Compute related rates

The Problem: Given an equation $F(x, y) = 0$, where $x = x(t)$ and $y = y(t)$, and given values of x, y and one of $x'(t)$ and $y'(t)$ (say $x'(t)$), we want to find the missing rate value (in this case is $y'(t)$).

(Step 1) View $x = x(t)$ and $y = y(t)$, apply chain rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to t. This will yield a new equation involving x, y, $x'(t)$ and $y'(t)$.
(Step 2) Solve the resulting equation from (Step 1) for $y'(t)$, assuming that the values of x, y and $x'(t)$ are given.

Remark: Please distinguish this problem with the implicit differentiation problem.

Example 1 A circular oil slick of uniform thickness is caused by a spill of 1 m^3 of oil. The thickness of the oil slick is decreasing at the rate of 0.1 cm/h. At what rate is the radius of the slick increasing when the radius is 8m?

Solution: Let r and h denote the radius and the thickness of the oil slick, respectively. Then both $r = r(t)$ and $h = h(t)$ are functions of the times t. That the volume of the slick is 1m^3 becomes

$$\pi r^2 h = 1.$$

View $r = r(t)$ and $h = h(t)$ and differentiating both sides of this equation with respect to t, we get

$$2\pi r r' h + \pi r^2 h' = 0.$$

We shall use meter as the unit for length. Therefore, $h'(t) = -0.001\text{m/h}$. When $r = 8$, we have $h = \frac{1}{8^2\pi} = \frac{1}{64\pi}$. Substitute all these in to equation involving the rates, we have

$$2\pi(8)r'(t)\frac{1}{64\pi} + \pi 64(-0.001) = 0, \text{ and so } r'(t) = \frac{4 \cdot 64\pi}{1000} = \frac{32\pi}{125}\text{m/h}.$$

Thus when the radius is 8m, the radius of the slick increasing at the rate of $\frac{32\pi}{125}$ m/h.

Example 2 The width of a rectangle is half its length. At what rate is its area increasing if its width is 10cm and is increasing at 0.5 cm/s^2?.

Solution: Let l and w denote the width and the length of the rectangle, respectively. Then both $l = l(t)$ and $w = w(t)$ are functions of the time t. Moreover, $2w = l$. Thus the area $A = lw = 2w^2$.

View $w = w(t)$ and differentiating $A(t)$ with respect to t, we get

$$A'(t) = 4ww'(t).$$

When $w = 10\text{cm}$ and $w'(t) = 0.5\ \text{cm/s}^2$, we have $A'(t) = 4(10)(0.5) = 20$ cm/s.