Compute integrals in spherical coordinates

Useful facts: With spherical coordinates, we have

$$dV = \rho^2 \sin\phi d\rho d\phi d\theta.$$

Suppose that $f(x, y, z)$ is continuous on a spacial region T, and T in spherical coordinates can be described as:

$$\rho_1(\phi, \theta) \le \rho \le \rho_2(\phi, \theta), \phi_1 \le \phi \le \phi_2, \text{ and } \theta_1 \le \theta \le \theta_2.$$

Then

$$\int\int\int_T f(x, y, z)dV = \int_{\theta_1}^{\theta_1}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\phi d\theta.$$

Example (1) Find the volume of the solid bounded above by the surface $\rho = 2a\sin\phi$, where $a > 0$ is a real number.

Solution: As θ is not shown up in the equation, θ can take all the values in an interval of length 2π. Which means that the solid can be obtained by rotating the curve $\rho = 2a\sin\phi$ in the $\rho\phi$-plane (viewed as polar coordinates) about the line throught the origin and perpendicular to the $\rho\phi$-plane.

Thus we can use $[-\pi, \pi]$ for the interval of θ, and so $-\pi \le \theta \le \pi$. Note that $0 \le \phi \le \pi$. For fixed θ and ϕ, ρ changes in the interval $[0, 2a\sin\phi]$. Hence the volume is

$$\begin{aligned}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{2a\sin\phi} \rho^2\sin\phi d\rho d\phi d\theta &= 2\pi\int_0^{\pi}\frac{(2a\sin\phi)^3}{3}\sin\phi d\phi = \frac{16a^3\pi}{3}\int_0^{\pi}\frac{1 - 2\cos(2\phi) + \cos^2 2\phi}{4}d\phi \\ &= \frac{4a^3\pi}{3}\left[\phi - \sin(2\phi) + \frac{\phi}{2} - \frac{\sin 4\phi}{8}\right]_0^{\pi} = \frac{4a^3\pi}{3}\frac{3\pi}{2} = 2a^3\pi^2.\end{aligned}$$

Example (2) Find the volume of the solid bounded above by the surface $\rho = 1 + \cos\phi$.

Solution: As θ is not shown up in the equation, θ can take all the values in an interval of length 2π. Which means that the solid can be obtained by rotating the curve $\rho = 1 + \cos\phi$ in the $\rho\phi$-plane (viewed as polar coordinates) about the line throught the origin and perpendicular to the $\rho\phi$-plane.

Thus we can use $[-\pi, \pi]$ for the interval of θ, and so $-\pi \le \theta \le \pi$. Note that $0 \le \phi \le \pi$. For fixed θ and ϕ, ρ changes in the interval $[0, 1 + \cos\phi]$. Hence the volume is

$$\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{1+\cos\phi} \rho^2\sin\phi d\rho d\phi d\theta = 2\pi\int_0^{\pi}\frac{(1 + \cos\phi)^3}{3}\sin\phi d\phi = \frac{2\pi}{3}\left[-\frac{(1 + \cos\phi)^4}{4}\right]_0^{\pi} = \frac{8\pi}{3}.$$