A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES: A COLLECTION OF HANDOUTS FOR REVIEW AND PREVIEW OF LINEAR THEORY INCLUDING FUNDAMENTALS OF LINEAR ALGEBRA

CHAPTER 1

Review of Elementary Algebra and Geometry in One, Two, and Three Dimensions

1. Linear Algebra and Vector Analysis

2. Fundamental Properties of the Real Number System

3. Introduction to Abstract Algebraic Structures: An Algebraic Field

4. An Introduction to Vectors

5. Complex Numbers, Geometry, and Complex Functions of a Complex Variable

As engineers, scientists, and applied mathematicians, we are interested in geometry for two reasons:

1. The engineering and science problems that we are interested in solving involve time and space. We model time by **R** and (three dimensional) space using $\mathbf{R}^3$. However some problems can be idealized as one or two dimensional problems. i.e., using **R** or $\mathbf{R}^2$. We assume familiarity with **R** as a model for one dimensional space (Descartes) and with $\mathbf{R}^2$ as a model for two dimensional space (i.e., a plane). We assume less familiarity with $\mathbf{R}^3$ as a model of three space, but we do assume some.
2. Even if the state variables of interest are not position in space (e.g., amounts of chemicals in a reactor, voltages, and currents), we may wish to draw graphs in two or three dimensions that represent these quantities. Visualization, even of things that are not geometrical, may provide some understanding of the process.

It is this second reason (i.e., that the number of state variables in an engineering system of interest may be greater than three) that motivates us to consider "**vector spaces**" of dimension greater than three. When speaking of a system, the term "**degrees of freedom**" is also used, but we will usually use the more traditional term "**dimension**" in a problem solving setting where no particular application has been selected.

This leads to several definitions for the word "vector";

1. A physical quantity having magnitude and direction. (What your physics teacher told you.)
2. A directed line segment in the plane or in 3-space.
3. An element in $\mathbf{R}^2$ or $\mathbf{R}^3$ (i.e., an ordered pair or an ordered triple of real numbers).
4. An element in $\mathbf{R}^n$ (i.e., an ordered n-tuple of real numbers).
5. An element in $\mathbf{K}^n$ where **K** is a field (an abstract algebraic structure e.g., **Q**, **R** and **C**).
6. An element of a vector space (another abstract algebraic structure).

The above considerations lead to two separate topics concerning "vectors".

1. Linear Algebra.
2. Vector Analysis

In **Linear Algebra** a "vector" in an n dimensional **vector space** may represent n state variables and thus a system having n **degrees of freedom**. Although pictures of such vectors in one, two, and three dimensions may provide insight into the behavior of our system, these graphs need have no geometrical meaning. A system having n degrees of freedom and requiring n state variables resides in an n dimensional vector space which has no real geometrical meaning (e.g., amounts of n different chemicals in a reactor, n voltages or currents, tension in n different elements in a truss, flow rates in n different connected pipes, or amounts of money in n different accounts) . This does not diminish the usefulness of these vector spaces in solving engineering, science, and economic problems. A typical course in linear algebra covers topics in three areas:

1. Matrix Algebra.
2. Abstract Linear Algebra (Vector Space Theory).
3. How to Solve a System of m Linear Algebraic Equations in n Unknowns.

Although a model for the equilibrium or steady state of a linear system with n state variables usually has n equations in n unknowns, little is gained and much is lost by a restriction to n equations in n unknowns. Matrix algebra and abstract linear algebra (or vector space theory) provide the tools for understanding the methods used to solve m equations in n unknowns. The big point is that **no geometrical interpretation of a vector is needed to make vectors in linear algebra useful for solving engineering, scientific, and economic problems**. A geometrical interpretation is useful in one, two and three dimensions to help you understand what can happen and the "why's" of the solution procedure, but **no geometrical interpretation of a vector in "n" space is needed to make vectors in linear algebra useful for solving engineering, scientific, and economic problems**.

In **Vector Analysis** (Vector Calculus and Tensor Analysis) we are very much interested in the geometrical interpretation of our vectors (and tensors). Providing a mathematical model of geometrical concepts is the central theme of the subject. Interestingly, the "vector" spaces of interest from a linear algebra perspective are usually **infinite dimensional** as we are interested in physical quantities at **every point in physical space** and hence have an infinite number of degrees of freedom and hence an infinite number of state variables for the system. These are often represented using functions rather than n-tuples of numbers as well as "vector" valued functions (e.g., three ordered functions of time and space) . We divide our system models into two categories based on the number of variables needed to describe a state for our system:

1. **Discrete System Models** having only a finite number of state variables. These are sometimes referred to as **lumped parameter systems** in engineering books. Examples are electric circuits, systems of springs and "stirred" chemical reactors.
2. **Continuum System Models** having an infinite number of state variables in one , two, or three physical dimensions. These are sometimes referred to as **distributed parameter systems** in engineering books. Examples are Electromagnetic Theory (Maxwell's equations), Elasticity, and Fluid Flow (Navier-Stokes equations).

One last point. Although the static or equilibrium problem for a discrete or lumped parameter problem requires you to find values for n unknown variables, the dynamics problem requires values for these n variables for all time and hence requires an infinite number of values. Again, these are usually given by n functions of time or equivalently by an n-dimensional time varying "vector". (Note that here the term vector means n-tuple. When we use "vector" to mean n-tuple, rather than an element in a vector space, we will put it in quotation marks.)

