Finding maximum volumes

Strategy to tackle the problem

(1) Identify the variable to be maximized (here will be the volume V), and assign symbols to other given quantities. (Give the variables names).
(2) Identify the relation that allow us to write down an equality (referred as the primary equation) expressing V in terms of other quantities. (Usually such a relationship can be found from facts in geometry).
(3) Reduce the primary equation into one that express V in terms of a single independent variable. (This may involve the use of a secondary equation). Determine the domain. (Where can the independent variable take values from?)
(4) Use calculus (derivative, critical numbers, comparison or appropriate discussions we learn from calculus).

Example 1 A manufacturer is to design an open top box having a square base and a surface area 108 square inches. What dimensions will produce a box with a maximum volume?

Solution: (1) Let V denote the volume and let x denote the length of one side of the square base. (2) Express the volume of this box in terms of the length of one side of the base square. From geometry, the primary equation is $V = x^2h$, where h is the height of the box. (3) We recognize that we need to express h in terms of x. As the surface has 4 rectangles (four sides) and a square (the base), it gives a secondary equation $108 = 4xh + x^2$. Solve $108 = 4xh + x^2$ for h to get $h = \frac{108-x^2}{4x}$. Substitute $h = \frac{108-x^2}{4x}$ into the primary equation, we have

$$V = V(x) = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}.$$

Since x is a length, $x \geq 0$. Since $x^2 \leq 4xh + x^2 = 108$, $x \leq \sqrt{108}$. Thus the domain of the function $V(x)$ is $0 \le x \le \sqrt{108}$, or $[0, \sqrt{108}]$.

(4) Now we have successfully model the problem into one that finds the absolute maximum of a function $V(x)$ on a closed interval $[0, \sqrt{108}]$. To apply calculus to find the maximum volume, we first compute the derivative $V'(x) = 27 - 3x^2/4$. Set $V'(x) = 0$ to get the critical numbers $x = 6$ (the other solution $x = -6$ is not in $[0, \sqrt{108}]$). Compare the values $V(0) = 0, V(6) = 108$ and $V(\sqrt{108}) = 0$, we conclude that $x = 6$ and $h = \frac{108-6^2}{4 \cdot 6} = 3$, and so the dimensions of the box are $6 \times 6 \times 3$.