# Linear Algebra Notes

by
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SPACE: THE FINAL FRONTIER

A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

# LINEAR CLASS NOTES:

# A COLLECTION OF HANDOUTS

# FOR

# REVIEW AND PREVIEW

# OF LINEAR THEORY

# INCLUDING FUNDAMENTALS

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# CONTENTS

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# CHAPTER 0

# Introductory Material

1. Teaching Objectives for the Linear Algebra Portion of the Course

2. Sample Lectures for Linear Algebra

TEACHING OBJECTIVES
for the
LINEAR ALGEBRA PORTION
of Math 251 (2-4 weeks)

1. Help engineering students to better understand how mathematicians do mathematics, including understanding the concepts of concrete and abstract spaces.
2. Know how to do matrix operations with real and complex matrices. This may require some remedial work in complex arithmetic including the conjugate of a complex number.
3. Require that students memorize the definition of the abstract concept of a Vector (or Linear) Space as an abstract algebraic structure and that of a subspace of a vector space. The definition of a vector space includes the eight axioms for Vector Space Theory. The general definition of a Subspace in mathematics is a set with the same structure as the space. For a Vector Space, this requires closure of vector addition and multiplication by a scalar.
4. Be able to solve $A\vec{x} = \vec{b}$ for all three cases (unique solution, no solution, and infinte number of solutions). Understand what a problem is in mathematics, especially scalar and vector equations.
5. Develop some understanding of the definitions of Linear Operator, Span, Linear Independence, Basis, and Dimension. This need not be indepth, but exposure to all of the definitions is important. Understanding begins with exposure to the definitions. This orients students toward $A\vec{x} = \vec{b}$ as a mapping problem and understanding dimension as an algebraic concept, helping them not to get stuck in 3-space.
6. Know how to compute determinants by both using the Laplace expansion and by using Gauss elimination.
7. Know how to compute the inverse of a matrix. A formula for a 2x2 can be derived.
8. Know the properties of a nonsingular (square) matrix and other appropriate theory (without proofs).

SAMPLE LECTURES FOR LINEAR ALGEBRA

We begin with an algebraic rather than a geometric approach to vectors. This will help the engineers not to get stuck in 3-space. If you start with geometry and then start talking about n-space, the engineers decide that what you are saying has nothing to do with engineering and quit listening. “Help I’m stuck in three space and I can’t get out.”

DO NOT PRESENT IN CLASS ANY OF THE MATERIAL IN CHAPTER 1. Have them read it on their own. This material provides background to help the students to convert from a geometric approach to an algebraic approach to vectors. This makes it easier to move on to 4,5,...n... dimensions but is not necessary to go over in class.

Lecture #1 “SPACE, THE FINAL FRONTIER”
Handout#1 Page 1 and 2 (on one sheet of paper) of any old exam.

To a mathematician, a **space** is a set plus structure. There are two kinds: **Concrete** and **Abstract**. The number systems, $\mathbf{N}\subseteq\mathbf{Z}\subseteq\mathbf{Q}\subseteq\mathbf{R}\subseteq\mathbf{C}$, are sets with structure and hence are spaces. However, we do not call them spaces, we call them number systems. We may begin with the Peano Postulates, and then construct successively the positive integers (natural numbers) **N**, the integers **Z**,the rational numbers **Q**, and the reals **R** and the complexes **C**. However, in a real analysis course, we follow Euclid and list the axioms that uniquely determine **R**. These fall into three categories: the field (algebraic) axioms, the order axioms, and the least upper bound axiom (LUB). These are sufficient to determine every real number so that **R** is concrete (as are all the number systems). If we instead consider only the field axioms, we have the axioms for the abstract algebraic structure we call a **field**. We can define a field informally as a number system where we can add, subtract, multiply, and divide (except by zero of course). (Do not confuse an algebraic field with a scalar or vector field that we will encounter later in Vector Calculus.) Since an algebraic field is an abstract space, we give three examples: **Q**, **R**, and **C**. **N** and **Z** are not fields. Why?

A **matrix** is an array of elements in a field. Teach them to do matrix algebra with **C** as well as **R** as entries. Begin with the examples on the old exam on page 1. That is, teach them how to compute all of the matrix operations on the old exam. You may have to first teach them how to do complex arithmetic including what the **conjugate** of a complex number is. Some will have had it in high school, some may not. The True/False questions on page 2 of the old exam provide properties of Matrix Algebra. You need not worry about proofs, but for false statements such as “Matrix multiplication of square matrices is commutative” you can provide or have them provide counter examples. DO NOT PRESENT THE MATERIAL IN CHAPTER 2 IN CLASS. It provides the proofs of the matrix algebra properties.

Lecture #2 VECTOR (OR LINEAR) SPACES
Handout#2 One sheet of paper. On one side is the definition of a vector space from the notes. On the other side is the definition of a subspace from the notes.

Begin by reminding them that a space is a set plus structure. Then read them the definition of a vector space. Tell them that they must memorize this definition. This is our second example of an **abstract algebraic space**. Then continue through the notes giving them examples of a vector space including $\mathbf{R}^n$, $\mathbf{C}^n$, matrices, and function spaces.

Then read them the definition of a subspace. Tell them that they must memorize this definition. Stress that a subspace is a vector space in its own right. Why? Use the notes as appropriate, including examples (maybe) of the null space of a Linear Operator.

Lecture #3&4 SOLUTION OF $A\vec{x} = \vec{b}$

Teach them how to do Gauss elimination. See notes. Start with the real 3x3 Example#1 on page 8 of Chapter 4 in the notes. This is an example with exactly one solution. Then do the example on page 15 with an arbitrary $\vec{b}$. Do both of the $\vec{b}$ 's. One has no solution; one has an infinite number of solutions. Note that for a 4x4 or larger, geometry is no help. They will get geometry when you go back to Stewart.

Lecture #5&6 REST OF VECTOR SPACE CONCEPTS

Go over the definitions of Linear Operators, Span, Linear Independence, Basis Sets, and Dimension. You may also give theorems, but no proofs (no time). Two examples of Linear operators are T:$\mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ and the derivative D:$\mathcal{A}(\mathbf{R},\mathbf{R}) \to \mathcal{A}(\mathbf{R},\mathbf{R})$ where if $f \in \mathcal{A}(\mathbf{R},\mathbf{R}) = \{f \in \mathcal{F}(\mathbf{R},\mathbf{R})$: f is analytic on $\mathbf{R}\}$ and $\mathcal{F}(\mathbf{R},\mathbf{R})$ is the set of all real valued functions of a real variable, and we define D by $D(f) = \frac{df}{dx}$ .

Lecture #7 DETERMINANTS AND CRAMER'S RULE

Teach them how to compute determinants and how to solve $A\vec{x} = \vec{b}$ using Cramer's Rule.

Lecture #8 INVERSES OF MATRICES

Teach them how to compute (multiplicative) inverses of matrices. You can do any size matrix by augmenting it with the (multiplicative) identity matrix I and using Gauss elimination. I do an arbitrary 2x2 to develop a formula for the inverse of a 2x2 which is easy to memorize.

This indicates two weeks of Lectures, but I usually take longer and this is ok. Try to get across the idea of the difference in a concrete and an abstract space. All engineers need Linear Algebra. Some engineers need Vector Calculus. If we have to short change someplace, it is better the Vector Calculus than the Linear Algebra. When you go back to Stewart in Chapter 10 and pick up the geometrical interpretation for 1,2, and 3 space dimensions, you can move a little faster so you can make up some time. My first exam covers up through 10.4 of Stewart at the end of the fourth week. Usually, by then I am well into lines and planes and beyond.