Geometrically, the real numbers **R** can be put in one-to-one correspondence with a line in space (Descartes). We call this the **real number line** and view **R** as representing one dimensional space. We also view **R** as being a fundamental model for time. However, we wish to also view **R** mathematically as a set with algebraic and analytic properties. In this approach, we postulate the existence of the set **R** and use axioms to establish the fundamental properties. Other properties are then developed using a theorem/proof/definition format. This removes the need to view **R** geometrically or temporally and allows us to algebraically move beyond three dimensions. **Degrees of freedom** is perhaps a better term, but **dimension** is standard in mathematics and we will use it. We note that once a mathematical mode has been developed, although sometimes useful for intuitive understanding, **no geometrical interpretation of R is needed in the solution process for engineering, scientific, and economic problems**. We organize the fundamental properties of the real number system into several groups. The first three are the standard **axiomatic properties** of **R**:

1) The **algebraic** (**field**) properties. (An **algebraic field** is an abstract algebraic structure. Examples are **Q**, **R**, and **C**. However, **N** and **Z** are not)
2) The **order** properties. (These give **R** its one dimensional nature.)
3) The **least upper bound property**. (This leads to the **completeness property** that insures that **R** has no "holes" and may be the hardest property of **R** to understand. **R** and **C** are complete, but **Q** is not complete)

All other properties of the real numbers follow from these axiomatic properties. Being able to "see" the geometry of the real number line may help you to intuit other properties of **R**. However, this ability is not necessary to follow the axiomatic development of the properties of **R**, and can sometimes obscure the need for a proof of an "obvious" property.

We consider other properties that can be derived from the axiomatic ones. To solve **equations** we consider

4)Additional algebraic (field) properties of **R** including **factoring** of polynomials.

To solve **inequalities**, we consider:

5) Additional order properties including the definition and properties of the **order relations** $<$, $>$, $\leq$, and $\geq$.

A **vector space** is another abstract algebraic structure. Application problems are often formulated and solved using the algebraic properties of fields and vector spaces. **R** is not only a field but also a (one dimensional) vector space. The last two groups of properties of interest show that **R** is not only a field and a vector space but also an **inner product space** and hence a **normed linear space**:

6) The **inner product** properties and
7) The **norm** (**absolute value or length**) properties.

These properties are of interest since the notion of length (norm) implies the notion of **distance apart** (**metric**), and hence provides topological properties and allows for **approximate solutions**. They can be algebraically extended to two dimensions ($\mathbf{R}^2$), three dimensions ($\mathbf{R}^3$), n dimensions ($\mathbf{R}^n$), and even infinite dimensions (**Hilbert Space**). There are many applications for vector spaces of dimension greater than three (e.g., solving m linear algebraic equations in n unknown variables) and infinite dimensional spaces (e.g., solving differential equations). In these applications, the term dimension does not refer to space or time, but to the **degrees of freedom** that the problem has. A physical system may require a finite or infinite number of **state variables** to specify the state of the system. Hence it would aid our intuition to replace the term **dimension** with the term **degrees of freedom**. On the other hand (OTOH), just as the real number line aids (and sometimes confuses) our understanding of the real number system, vector spaces in one two, and three dimensions can aid (and confuse) our understanding of the **algebraic** properties of vector spaces.

ALGEBRAIC (FIELD) PROPERTIES. Since the set **R** of real numbers along with the **binary operations** of addition and multiplication satisfy the following properties, the system denoted by the 5-tuple ($\mathbf{R}$, +, $\cdot$.0,1) is an example of a **field** which is an abstract algebraic structure.

A1.	$x+y = y+x \quad \forall x,y \in \mathbf{R}$	Addition is Commutative
A2.	$(x+y)+z = x+(y+z) \quad \forall x,y\ z \in \mathbf{R}$	Addition is Associative
A3.	$\exists\ 0 \in \mathbf{R}$ s.t. $x+0 = x \quad \forall x \in \mathbf{R}$	Existence of a Right Additive Identity
A4.	$\forall x \in \mathbf{R}\ \exists\ w$ s.t. $x+w = 0$	Existence of a Right Additive Inverse
A5.	$xy = yx \quad \forall x,y \in \mathbf{R}$	Multiplication is Commutative
A6.	$(xy)z = x(yz) \quad \forall x,y\ z \in \mathbf{R}$	Multiplication is Association
A7.	$\exists\ 1 \in \mathbf{R}$ s.t. $x \cdot 1 = x \quad \forall x \in \mathbf{R}$	Existence of a Right Multiplicative Identity
A8.	$\forall x \in \mathbf{R}$ s.t. $x \neq 0\ \exists\ w \in \mathbf{R}$ s.t. $xw=1$	Existence of a Right Multiplicative Inverse
A9.	$x(y+z) = xy + xz \quad \forall x,y\ z \in \mathbf{R}$	Multiplication Distributes over Addition

There are other algebraic (field) properties which follow from these nine fundamental properties. Some of these additional properties (e.g., cancellation laws) are listed in high school algebra books and calculus texts. Check out the above properties with specific values of x, y, and z. For example, check out property A3 with $x = \pi$. Is $\pi + 0 = \pi$?

ORDER PROPERTIES. There exists the subset $P_{\mathbf{R}} = \mathbf{R}^+$ of positive real numbers that satisfies the following:

O1. $x,y \in P_{\mathbf{R}}$ implies $x+y \in P_{\mathbf{R}}$
O2. $x,y \in P_{\mathbf{R}}$ implies $xy \in P_{\mathbf{R}}$
O3. $x \in P_{\mathbf{R}}$ implies $-x \notin P_{\mathbf{R}}$
O4. $x \in \mathbf{R}$ implies exactly one of $x = 0$ or $x \in P_{\mathbf{R}}$ or $-x \in P_{\mathbf{R}}$ holds (trichotomy).

Note that the order properties involve the binary operations of addition and multiplication and are therefore linked to the field properties. There are other order properties which follow from the four fundamental properties. Some of these are listed in high school algebra books and calculus texts. The symbols <, >, $\leq$, and $\geq$ can be defined using the set $P_{\mathbf{R}}$. For example, $a \leq b$ if (and only if) $b-a \in P_{\mathbf{R}} \cup \{0\}$. The properties of these relations can then be established. Check out the above properties with specific values of x and y. For example, check out property O1 with $x = \sqrt{2}$ and $y = \pi$. Is $\sqrt{2} + \pi$ positive?

LEAST UPPER BOUND PROPERTY: The **least upper bound property** leads to the **completeness property** that assures us that the real number line has no holes. This is perhaps the most difficult concept to understand. It means that we must include the irrational numbers (e.g. π and $\sqrt{2}$) with the rational numbers (fractions) to obtain all of the real numbers.

