Application of Derivatives to Optimization Problems

Facts: Let $f(x)$ be a differentiable function on an interval I.

(1) (Where to seek optimal solutions?) Any local extrema must occur at a critical point.
(2) (How does $f'(x)$ play a role?) The sign of $f'(x)$ can predict whether $f(x)$ is increasing or decreasing.

Strategy to find optimal solutions: Firstly we determine the critical points, and the sign of $f'(x)$ in the subintervals determined by the critical points. Then interpret the sign of $f'(x)$ to discuss whether an optimal solution occurs at a given critical point.

Example 1 A three side fence is to be built next to a straight section of river, which forms the forth side of a rectangular region. The enclosed area is to equal 1800 ft^2. Find the minimum perimeter and the dimensions of the corresponding enclosure.

Solution: Let the length of the two sides of the rectangular region be x and y, where y represents the length of the side parallel to the river bank. Then as $xy = 1800$, $y = \frac{1800}{x}$. The perimeter of the region is a function of x:

$$f(x) = 2x + y = 2x + \frac{1800}{x}.$$

As x is a length, $x > 0$. Therefore, the domain of $f(x)$ is $(0, \infty)$.

Compute $f'(x) = 2 - \dfrac{1800}{x^2}$. Set $f'(x) = 0$ to get $x = \pm 30$. Only $x = 30$ is in the domain. Moreover, $x = 30$ partitions the domain $(0, \infty)$ into two subintervals: $(0, 30)$ and $(30, \infty)$. As $f'(1) < 0$ and $f'(40) > 0$, $f(x)$ is decreasing in $(0, 30)$ and increasing in $(30, \infty)$. The shape of graph of $y = f(x)$ suggests that $f(x)$ has an absolute minimum at $x = 30$. Thus the minimum perimeter and the corresponding dimensions are, respectively,

$$f(30) = 2 \cdot 30 + \frac{1800}{30} = 120, x = 30, y = \frac{1800}{30} = 60.$$

Example 2 Find the point on the curve $y = x^2$ closest to the point $Q = (0, 1)$.

Solution: Let P be a point of the curve $y = x^2$. Then $P = (x, x^2)$. The distance between P and Q is a function of x:

$$f(x) = \sqrt{(x-0)^2 + (x^2-1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} = \sqrt{x^4 - x^2 + 1} = (x^4 - x^2 + 1)^{\frac{1}{2}}.$$

As $x^4 - x^2 + 1 > 0$ for all x, the domain of $f(x)$ is $(-\infty, \infty)$. Compute $f'(x)$ by using the chain rule

$$f'(x) = \frac{1}{2} \frac{4x^3 - 2x}{\sqrt{x^4 - x^2 + 1}} = \frac{x(2x^2 - 1)}{\sqrt{x^4 - x^2 + 1}}.$$

Set $f'(x) = 0$. Then the numerator must equal zero. Thus the critical numbers are

$$x = 0, x = \frac{1}{\sqrt{2}}, x = -\frac{1}{\sqrt{2}}.$$

These points partition the domain $(-\infty, \infty)$ into four intervals:

$$(-\infty, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, 0), (0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \infty).$$

Compute $f'(-10) < 0$, $f'(-\frac{1}{2}) > 0$, $f'(\frac{1}{2}) < 0$ and $f'(10) > 0$, (note that the denominator is always positive). Therefore, $f(x)$ is increasing in $(-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$, and decreasing in $(-\infty, -\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$.

From the shape of the graph $y = f(x)$, $f(x)$ has local minimum at $x = \pm\frac{1}{\sqrt{2}}$, and at least one of them gives the absolute minimum. Compute

$$f(\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}) = \sqrt{\frac{3}{4}}.$$

Therefore, there are two points $(\pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ which are on $y = x^2$ and are closest to $(0, 1)$.

Example 3 Determine two real numbers with difference 20 and minimum possible product.

Solution: Let x and y denote these two real numbers with $x \leq y$. Then $y = 20 + x$. Thus we want to minimize

$$f(x) = xy = x(20 + x) = 20x + x^2,$$

in $(-\infty, \infty)$. Note that $f'(x) = 2x + 20$, and so $x = -10$ is the only critical number. As $f'(x) < 0$ when $x < -10$ and $f'(x) > 0$ when $x > -10$. Therefore $f(-10)$ is the only local minimum value of $f(x)$ in $(-\infty, \infty)$, and so it is also the absolute minimum value of $f(x)$ in its domain. Hence $x = -10$ and $y = 10$ are the two numbers we want.