underset{mxn}{A}$, the matrix $-\underset{mxn}{A}$ has the properties

$\underset{mxn}{A} + (-\underset{mxn}{A}) = -\underset{mxn}{A} + \underset{mxn}{A} = \underset{mxn}{\mathbf{O}}$. (-A is a (right and left) **additive inverse**).

Furthermore, this right and left inverse is unique.

INVERSE OPERATION. We define subtraction of matrices by $A - B = A + (-B)$. Thus to compute A) B, we first find the additive inverse matrix for B (i.e., C =) B where if $C = [c_{ij}]$ and $B = [b_{ij}]$, then $c_{ij} =$) b_{ij}). Then we add A to C =) B. Computationally, if $A = [a_{ij}]$ and $B = [b_{ij}]$ then

$$\underset{m\mathrm{x}n}{A} - \underset{m\mathrm{x}n}{B} = \begin{bmatrix} a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} & \cdot & \cdot & \cdot & a_{1,n} - b_{1,n} \\ a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} & \cdot & \cdot & \cdot & a_{2,n} - b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} - b_{m,1} & a_{m,1} - b_{m,1} & \cdot & \cdot & \cdot & a_{m,n} - b_{m,n} \end{bmatrix} \in \mathbf{K}^{m\mathrm{x}n}$$

<u>THEOREM #7.</u> Let $A, B \in \mathbf{K}^{m\times n}$. Then $(A + B)^T = A^T + B^T$.

<u>THEOREM #8.</u> Let $A, B \in \mathbf{C}^{m\times n}$. Then the following hold:
1) $(\overline{A+B}) = \overline{A} + \overline{B}$
2) $(A + B)^* = A^* + B^*$

EXERCISES on Matrix Addition

<u>EXERCISE #1</u>. If possible, add A to B (i.e., find the sum A + B) if

a) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$ b) A = [1,2] , B = [1,2,3]

c) $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ 2 & 0 \\ 0 & 1-i \end{bmatrix}$ d) $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$, $B = \begin{bmatrix} 2+2\sqrt{2}i \\ 0 \\ 1-2\pi i \end{bmatrix}$

<u>EXERCISE #2</u>. Compute A+(B+C) (that is, first add B to C and then add A to the sum obtained) and (A+B)+C (that is, add A+B and then add this sum to C) and show that you get the same answer if

a) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1+i \\ 1 & 1-i \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ 2 & 0 \\ 0 & 1-i \end{bmatrix}$ $C = \begin{bmatrix} 2 & 2+i \\ 3e & 3 \\ 3-i & i \end{bmatrix}$,

$A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$, $B = \begin{bmatrix} 2+2\sqrt{2}i \\ 0 \\ 1-2\pi i \end{bmatrix}$ $C = \begin{bmatrix} 1+\sqrt{2}i \\ 0 \\ 2-\pi i \end{bmatrix}$

<u>EXERCISE #3</u>. Compute A+B (that is, add A to B) and B+A (that is, add B to A) and show that you get the same answer if:

a) $A = \begin{bmatrix} 1 & 1+i \\ 2 & 2-i \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2i \\ 3 & 1-i \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1+i \\ 2e & 3 \\ 1-i & i \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ 2 & 0 \\ 0 & 1-i \end{bmatrix}$ $A = \begin{bmatrix} 2+\sqrt{2}i \\ 0 \\ 1-\pi i \end{bmatrix}$, $B = \begin{bmatrix} 2+2\sqrt{2}i \\ 0 \\ 1-2\pi i \end{bmatrix}$

<u>EXERCISE #4</u>. Can you explain in one sentence why both commutativity and associativity hold for matrix addition? (Hint: They follow because of the corresponding properties for _________.)
(Fill in the blank)

Now elaborate.

EXERCISE #5. Find the additive inverse of A if

a) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$; b) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ c) A = [i,e,π]

EXERCISE # 6. Write a proof of Theorem #1. Since these are identities, use the standard form for writing proofs of identities. Begin by defining arbitrary matrices A and B. Represent them using at least the four corner entries. Then start with the left hand side and go to the right hand side, justifying every step.

EXERCISE # 7. Write a proof of Theorem #2. Since it is an identity, you can use the standard form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using at least the four corner entries. Then start with the left hand side and go to the right hand side, justifying every step.