DEFINITION. If S is a set of real numbers, we say that b is an **upper bound** for S if for each $x \in S$ we have $x \le b$. A number c is called a **least upper bound** for S if it is an upper bound for S and if $c \le b$ for each upper bound b of S.

LEAST UPPER BOUND AXIOM. Every nonempty set S of real numbers that has an upper bound has a least upper bound.

Check this property out with the set $S = \{x \in \mathbf{R}: \ x^2 < 2\}$. Give three different upper bounds for the set S. What real number is the least upper bound for the set S? Is it in the set S?

ADDITIONAL ALGEBRAIC PROPERTIES AND FACTORING. Additional algebraic properties are useful in solving for the **roots** of the **equation** $f(x) = 0$ (i.e., the **zeros** of f) where f is a real valued function of a real variable. Operations on equations can be defined which result in **equivalent equations** (i.e., ones with the same solution set). These are called **equivalent equation operations** (EEO's). An important property of **R** states that if the product of two real numbers is zero, then at least one of them is zero. Thus if f is a polynomial that can be factored so that $f(x) = g(x)\,h(x) = 0$, then either $g(x) = 0$ or $h(x) = 0$ (or both since in logic we use the **inclusive or**). Since the degrees of g and h will both be less than that of f, this reduces a hard problem to two easier ones. When we can repeat the process to obtain a product of linear factors, we can obtain all of the zeros of f (i.e., the roots of the equation).

ADDITIONAL ORDER PROPERTIES. Often application problems require the solution of inequalities. The symbols $<$, $>$, $\le$, and $\ge$ can be defined in terms of the set of positive numbers $P_{\mathbf{R}} = \mathbf{R}^+ = \{x \in \mathbf{R}: x>0\}$ and the binary operation of addition. As with equalities, operations on inequalities can be developed that result in **equivalent inequalities** (i.e., ones that have the same solution set). These are called **equivalent inequality operations** (EIO's).

ADDITIONAL PROPERTIES. There are many additional properties of **R**. We consider here only the important absolute value property which is basically a function.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

This function provides a norm, a metric, and a topology for **R**. Thus we can define limits, derivatives and integrals over intervals in **R**.

To introduce the notion of an **abstract algebraic structure** we consider (algebraic) **fields**. (These should not to be confused with vector and scalar fields in vector analysis.) Loosely, an algebraic field is a number system where you can add, subtract, multiply, and divide (except for dividing by zero of course). Examples of fields are **Q**, **R**, and **C**. Examples of other abstract algebraic structures are: Groups, Rings, Integral Domains, Modules and Vector Spaces.

DEFINITION. Let F be a set (of numbers) together with two **binary operations** (which we call addition and multiplication), denoted by + and · (or juxtaposition), which satisfy the following list of properties.:

F1) $\forall$ x,y $\in$ F	$x + y = y + x$	Addition is commutative
F2) $\forall$ x,y $\in$ F	$x + (y + z) = (x + y) + z$	Addition is associative
F3) $\exists$ an element $0 \in F$ such that. $\forall$ x $\in$ F, $x + 0 = x$		Existence of a right additive identity
F4) $\forall$ x $\in$ F, $\exists$ a unique $y \in F$ s.t. $x + y = 0$ We usually denote y by -x for each x in F.		Existence of a right additive inverse for each element
F5) $xy = yx$ $\forall$ x,y $\in$ F		Multiplication is commutative
F6) $x(yz) = (xy)z$ $\forall$ x,y,z $\in$ F		Multiplication is associative
F7) $\exists$ an element $1 \in F$ such that $1 \neq 0$ and $\forall$ x $\in$ F, $x1 = x$		Existence of a right multiplicative identity
F8) $\forall$ x s.t. $x \neq 0$, $\exists$ a unique $y \in F$ s.t. $xy = 1$ We usually denote y by (x^{-1}) or $(1/x)$ for each nonzero x in F.		Existence of a right multiplicative inverse for each element except 0
F9) $x(y + z) = xy + xz$ $\forall$ x,y,z $\in$ F		(Multiplication distributes over addition)

Then the ordered 5-tuple consisting of the set F and the structure defined by the two operations of addition and multiplication as well as two identity elements mentioned in the definition, $K = (F,+,\cdot,0,1)$, is an algebraic **field**.

Although technically not correct, we often refer to the set F as the (algebraic) field. The elements of a field (i.e. the elements in the set F) are often called **scalars**. Since the letter F is used a lot in mathematics, the letter K is also used for a field of scalars. Since the rational numbers **Q**, the real numbers **R**, and the complex numbers **C** are examples of fields, it will be convenient to use the notation **K** to represent (the set of elements for) any field $K = (\mathbf{K},+,\cdot,0,1)$.

The properties in the definition of a field constitute the fundamental axioms for **field theory**. Other field properties can be proved based only on these properties. Once we proved that **Q**, **R**, and **C** are fields (or believe that someone else has proved that they are) then, by the **principle of abstraction**, we need not prove these properties for these fields individually, one proof has done the work of three. In fact, for every (concrete) structure that we can establish as a field, all of the field properties apply.

We prove an easy property of fields. As a **field property**, this property must hold in every algebraic field. It is an identity and we use the standard form for proving identities.

THEOREM #1. Let $K = (\mathbf{K},+,\cdot,0,1)$ be a field.. Then $\forall x \in \mathbf{K}$, $0 + x = x$. (In English, this property states that the right identity element 0 established in F3 is also a left additive identity element.)
Proof. Let $x \in \mathbf{K}$. Then

STATEMENT	REASON
$0 + x = x + 0$	F1. Addition is commutative
$= x$	F3 0 is a rightr additive identity element. (and properties of equality)

Hence $\forall x \in \mathbf{K}$, $0 + x = x$.

Q.E.D.

Although this property appears "obvious" (and it is) a formal proof can be written. This is why mathematicians may disagree about what axioms to select, but once the selection is made, they rarely disagree about what follows logically from these axioms. Actually, the first four properties establish a field with addition as an **Abelian (or commutative) group** (another abstract algebraic structure). **Group Theory**, particularly for Abelian groups have been well studied by mathematicians. Also, if we delete 0, then the nonzero elements of **K** along with multiplication also form an Abelian group. We list some (Abelian) group theory properties as they apply to fields. Some are identities, some are not.

