Finding minimum distances

Strategy to tackle the problem

(1) Identify the variable to be minimized (here will be the distance D), and assign symbols to other given quantities. (Give the variables names).
(2) Identify the relation that allow us to write down an equality (referred as the primary equation) expressing D in terms of other quantities. (Usually such a relationship can be found from facts in geometry).
(3) Reduce the primary equation into one that express D in terms of a single independent variable. (This may involve the use of a secondary equation). Determine the domain. (Where can the independent variable take values from?)
(4) Use calculus (derivative, critical numbers, comparison or appropriate discussions we learn from calculus).

Example 1 Which points on the graph $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution: (1) Let D denote the distance from a point (x, y) on the graph to the on given point $(0, 2)$. (2) Express the distance in terms of x and y. From the distance formula, the primary equation is $D = \sqrt{(x-0)^2 + (y-2)^2}$. (3) We recognize that we need to express y in terms of x. As (x, y) is a point on the graph $y = 4 - x^2$, which also serves as a secondary equation. Substitute $y = 4 - x^2$ into the primary equation, we have

$$D = D(x) = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(x-0)^2 + (4-x^2-2)^2} = \sqrt{x^4 - 3x^2 + 4}.$$

Since x is a length, $x > 0$. Thus the domain of the function $D(x)$ is $x > 0$, or $[0, \infty)$.

(4) Now we have successfully model the problem into one that finds the absolute maximum of a function $D(x)$ on a closed interval $[0, \infty)$. We could directly apply

the calculus skills to complete the story. However, we can make the computation simpler in this problem. Note that $D(x) \geq 0$, and so $f(x) = (D(x))^2 = x^4 - 3x^2 + 4$ will reach the minimum at the same places as $D(x)$. As it is easier to deal with polynomials than the square root functions, we find the minimum of $f(x)$ instead. Compute $f'(x) = 4x^3 - 6x = 2x(2x^2 - 3)$. Set $f'(x) = 0$ to get the critical numbers $x = 0$, $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$. Apply the first derivative test to conclude that at $x = 0$, $f(x)$ has a local maximum, and at $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$, $f(x)$ reaches its absolute minimum. Thus the closest points are $(\sqrt{3/2}, 5/2)$ and $(-\sqrt{3/2}, 5/2)$. (We need to compute that $D(\pm\sqrt{3/2}) = 5/2$).