EXERCISE # 8. Finish the proof of Theorem #3. Theorem #2 claims that **O** is a right additive identy, i.e., $\forall A \in \mathbf{K}^{m \times n}$ we have the identity A+**O** = A. Thus we can use the standard form for writing proofs of identities to show that O is also a left additive identity. Begin by defining the arbitrary matrix A. Represent it using at least the four corner entries. Then define the additive inverse element **O** using at least the four corners. Then start with the left hand side and go to the right hand side, justifying every step. Hence **O** is an additive identity. We have shown uniqueness so that **O** is the additive identity element.

EXERCISE # 9. Write a proof of Theorem #4. Since it is an identity, you can use the standard form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using at least the four corner entries. Then define the additive inverse element $B = -A = [-a_{ij}]$ using at least the four corners. Then start with the left hand side and go to the right hand side, justifying every step.

EXERCISE # 10. Write a proof of Corollary #5. You can use the standard form for writing proofs of identities to show that $-A$ is a right additive inverse. Then show uniqueness.

EXERCISE # 11. Write a proof of Theorem #6. Since it is an identity, you can use the standard form for writing proofs of identities.

EXERCISE # 12. Write a proof of Theorem #7. Since it is an identity, you can use the standard form for writing proofs of identities.

EXERCISE # 13. Write a proof of Theorem #8. Since these are identities, you can use the standard form for writing proofs of identities.

Componentwise multiplication is **not** what is usually called matrix multiplication. Although not usually developed in elementary matrix theory, it does not deserve to be ignored.

DEFINITION #1. Let A and B be $m \times n$ matrices in $\mathbf{K}^{m\times n}$ defined by

$$\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} = [a_{ij}], \quad \underset{mxn}{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdot & \cdot & \cdot & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdot & \cdot & \cdot & b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ b_{m,1} & b_{m,1} & \cdot & \cdot & \cdot & b_{m,n} \end{bmatrix} = [b_{ij}] \qquad (1)$$

Then we define the **componentwise product** of A and B by

$$\underset{mxn}{A} \otimes \underset{mxn}{B} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & \cdot & \cdot & \cdot & a_{1,n}b_{1,n} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & \cdot & \cdot & \cdot & a_{2,n}b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1}b_{m,1} & a_{m,1}b_{m,1} & \cdot & \cdot & \cdot & a_{m,n}b_{m,n} \end{bmatrix} \in \mathbf{K}^{mxn} \qquad (2)$$

that is, $A \otimes B \equiv C \equiv [c_{ij}]$ where $c_{ij} = a_{ij}\, b_{ij}$ for $i = 1,\dots, m$ and $j = 1,\dots, n$.

We say that we multiply A and B <u>componentwise</u>. Note that it takes less space (but is not as graphic) to define the $i,j^{\underline{th}}$ component (i.e., $c_{i\,j}$) of $C = A \otimes B$ then to write out the product in rows and columns. Even in writing out the entries we were terse since we wrote only nine entries of the array. Again, since this nine element technique often gives a more graphic depiction of a concept, it will often be used to help with the visualization of matrix computations.

<u>PROPERTIES</u>. Since the entries in a matrix are elements of a field, the field properties can be used to prove corresponding properties for componentwise matrix multiplication.

<u>THEOREM #1</u>. Let A, B, and C be $m \times n$ matrices. Then

a. $\underset{mxn}{A} \otimes (\underset{mxn}{B} \otimes \underset{mxn}{C}) = (\underset{mxn}{A} \otimes \underset{mxn}{B}) \otimes \underset{mxn}{C}$ (componentwise matrix mult. is associative) (3)

b. $\underset{mxn}{A} \otimes \underset{mxn}{B} = \underset{mxn}{B} \otimes \underset{mxn}{A}$ (componentwise matrix mult. is commutative) (4)

THEOREM #2. There exists a unique $m \times n$ matrix $\mathbf{1}$ such that for all $m \times n$ matrices

$$\underset{m\mathrm{x}n}{A} \otimes \underset{m\mathrm{x}n}{\mathbf{1}} = \underset{m\mathrm{x}n}{A} \otimes \underset{m\mathrm{x}n}{\mathbf{1}} = \underset{m\mathrm{x}n}{A}$$

(existence of a **componentwise multiplicative identity**).