THEOREM #2. Let $K = (\mathbf{K},+,\cdot,0,1)$ be a field. Then

1. The identity elements 0 and 1 are unique.
2. For each nonzero element in **K**, its additive and multiplicative inverse element is unique.
3. 0 is its own additive inverse element (i.e., $0 + 0 = 0$) and it is unique, but it has no multiplicative inverse element.
4. The additive inverse of an additive inverse element is the element itself. (i.e., if $-a$ is the additive inverse of a, then $-(-a) = a$).
5. $-(a + b) = -a + -b$. (i.e., the additive inverse of a sum is the sum of their additive inverses.)
6. The multiplicative inverse of a multiplicative inverse element is the element itself. (i.e., if a^{-1} is the multiplicative inverse of a, then $(a^{-1})^{-1} = a$).
7. $(a\,b)^{-1} = a^{-1}\,b^{-1}$. (i.e., the multiplicative inverse of a product is the product of their multiplicative inverses.)
8. Sums and products can be written in any order you wish.
9. If $a + b = a + c$, then $b = c$. (Left Cancellation Law for Addition)
10. If $ab = ac$ and $a \neq 0$, then $b = c$. (Left Cancellation Law for Multiplication.)

Traditionally the concept of a **vector** is introduced by physicists as a quantity such as force or velocity that has magnitude and direction. Vectors are then represented geometrically by **directed line segments** and are thought of geometrically. Modern mathematical treatments introduce vectors algebraically as elements in a **vector space**. As an introductory compromise we introduce two dimensional vectors algebraically and then examine the correspondence between algebraic vectors and directed line segments in the plane. To define a **vector space** algebraically, we need a set of <u>vectors</u> and a set of <u>scalars</u>. We also need to define the algebraic operations of **vectors addition** and **scalar multiplication**. Since we associate the analytic set $\mathbf{R}^2 = \{(x,y): x,y \in \mathbf{R}\}$ with the geometric set of <u>points in a plane</u>, we will use another notation, $\mathbb{R}^2$, for the set of <u>two dimensional **algebraic** vectors</u>.

<u>SCALARS AND VECTORS</u>. We define our set of scalars to be the set of real numbers **R** and our set of vectors to be the set of ordered pairs $\mathbb{R}^2 = \{[x,y]^T: x,y \in \mathbf{R}\}$. We use the matrix notation $[x,y]^T$ (i.e.[x,y] transpose, see Chapter 2) to indicate a **column vector** to save space. (We could use row vectors, [x,y], but when we solve liear algebraqic equations we prefer column vectors.) When writing homework papers it is better to use column vectors explicitly. We write $\vec{x} = [x,y]^T$ and refer to x and y as the **components** of the vector $\vec{x}$.

<u>VECTOR ADDITION</u>. We define the **sum** of the vectors $\vec{x}_1 = [x_1,y_1]^T$ and $\vec{x}_2 = [x_2,y_2]^T$ as the vector $\vec{x}_1 + \vec{x}_2 = [x_1 + x_2, y_1 + y_2]^T$. For example, $[2,1]^T + [5,-2]^T = [7,-1]^T$. Thus we add vectors <u>component wise</u>.

<u>SCALAR MULTIPLICATION</u>. We define **scalar multiplication** of a vector $\vec{x} = [x,y]^T$ by a scalar $\alpha \in \mathbf{R}$ by $\alpha\,\vec{x} = [\alpha\, x, \alpha\, y]^T$. For example, $3[2,-1]^T = [6,-3]^T$. Thus we multiply each component in $\vec{x}$ by the scalar α. We define $\hat{i} = [1, 0]^T$ and $\hat{j} = [0, 1]^T$ so that every vector $\vec{x} = [x, y]^T$ may be written uniquely as $\vec{x} = x\,\hat{i} + y\,\hat{j}$.

<u>GEOMETRICAL INTERPRETATION</u>. Recall that we associate the analytic set $\mathbf{R}^2 = \{(x,y): x,y \in \mathbf{R}\}$ with the geometric set of points in a plane. Temporarily, we use $\tilde{x} = (x,y)$ to denote a point in $\mathbf{R}^2$. We might say that the points in a plane are a geometric interpretation of $\mathbf{R}^2$. We can establish a **one-to-one correspondence** between the analytic set $\mathbf{R}^2$ and <u>geometrical vectors</u> (**directed line segments**). First consider only **directed line segments** which are **position vectors**; that is, have their "tails" at the origin (i.e. at the point $\tilde{0} = (0,0)$ and their heads at some other point, say, the point $\tilde{x} = (x,y)$ in the plane $\mathbf{R}^2 = \{(x,y): x, y \in \mathbf{R}\}$. Denote this set by **G**.

Position vectors are said to be "based at $\vec{0}$". If $\tilde{x} \in \mathbf{R}^2$ is a point in the plane, then we let $\overrightarrow{0\,x}$ denote the position vector from the origin $\tilde{0}$ to the point $\tilde{x}$. "Clearly" there exist a <u>one-to-one correspondence</u> between **G** and the set $\mathbf{R}^2$ of points in the plane; that is, we can readily identify exactly one vector in **G** with exactly one point in $\mathbf{R}^2$. Now "clearly" a <u>one-to-one correspondence</u> also exists between the set of <u>points</u> in $\mathbf{R}^2$ and the set of <u>algebraic vectors</u> in $\mathbb{R}^2$. Hence a <u>one-to-one correspondence</u> exists between the set of algebraic vectors $\mathbb{R}^2$ and the set of geometric vectors **G**. In a traditional treatment, <u>vector addition</u> and <u>scalar multiplication</u> are defined <u>geometrically</u> for <u>directed line segments</u> in **G**. We must then prove that the <u>geometric definition</u> of the addition of two directed line segments using the **parallelogram law** corresponds to algebraic addition of the corresponding vectors in $\mathbb{R}^2$. Similarly for scalar multiplication. We say that we must prove that the two structures are **isomorphic**. It is somewhat simpler to define vector addition and scalar multiplication algebraically on $\mathbb{R}^2$ and think of **G** as a geometric interpretation of the two dimensional vectors. One can then develop the **theory of vectors** without getting bogged down in the geometric proofs to establish the isomorphism. We can extend this isomorphism to $\mathbf{R}^2$. However, although we readily accept adding geometric vectors in **G** and will accept with coaxing adding algebraic vectors in $\mathbb{R}^2$, we may baulk at the idea of adding points in $\mathbf{R}^2$. However, the distinction between these sets with structure really just amounts to an interpretation of ordered pairs rather than an inherent difference in the mathematical objects being considered. Hence from now on, for any of these we will use the symbol $\mathbf{R}^2$ unless there is a philosophical reason to make the distinction. Reviewing, we have

