Compute the line integrals

Useful facts: Let $(x(t), y(t), z(t))$, $a \le t \le b$ denote a parametric equations of a spacial curve C. Suppose that the function $f(x,y,z)$ is continuous on curve C.

(1) The **line integral** of f **with respect to arc length** along the curve C is

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + y'(t)]^2 + z'(t)]^2}dt.$$

(2) If C is a wire and the density function of this aire is $\delta(x,y,z)$, then the **mass** of the wire is

$$m = \int_C \delta(x,y,z)ds,$$

and the **centroid** $(\overline{x}, \overline{y}, \overline{z})$ is

$$\overline{x} = \frac{1}{m}\int_C x\delta(x,y,z)ds, \ \overline{y} = \frac{1}{m}\int_C y\delta(x,y,z)ds, \ \overline{z} = \frac{1}{m}\int_C z\delta(x,y,z)ds.$$

(3) The **line integral** of f **with respect to x**, and to y, to z, along the curve C are, respectively

$$\begin{aligned}\int_C f(x,y,z)dx &= \int_a^b f(x(t),y(t),z(t))x'(t)dt.\\ \int_C f(x,y,z)dy &= \int_a^b f(x(t),y(t),z(t))y'(t)dt.\\ \int_C f(x,y,z)dz &= \int_a^b f(x(t),y(t),z(t))z'(t)dt.\end{aligned}$$

(4) Reverse of the direction of the curve.

$$\int_{-C} f(x,y,z)ds(dx, dy, \text{ or } dz, \text{ repsectively}) = -\int_C f(x,y,z)ds(dx, dy, \text{ or } dz, \text{ repsectively})$$

(5) Suppose that $\mathbf{F} = (P,Q,R)$ is a continuous vector function, and $\mathbf{T}$ is the unit tangent vector to the smooth curve C. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C Pdx + Qdy + Rdz.$$

(6) The Fundamental Theorem of Calculus: If for a continuous function $f(x,y,z)$, $\mathbf{F} = \nabla f$, and if the curve C is from a point (x_1,y_1,z_1) (when $t = a$)), to a point (x_2,y_2,z_2) (when $t = b$)), then f is a **potential function** of $\mathbf{F}$, and

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_a^b \left[\frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial z}z'(t)\right]dt = \int_a^b \frac{df}{dt}dt = f(x_2,y_2,z_2) - f(x_1,y_1,z_1).$$

Example (1) Give $f(x,y) = 2x - y$, and the curve C with parametric equations $x = \sin t$, $y = \cos t$, $0 \le t \le \pi/2$. Evaluate the line integrals.

$$\int_C f(x,y)ds, \int_C f(x,y)dx, \int_C f(x,y)dy.$$

Solution: Compute to get $x'(t) = \cos t$ and $y'(t) = sint$. Thus $ds = \sqrt{\cos^2 t + (-sint)^2} = 1$.

$$\begin{aligned}
\int_C f(x,y)ds &= \int_0^{\pi/2} (2\sin t - \cos t)dt = 1. \\
\int_C f(x,y)dx &= \int_0^{\pi/2} (2\sin t - \cos t)\cos t dt = \int_0^{\pi/2} \left(2\sin t\cos t - \frac{1+\cos 2t}{2}\right) dt = \frac{4-\pi}{4}. \\
\int_C f(x,y)dy &= \int_0^{\pi/2} (2\sin t - \cos t)(-\sin t)dt = \int_0^{\pi/2} \left((-2)\frac{1-\cos 2t}{2} + \sin t\cos t\right) dt = \frac{1-\pi}{2}.
\end{aligned}$$

Example (2) Give $P(x,y) = y\sqrt{x}Q(x,y) = x\sqrt{y}$, and the curve C is the part of the graph of $y^2 = x^3$ from $(1,1)$ to $(4,8)$. Evaluate the line integral

$$\int_C Pdx + Qdy.$$

Solution: The first and important step is to find a parametric form of C. Set $t = x$. Then C can have parametric equations $x = t^2$ and $y = t^3$ with $1 \le t \le 2$. Note that $dx = 2xdt$ and $dy = 3t^2dt$. Thus

$$\int_C Pdx + Qdy = \int_1^2 2t^5dt + \int_1^2 3t^5dt = 5\int_1^2 t^5dt = \left[\frac{5}{6}t^6\right]_1^2 = \frac{105}{2}.$$

Example (3) Give $\mathbf{F} = (yz, xz, xy)$, and the curve C is the straight line segment from $(2,-1,3)$ to $(4,2,-1)$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{T}ds$$

Solution 1: The first and important step is to find a parametric form of C. For a straight line from a point (x_1,y_1,z_1) to (x_2,y_2,z_2) can always have the parametric equations $x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$, and $z = z_1 + t(z_2 - z_1)$, with $0 \le t \le 1$. In this case, $x = 2 + 2t$, $y = -1 + 3t$ and $z = 3 - 4t$. Note that $dx = 2dt, dy = 3dt$ and $dz = -4dt$. Thus

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^1 [2(-1+3t)(3-4t) + 3(2+2t)(3-4t) - 4(2+2t)(-1+3t)]dt = -2.$$

Solution 2: (Need the help of potential functions). Note that if $f(x,y,z) = xyz$, then $\nabla f = \mathbf{F}$. Therefore, by the Fundamental Theorem of Calculus (on the curves)

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = f(4,2,-1) - f(2,-1,3) = -2.$$

Example (4) Give $\mathbf{F} = (2x+3y, 3x+2y, 3z^2)$, and the curve C is the path from $(0,0,0)$ to $(4,2,3)$. that consists of three line segments parallel to the x-axis, y-axis, and z-axis in that order. Evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{T}ds$$

Solution 1: The first and important step is to find a parametric form of C. We can break the curve C into three pieces $C_1 : 0 \le x \le 4$, $y = 0$ and $z = 0$; $C_2 : 0 \le y \le 2$, $x = 4$ and $z = 0$; and $C_3 : 0 \le z \le 3$, $x = 4$ and $y = 2$. Note that on C_1, $dy = dz = 0$, on C_2, $dx = dz = 0$ and on C_3, $dx = dy = 0$. Thus

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_{C_1} + \int_{C^2} + \int_{C_3} = \int_0^4 2xdx + \int_0^2 (12+2y)dy + \int_0^3 3z^2dz = 71.$$

Solution 2: Note that if $f(x,y,z) = x^2 + 3xy + y^2 + z^3$, then $\nabla f = \mathbf{F}$. Therefore, by the Fundamental Theorem of Calculus (on the curves)

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = f(4,2,3) - f(0,0,0) = 71.$$