We call the componentwise multiplicative identity matrix the **one matrix** since it can be shown to be the $m \times n$ matrix whose entries are all ones. To distinguish the one matrix from the number one we have denoted it by **1**; that is

$$\underset{m\mathrm{x}n}{\mathbf{1}} = \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \end{bmatrix} = [\,1\,] \in \mathbf{K}^{m\mathrm{x}n} \qquad (5)$$

THEOREM #3. For every matrix $\underset{m\mathrm{x}n}{A} = [a_{ij}]$ such that $a_{ij} \neq 0$ for $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, there exists a unique matrix, call it $\underset{m\mathrm{x}n}{B}$ such that

$$\underset{m\mathrm{x}n}{A} \otimes \underset{m\mathrm{x}n}{B} = \underset{m\mathrm{x}n}{B} \otimes \underset{m\mathrm{x}n}{A} = \underset{m\mathrm{x}n}{\mathbf{1}}$$

(existence of a unique **componentwise multiplicative inverse**).

It is easily shown that B is the matrix containing the multiplicative inverses of the entries in A. That is, if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $b_{ij} = 1/a_{ij}$. Hence we denote B by $1/A$.

INVERSE OPERATION. We define componentwise division of matrices by $A / B = A \otimes (1/B)$. Thus to compute A / B, we first find the componentwise multiplicative inverse matrix for B (i.e., $C = 1/B$ where if $C = [c_{ij}]$ and $B = [b_{ij}]$, then $c_{ij} = 1/b_{ij}$). Then we componentwise multiply A by $C = 1/B$. Computationally, if $A = [a_{ij}]$ and $B = [b_{ij}]$ then

$$\underset{m\mathrm{x}n}{A} / \underset{m\mathrm{x}n}{B} = \begin{bmatrix} a_{1,1}/b_{1,1} & a_{1,2}/b_{1,2} & \cdot & \cdot & \cdot & a_{1,n}/b_{1,n} \\ a_{2,1}/b_{2,1} & a_{2,2}/b_{2,2} & \cdot & \cdot & \cdot & a_{2,n}/b_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1}/b_{m,1} & a_{m,1}/b_{m,1} & \cdot & \cdot & \cdot & a_{m,n}/b_{m,n} \end{bmatrix} \in \mathbf{K}^{m\mathrm{x}n} \qquad (6)$$

EXERCISES on Componentwise Matrix Multiplication

<u>EXERCISE #1</u>. If possible, multiply componentwise A to B (i.e., find the componentwise product $A \circ_{cm} B$) if

a) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ b) A = [1,2] , B = [1,2,3]

c) $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & -3 \end{bmatrix}$ d) $\mathbf{A} = \begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1+\pi \\ 2e \\ 3+\sqrt{2} \end{bmatrix}$

<u>EXERCISE #2</u>. Compute $A \circ_{cm} (B \circ_{cm} C)$ (that is, first componentwise multiply B and C and then componentwise multiply A and the product obtained) and $(A \circ_{cm} B) \circ_{cm} C$ (that is, first componentwise multiply A and B and then componentwise multiply the product obtained with C) and show that you get the same answer if

a) $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -5 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

b) $\mathbf{A} = [\pi, 2]$, $\mathbf{B} = \left[1+3\sqrt{2}, \sqrt{2}\right]$, $\mathbf{C} = \left[0, 1+\sqrt{2}\right]$

<u>EXERCISE #3</u>. Compute $A \otimes B$ and $B \otimes A$ and show that you get the same answer if:

a) $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ b) $\mathbf{A} = \begin{bmatrix} 2 & 2i \\ 0 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$

<u>EXERCISE #4</u>. Can you explain in one sentence why commutativity and associativity hold for componentwise matrix multiplication? (Hint: They follow because of the corresponding properties for ____.)

(Fill in the blank, then elaborate)

<u>EXERCISE #5</u>. If possible, find the componentwise multiplicative inverse of A if

a) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$; b) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ c) $A = [i, e, \pi]$

<u>EXERCISE # 6</u>. Write a proof of Theorem #1. Since these are identities, use the standard form for writing proofs of identities. Begin by defining arbitrary matrices A and B. Represent them using at least the four corner entries. Then start with the left hand side and go to the right hand side, justifying every step.

<u>EXERCISE # 7</u>. Write a proof of Theorem #2. Since it is an identity, you can use the standard

form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using at least the four corner entries. Then start with the left hand side and go to the right hand side, justifying every step.

EXERCISE # 8. Write a proof of Theorem #3. Since it is an identity, you can use the standard form for writing proofs of identities. Begin by defining the arbitrary matrix A. Represent it using at least the four corner entries. Then define the additive inverse element $B = -A = [-a_{ij}]$ using at least the four corners. Then start with the left hand side and go to the right hand side, justifying every step.