$$\mathbb{R}^2 \cong \mathbf{G} \cong \mathbf{R}^2$$

where we have used the symbol $\cong$ to denote that there is an isomorphism between these sets with structure.

MAGNITUDE AND DIRECTION. Normally we think of directed line segments as being "free" vectors; that is, we identify any directed line segment in the plane with the directed line segment in **G** which has the same **magnitude** (length) and **direction**. The magnitude of the algebraic vector $\vec{x} = [x,y]^T \in \mathbb{R}^2$ is defined to be $||\vec{x}|| = \sqrt{x^2 + y^2}$ which, using Pythagoras, is the length of the directed line segment in **G** which is associated with $\vec{x}$. It is easy to show that the property (i.e. prove the theorem) $||\alpha\,\vec{x}|| = |\alpha|\;||\vec{x}||$ holds for all $\vec{x} \in \mathbb{R}^2$ and all scalars α. It is also easy to show that if $\vec{x} \neq 0$ the vector $\vec{x}\,/||\vec{x}|| = (1/||\vec{x}||)\,\vec{x}$ has magnitude equal to one (i.e. is a **unit vector**). Examples of unit vectors are $\hat{i} = [1, 0]^T$ and $\hat{j} = [0, 1]^T$. "Obviously" any nonzero vector $\vec{x} = [x,y]^T$ can be written as $\vec{x} = ||\vec{x}||\;\hat{u}$ where $\hat{u} = \vec{x}\,/||\vec{x}||$ is a unit vector in the same direction as $\vec{x}$. That is, $\hat{u}$ gives the direction of $\vec{x}$ and $||\vec{x}||$ gives its magnitude. For example, $\vec{x} = [3, 4]^T = 5\,[3/5, 4/5]^T$ where $\hat{u} = [3/5, 4/5]^T$ and $||\vec{x}|| = 5$. To make vectors in $\mathbb{R}^2$ more applicable to two dimensional geometry we can introduce the concept of an **equivalence relation** and **equivalence classes**. We say that an arbitrary directed line segment in the plane is **equivalent** to a geometrical vector in **G** if it has the same direction and magnitude. The set of all directed line segments equivalent to a given vector in **G** forms an equivalence class. Two directed line segments are **related** if they are in the same equivalence classes. This relation is called an **equivalence relation** since all directed line segments that are related can be thought of as being the same. The equivalence classes partition the set of all directed line segments into sets that are mutually exclusive whose union is all of the directed line segments.

VECTORS IN $\mathbf{R}^3$. Having established an isomorphism between $\mathbf{R}^2$, $\mathbb{R}^2$, and G, we make no distinction in the future and will usually use $\mathbf{R}^2$ for the set of vectors in the plane, the set of points in the plane and the set of geometrical vectors. The context will explain what is meant. A similar development can be done algebraically and geometrically for vectors in 3-space. There is a technical difference between the sets $\mathbf{R}\times\mathbf{R}\times\mathbf{R} = \{(x,y,z): x, y, z\in\mathbf{R}\}$, $\mathbf{R}^2\times\mathbf{R} = \{((x,y),z): x, y, z\in\mathbf{R}\}$, and $\mathbf{R}\times\mathbf{R}^2 = \{(x,(y,z)): x, y, z\in\mathbf{R}\}$, but they are all isomorphic and we will usually consider them to be the same and denote all three by $\mathbf{R}^3$. Furthermore, similar to 2 dimensions, we will use $\mathbf{R}^3$ to represent: the set of **ordered triples**, the set of points in 3 dimensional space, the set of 3 dimensional position vectors given by $\vec{x} = [x,y,z]^T = x\hat{i} + y\hat{j} + z\hat{k}$ where $\hat{i} = [1, 0, 0]^T$, $\hat{j} = [0, 1, 0]^T$, and $\hat{k} = [0, 0, 1]^T$, and the set of any three dimensional geometric vectors denoted by directed line segments.

The same analytic (but not geometric) development can be done for 4,5,..., n to obtain the set $\mathbf{R}^n$ of n-dimensional vectors. Again, no "real" geometrical interpretation is available. This does not diminish the usefulness of **n-dimensional space** for engineering since, for example, it plays a central role in the theory behind solving <u>linear algebraic equations</u> (m equations in n unknowns) and the theory behind any system which has n <u>state variables</u> (e.g., springs, circuits, and stirred tank reactors). The important thing to remember is that, although we may use geometric language for $\mathbf{R}^n$, we are doing algebra (and/or analysis), and not geometry.

<u>PROPERTIES THAT CAN BE EXTENDED TO HIGHER DIMENSIONS</u>. There are additional <u>algebraic, analytic, and topological properties</u> for $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$ that deserve to be listed since they are easily extended from $\mathbf{R}$, $\mathbf{R}^2$,$\mathbf{R}^3$ to $\mathbf{R}^4$,..., $\mathbf{R}^n$,... and, indeed to the (countably) infinite dimensional <u>Hilbert space</u>. We list these in four separate categories: inner product, norm, metric, and topology. Topologists may start with the definition of a topology as a collection of open sets. However, we are only interested in topologies that come from metrics which come from norms.

