Compute area and volume by evaluating double integrals

Useful facts: Suppose that $f(x, y)$ is continuous on a region R in the plane $z = 0$.
(1) The area A of the region R is

$$A = \int\int_R dA.$$

(2) The volume V of the solid that lies below the surface $z = f(x, y)$ and above the region is (assuming that this integral exists)

$$V = \int\int_R f(x, y)dA.$$

(3) The volume V of the solid that lies below the surface $z = z_{\text{top}} = z_2(x, y)$ and above the surface $z = z_{\text{bottom}} = z_1(x, y)$ is (assuming that this integral exists)

$$V = \int\int_R [z_{\text{top}} - z_{\text{bottom}}]dA = \int\int_R [z_2(x, y) - z_1(x, y)]dA.$$

Example (1) Find the area of the region R on the plane $z = 0$ bounded by the curves $y = 2x+3$ and $y = 6x - x^2$ by evaluate a double integral.

Solution: View this region as a vertically simple one. Then solve the system of equations $y = 2x + 3$ and $y = 6x - x^2$ for x to get the x-bounds.

Substitute $y = 2x + 3$ in $y = 6x - x^2$ to get $x^2 - 4x + 3 = 0$, and so $x = 1$ and $x = 3$. Therefore, the x-bounds are $a = 1$ and $b = 3$. Thus

$$A = \int_1^3 \int_{2x+3}^{6x-x^2} dydx = \int_1^3 (4x - x^2 - 3)dx = \frac{4}{3}.$$

Example (2) Find the volume of the solid that is below the surface $z = 3 + \cos x + \cos y$ over the region R on the plane $z = 0$ bounded by the curves $x = 0$, $x = \pi$, $y = 0$ and $y = \pi$ by evaluate a double integral.

Solution: Set up the double integral and evaluate it:

$$V = \int_0^\pi \int_0^\pi (3 + \cos x + \cos y)dxdy = \int_0^\pi (3\pi + \pi \cos x)dx = 3\pi^2$$

Example (3) Find the volume of the solid that is below the surface $z = 3x+2y$ over the region R on the plane $z = 0$ bounded by the curves $x = 0$, $y = 0$ and $x + 2y = 4$ by evaluate a double integral.

Solution: Set up the double integral and evaluate it:

$$V = \int_0^2 \int_0^{4-2y} (3x + 2y)dxdy = \int_0^2 \left[\frac{3}{2}x^2 + 2xy\right]_0^{4-2y} dx = \int_0^2 (24 - 16y) + 2y^2)dy = \frac{64}{3}.$$

Example (4) Find the volume of the solid bounded by the planes $y = 0$, $z = 0$, $y = 2x$ and $4x + 2y + z = 8$.

Solution: Study the solid to understand that it is above $z = 0$ and below $z = 8 - 4x - 2y$, over the region R on the $z = 0$ plane which is bounded by the lines $4x + 2y = 8$ $(z = 0)$, $y = 2x$ and $y = 0$.

$$V = \int_0^2 \int_{\frac{y}{2}}^{\frac{4-y}{2}} (8 - 4x - 2y)dxdy = \int_0^2 (9 - 8y + 2y^2)dy = \frac{16}{3}.$$

Example (5) Find the volume of the first octant part of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

Solution: Study the solid to understand that it is above $z = 0$ and below $z = \sqrt{1 - y^2}$, over the region R on the $z = 0$ plane which is bounded by the lines $x = 0$, $y = 0$ and $x^2 + y^2 = 1$. Note that in R, $x \geq 0$ and $y \geq 0$. Thus the integral is obtained below.

$$V = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - y^2}dxdy = \int_0^1 (1 - y^2)dy = \frac{2}{3}.$$

Example (6) Find the volume of a sphere of radius a by double integration.

Solution: We can view that the center of the sphere is at the origin $(0, 0, 0)$, and so the equation of the sphere is $x^2 + y^2 + z^2 = a^2$. We then can compute the volume of the upper half part of the sphere and multiply our answer by 2 (or the portion in the first octant and multiply the answer by 8).

$$V = 8 \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}dxdy = \frac{4}{3}\pi a^3.$$

(How do we compute this integral? We can read the next Tip: Compute double integrals in polar coordinates).

Example (7) Find the volume of the solid bounded below by the plane $z = 0$ and above by the paraboloid $z = 25 - x^2 - y^2$.

Solution: Study the solid to understand that it is over the region R on the $z = 0$ plane which is bounded by the circle $x^2 + y^2 = 25$.

$$V = 8 \int_0^5 \int_0^{\sqrt{25-y^2}} (25 - x^2 - y^2)dxdy = \frac{625}{2}\pi.$$

Example (8) Find the volume removed when a vertical square hole of edge length r is cut directly through the center of a long horizontal solid cylinder of radius r.

Solution: Set the coordinate system so that the center of the vertical square hole is the y-axis and the center of the long horizontal solid cylinder is the x-axis. