Compute the mass and the centroid and the moments of a solid

Useful facts: Suppose that $\delta(x,y,z)$ denotes the density function of a solid T. Let m denote the **mass** of T, and $\overline{x},\overline{y},\overline{z})$ denote the coordinates the **centroid** of T. and I_x, I_y and I_z denote the **moments** of T around the x-axis, the y-axis and the z-axis, respectively. Then

$$\begin{aligned}
m &= \int\int\int_T \delta(x,y,z)dV. \\
\overline{x} &= \frac{1}{m}\int\int\int_T x\delta(x,y,z)dV. \\
\overline{y} &= \frac{1}{m}\int\int\int_T y\delta(x,y,z)dV. \\
\overline{z} &= \frac{1}{m}\int\int\int_T z\delta(x,y,z)dV. \\
I_x &= \int\int\int_T (y^2+z^2)\delta(x,y,z)dV. \\
I_y &= \int\int\int_T (x^2+z^2)\delta(x,y,z)dV. \\
I_z &= \int\int\int_T (y^2+x^2)\delta(x,y,z)dV.
\end{aligned}$$

Example (1) Find the centroid of the solid T bounded by $z = x^2$, $y + z = 4$, $y = 0$ and $z = 0$, given the density function $\delta \equiv 1$.

Solution: As the density is 1, this is the same as to find the volume. View the y-axis as the vertical axis. Then T lies between $y = 0$ and $y = 4 - z$. The region R on the xz-plane is bounded by $z = x^2$ and $z = 4$ (obtained by substituting $y = 0$ in $y + z = 4$). Therefore, the mass is

$$\begin{aligned}
m &= \int\int_R \left(\int_0^{4-z} dy\right) dA = \int_{-2}^{2}\int_{x^2}^{4}(4-z)dzdx = \int_{-2}^{2}\left[4z - \frac{z^2}{2}\right]_{x^2}^{4} dx \\
&= \int_{-2}^{2}\left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \left[8x - \frac{4x^3}{3} + \frac{x^5}{10}\right]_{-2}^{2} = 64\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = \frac{256}{15}.
\end{aligned}$$

The coordinates of the centroid is

$$\begin{aligned}
\overline{x} &= \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-z} xdydzdx = \frac{15}{256}\int_{-2}^{2}\int_0^{x^2}(4-z)xdzdx = \int_{-2}^{2}\left(8x - 4x^3 + \frac{x^5}{2}\right) dx = 0 \\
\overline{y} &= \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-z} ydydzdx = \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\frac{(4-z)^2}{2}dzdx = \frac{15}{256}\int_{-2}^{2}\frac{1}{2}\int_{x^2}^{4}(16-8z+z^2)dzdx \\
&= \frac{15}{512}\int_{-2}^{2}\left[16z - 4z^2 + \frac{z^3}{3}\right]_{x^2}^{4} dx = \frac{15}{512}\int_{-2}^{2}\left[\frac{64}{3} - 16x^2 + 4x^4 - \frac{x^6}{3}\right] dx \\
&= \frac{15}{512}\left[\frac{64x}{3} - \frac{16x^3}{3} + \frac{4x^5}{5} - \frac{x^7}{213}\right]_{-2}^{2} = \frac{15}{512}\left[\frac{256}{3} - \frac{256}{3} + \frac{256}{5} - \frac{256}{21}\right] = \frac{15}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{8}{7}. \\
\overline{z} &= \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\int_0^{4-z} zdydzdx = \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}(4z - z^2)dzdx = \frac{15}{256}\int_{-2}^{2}\left[2z - \frac{z^3}{3}\right]_{x^2}^{4} dx \\
&= \frac{15}{256}\int_{-2}^{2}\left[\frac{32}{3} - 2x^4 + \frac{x^6}{3}\right] dx = \frac{15}{256}\left[\frac{32x}{3} - \frac{2x^5}{5} + \frac{x^7}{21}\right]_{-2}^{2} \\
&= \frac{15}{256}\left[\frac{128}{3} - \frac{128}{5} + \frac{256}{21}\right] = \frac{12}{7}.
\end{aligned}$$