<u>INNER (DOT) PRODUCT PROPERTIES</u>. For $x_1, x_2 \in \mathbf{R}$, define $<x_1,x_2> = x_1x_2$. For $\vec{x}_1 = [x_1,y_1]^T = x_1\hat{i} + y_1\hat{j}$, $\vec{x}_2 = [x_2,y_2]^T = x_2\hat{i} + y_2\hat{j} \in \mathbf{R}^2$, define $<\vec{x}_1,\vec{x}_2> = x_1x_2 + y_1y_2$. For for $\vec{x}_1 = [x_1,y_1,z_1]^T = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{x}_2 = [x_2,y_2,z_2]^T = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \in \mathbf{R}^3$ define $<\vec{x}_1,\vec{x}_2> = x_1x_2 + y_1y_2 + z_1z_2$. Then in $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$ and, indeed, in any (real) inner product space V, we have the following (abstract) properties of an inner product:

IP1) $<\vec{x},\vec{y}> = <\vec{y},\vec{x}> \quad \forall \vec{x},\vec{y} \in V$

IP2) $<\alpha\,\vec{x} + \beta\,\vec{y},\vec{z}> = \alpha<\vec{x},\vec{z}> + \beta<\vec{y},\vec{z}> \quad \forall x,y,z \in V, \quad \forall \alpha,\beta \in \mathbf{R}$

IP3) $<\vec{x},\vec{x}> \; > 0 \;$ if $\vec{x} \neq \vec{0}$

$<\vec{x},\vec{x}> \; = 0 \;$ if $\vec{x} = \vec{0}$

Check these properties out using specific values of x, y, z, α, β. For example, check out property IP2 with $\alpha = 2$, $\beta = 4$, $x = 1$, $y = 3$ and $z = 5$. First calculate the left hand side (LHS), then the right hand side (RHS). Are they the same? Is the property true for these values? Now do the same when $\vec{x}_1 = [2,3]^T = 2\hat{i} + 3\hat{j}$, $\vec{x}_2 = [1,4]^T = \hat{i} + 4\hat{j}$ and $\alpha = 2$ and $\beta = 3$. Also try some three dimensional vectors.

The inner or dot product of two vectors in two and three dimensions is more often denoted by $\vec{x}\cdot\vec{y}$. We have used the "physics" notation $<\vec{x},\vec{y}>$ in 2 and 3 dimensions since (x,y) is used to denote the coordinates of a point in the plane $\mathbf{R}^2$ and (x,y,z) for a point in 3 space. Note that we do not use arrows over the "vectors" when we are in one dimension. In $\mathbf{R}^2$ and $\mathbf{R}^3$, the geometric definition of inner (dot) product is $<\vec{x}_1,\vec{x}_2> = \|\vec{x}_1\|\|\vec{x}_2\|\cos\theta$ where $\|\cdot\|$ is the norm (magnitude) (see below) of the vector and θ is the angle between the two vectors. In $\mathbf{R}^n$, the definition of the inner product is $(\vec{x},\vec{y}) = \sum_{i=1}^{n} x_i y_i$ where $\vec{x} = (x_1, x_2, ..., x_n) = \{x_i\}_{i=1}^{n}$ and

$\vec{y} = \{y_i\}_{i=1}^{n}$. Note we have moved to algebraic notation (no geometric interpretation is available).

NORM (ABSOLUTE VALUE OR LENGTH) PROPERTIES. For $x \in \mathbf{R}$, define $\|x\| = \sqrt{<x,x>} = \sqrt{x \cdot x} = \sqrt{x^2} = |x|$, i.e., the absolute value of the number. For $\vec{x} = [x,y]^T = x\,\hat{i} + y\,\hat{j} \in \mathbf{R}^2$, define $\|\vec{x}\| = \sqrt{x^2 + y^2}$. For for $\vec{x} = [x,y,z]^T = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ define $\|\vec{x}\| = \sqrt{x^2 + y^2 + z^2}$. Then in $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$ and, indeed, in any normed linear space V, we have the following (abstract) properties of a norm:

N1) $\|\vec{x}\| > 0$ if $\vec{x} \neq \vec{0}$

$\|\vec{x}\| = 0$ if $\vec{x} = \vec{0}$ = 0 if x = 0

N2) $\|\alpha\,\vec{x}\| = |\alpha|\,\|\vec{x}\| \quad \forall \vec{x} \in V, \quad \forall \alpha \in \mathbf{R}$

N3) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad \forall \vec{x}, \vec{y} \in V \qquad \forall\, x, y \in \mathbf{R}$ (Triangle Inequality)

Check these properties out using specific values of x and α. For example check out property 2 with α = -3 and x = 2. That is, calculate the left hand side (LHS), then the right hand side (RHS). Are they the same? Is the property true for these values? Now check out these properties for two and three dimensional vectors.

In $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$, the norm is the geometric magnitude or length of the vector. In $\mathbf{R}^n$, the norm of the vector $\vec{x} = \{x_i\}_{i=1}^{n}$ is defined to be $\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^{\,2}}$.

METRIC. For $x_1, x_2 \in \mathbf{R}$, define $\rho(x_1,x_2) = |x_1 - x_2|$. For $\vec{x}_1 = [x_1,y_1]^T = x_1\,\hat{i} + y_1\,\hat{j}$, $\vec{x}_2 = [x_2,y_2]^T = x_2\,\hat{i} + y_2\,\hat{j} \in \mathbf{R}^2$, define $\rho(\vec{x}_1,\vec{x}_2) = \|\vec{x}_1 - \vec{x}_2\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. For $\vec{x}_1 = [x_1,y_1,z_1]^T = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$, $\vec{x}_2 = [x_2,y_2,z_2]^T = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k} \in \mathbf{R}^3$ define $\rho(\vec{x}_1,\vec{x}_2) = \|\vec{x}_1 - \vec{x}_2\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$. Then in $\mathbf{R}$, $\mathbf{R}^2$, and $\mathbf{R}^3$, and indeed, in any metric space V, we have the following (abstract) properties of a metric:

M1 $\rho(\vec{x},\vec{y}) > 0$ if $\vec{x} \neq \vec{y}$

$\rho(\vec{x},\vec{y}) = 0$ if $\vec{x} = \vec{y}$

M2 $\rho(\vec{x},\vec{y}) = \rho(\vec{y},\vec{x}) \quad \forall \vec{x},\vec{y} \in V$

M3 $\rho(\vec{x},\vec{z}) \leq \rho(\vec{x},\vec{y}) + \rho(\vec{y},\vec{z}) \quad \forall \vec{x},\vec{y},\vec{z} \in V$

In $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$, the metric is the geometric distance between the tips of the position vectors. In $\mathbf{R}^n$, the metric of the vectors $\vec{x} = \{x_i\}_{i=1}^{n}$, $\vec{y} = \{y_i\}_{i=1}^{n} \in \mathbf{R}^n$ is defined to be $\rho(\vec{x},\vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. A metric yields a definition of limit. Although in $\mathbf{R}^n$ with n > 3, this in not geometric, it does provides a measure for evaluating approximate solutions to engineering problems having n state variables. This also allows for the concept of a sequence of approximate solutions that converges to the exact solution in engineering problems.

TOPOLOGY. A topology is a collection of (open) sets having certain properties. Intuitively, an open set is one that does not contain any of its boundary points. For example, open intervals are open sets in the usual topology for **R**. Disks that do not contain any boundary points are open sets in $\mathbf{R}^2$. Balls which do not contain any boundary points are open sets in $\mathbf{R}^3$. Similarly for $\mathbf{R}^n$, but no geometry. Topologists characterize continuity in terms of open sets.

Handout # 5 COMPLEX NUMBERS, GEOMETRY, AND COMPLEX FUNCTIONS OF A COMPLEX VARIABLE Prof. Moseley

To solve $z^2 + 1 = 0$ we "invent" the number i with the defining property $i^2 =$) 1. (Electrical engineers use j instead if i.) We then "define" the set of complex numbers as $\mathbf{C} = \{x+iy : x,y\in\mathbf{R}\}$. Then $z = x+iy$ is known as the **Euler form** of z and $z = (x,y)\in\mathbf{C} = \{(x,y):x,y\in\mathbf{R}\}$ is the **Hamilton form** of z where we identify **C** with $\mathbf{R}^2$ (i.e., with points in the plane). If $z_1 = x_1+iy$, and $z_2 = x_2+iy_2$, then $z_1+z_2 =_{df} (x_1+x_2) + i(y_1+y_2)$ and $z_1z_2 =_{df} (x_1x_2\text{-}y_1y_2) + i(x_1y_2+x_2y_2)$. Using these definitions, the nine properties of addition and multiplication in the definition of an abstract algebraic field can be proved. Hence the system $(\mathbf{C}, +, \cdot, 0, 1)$ is an example of an abstract algebraic field. Computation of the product of two complex numbers is made easy using the Euler form and your knowledge of the algebra of **R**, the FOIL (first, outer, inner, last) method, and the defining property $i^2 = -1$. Hence we have $(x_1+iy_1)(x_2+iy_2) = x_1x_2+x_1iy_2 + iy_1y_2 + i^2y_1y_2 = x_1x_2 + i\,(x_1y_2 + y_1y_2) - y_1y_2 = (x_1x_2 - y_1y_2) + i\,(x_1y_2+x_2y_1)$. This makes evaluation of polynomial functions easy.

EXAMPLE #1. If $f(z) = (3 + 2i) + (2 + i)z + z^2$, then

$$f(1+i) = (3 + 2i) + (2 + i)(1 + i) + (1 + i)^2 = (3 + 2i) + (2 + 3i + i^2) + (1 + 2i + i^2)$$
$$= (3 + 2i) + (2 + 3i - 1) + (1 + 2i - 1) = (3 + 2i) + (1 + 3i) + (\ 2i\ ) = 4 + 7i$$

Division and evaluation of rational functions is made easier by using the **complex conjugate**. We also define the magnitude of a complex number as the distance to the origin in the complex plane.

DEFINITION #1. If $z = x+iy$, then the **complex conjugate** of z is given by $\overline{z} = x - iy$. Also the **magnitude** of z is $|z| = \sqrt{x^2 + y^2}$.

THEOREM #1. If $z,z_1,z_2\in\mathbf{C}$, then a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, c) $|z|^2 = z\,\overline{z}$, d) $\overline{\overline{z}} = z$.

REPRESENTATIONS. Since each complex number can be associated with a point in the (complex) plane, in addition to the **rectangular representation** given above, complex numbers can be represented using **polar coordinates**.

$$z = x + iy = r\cos\theta + i\sin\theta = r\,(\cos\theta + i\sin\theta)$$
$$= r\,\underline{\angle\,\theta}\ \text{(polar representation)}$$

Note that $r = |z|$. You should be able to convert from Rectangular form to Polar form and vice versa. For example, $2\,\underline{\angle\pi/4} = \sqrt{2} + \sqrt{2}\,i$ and $1 + \sqrt{3}\,i = 2\,\underline{\angle\,\pi/3}$. Also, if $z_1 = 3 + i$ and $z_2 = 1 + 2i$, then $\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2}\,\dfrac{\overline{z_2}}{\overline{z_2}} = \dfrac{z_1\overline{z_2}}{|z_2|^2} = \dfrac{(3+i)(1+2i)}{(1+2^2)} = \dfrac{(3+6i+i+2i^2)}{(1+4)} = \dfrac{(3-2+7i)}{5} = \dfrac{1}{5} + \dfrac{7}{5}i$

EXAMPLE # 2 If $f(z) = \dfrac{z+(3+i)}{z+(2+i)}$, evaluate f(4+i).

Solution. $f(4+i)) = \dfrac{(4+i)+(3+i)}{(4+i)+(2+i)} = \dfrac{(7+2i)}{(6+3i)} = \dfrac{(7+2i)}{(6+3i)}\ \dfrac{(6-3i)}{(6-3i)}$

$$= \frac{(42+6)+(12-21)i}{(36+9)} = \frac{48-9i}{45} = \frac{48}{45} - \frac{9}{45}i = \frac{16}{15} - \frac{1}{5}i$$

<u>THEOREM #2</u>. If $z_1 = r_1\angle\underline{\theta_1}$ and $z_2 = r_2\angle\theta_2$, then a) $z_1 z_2 = r_1 r_2 \angle\underline{\theta_1 + \theta_2}$, b) If $z_2 \neq 0$, then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\ \angle\underline{\theta_1 - \theta_2}$, c) $z_1^{\,2} = r_1^{\,2}\angle\underline{2\theta_1}$, d) $z_1^{\,n} = r_1^{\,n}\angle\underline{n\theta_1}$.

<u>EULER'S FORMULA</u>. By definition $e^{i\theta} =_{df} \cos\theta + i \sin\theta$. This gives another way to write complex numbers in polar form.

$z = 1 + i\sqrt{3} = 2(\cos \pi/3 + i \sin \pi/3) = 2\ \angle\underline{\pi/3} = 2e^{i\,\pi/3}$

$z = \sqrt{2} + i\sqrt{2} = 2(\cos \pi/4 + i \sin \pi/4) = 2\ \angle\underline{\pi/4} = e^{i\,\pi/4}$

More importantly, it can be shown that this definition allows the extension of exponential, logarithmic, and trigonometric functions to complex numbers and that the standard properties hold. This allows you to determine what these extensions should be and to evaluate these functions.

<u>EXAMPLE #6</u>. If $f(z) = (2 + i)\, e^{(1+i)z}$, find $f(1 + i)$.
<u>Solution</u>. First $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Hence
$f(1 + i) = (2 + i)\, e^{2i} = (2 + i)(\cos 2 + i \sin 2) = 2 \cos 2 + i(\cos 2 + 2 \sin 2) + i^2 \sin 2$
$= 2 \cos 2 - \sin 2 + i(\cos 2 + 2 \sin 2)$ (exact answer)
$\approx 2(-0.4161468) - (0.9092974) + i(-0.4161468 + 2(0.9092974))$
$= -1.7415911 + i(1.4024480)$ (approximate answer)
How do you know that $-1.7415911 + i(1.4024480)$ is a good approximation to $2 \cos 2 - \sin 2 + i(\cos 2 + 2 \sin 2)$? Can you give an expression for the distance between these two numbers?

In the complex plane, e^z is not one-to-one. Restricting the domain of e^z to the strip $\mathbf{R}\times(-\pi,\pi]$) $\{(0,0)\}$, in the complex plane, we can define Ln z as the (compositional) inverse function of e^z with this restricted domain. This is similar to restricting the domain of sin x to $[-\pi/2, \pi/2]$ to obtain $\sin^{-1} x$ (Arcsin x) as its (compositional) inverse function.

<u>EXAMPLE #7</u>. If $f(z) = \text{Ln}[(1 + i) z]$, find $f(1 + i)$.
<u>Solution</u>. First $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Now let $w = x + iy = f(1 + i)$ where $x,y \in \mathbf{R}$. Then $w = x + iy = \text{Ln}\, 2i$. Since Ln z is the (compositional) inverse of the function of e^z, we have that $2i = \exp(x + iy) = e^x e^{iy} = e^x(\cos y + i \sin y)$. Equating real and imaginary parts we obtain $e^x \cos y = 0$ and $e^x \sin y = 2$ so that $\cos y = 0$ and hence $y = \pm\pi/2$. Now $e^x \sin \pi/2 = 2$ yields $x = \ln 2$, but $e^x \sin(-\pi/2) = 2$ implies $e^x = -2$. Since this is impossible, we have the unique solution $f(1 + i) = \ln 2 + i(\pi/2)$. We check by direct computation using Euler's formula: $\exp[\ln 2 + i(\pi/2)] = e^{\ln 2}\ e^{i(\pi/2)} = 2(\cos(\pi/2) + i \sin(\pi/2)) = 2i$.
This is made easier by noting that if $z = x + iy = re^{i\theta}$, then $\text{Ln}\, z = \text{Ln}(r\, e^{i\theta}) = \ln r + i\theta$.

Letting $e^{iz} = \cos z + i \sin z$ we have $e^{-iz} = \cos z + i \sin(-z) = \cos z - i \sin z$. Adding we obtain $\cos z = (e^{iz} + e^{-iz})/2$ which extends cosine to the complex plane. Similarly $\sin z = (e^{iz} - e^{-iz})/2i$.

EXAMPLE #8. If $f(z) = \cos[(1 + i)z]$, find $f(1 + i)$.
Solution. First recall $(1 + i)(1 + i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$. Hence
$f(1 + i) = \cos(2i) = (e^{i\,2i} + e^{i(-2i)})/2 = (e^{-2} + e^{2})/2 = \cosh 2$ (exact answer)
≈ 3.762195691. (approximate answer)
How do you know that 3.762195691 is a good approximation to cosh 2 ? Can you give an expression for the distance between these two numbers (i.e., cosh 2 and 3.762195691)?

It is important to note that we represent **C** geometrically as points in the (complex) plane. Hence the domain of f:**C**→**C** requires two dimensions. Similarly, the codomain requires two dimensions. Hence a "picture" graph of a complex valued function of a complex variable requires four dimensions. We refer to this a s a geometric depiction of f of Type 1. Hence we can not visualize complex valued function of a complex variable the same way that we do real valued function of a real variable. Again, geometric depictions of complex functions of a complex variable of Type 1 (picture graphs) are not possible. However, other geometric depictions are possible. One way to visualize a complex valued function of a complex variable is what is sometimes called a “set theoretic” depiction which we label as Type 2 depictions. We visualize how regions in the complex plane (domain) get mapped into other regions in the complex plane (codomain). That is, we draw the domain as a set (the complex z-plane) and then draw the w = f(z) plane. Often, points in the z-plane are labeled and their images in the w-plane are labeled with the same or similar notation. Another geometric depiction (of Type 3) is to let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and sketch the real and imaginary parts of the function separately.
Instead of drawing three dimensional graphs we can draw level curves which we call a Type 4 depiction. If we put these level curves on the same z-plane, we refer to it as a Type 5 depiction.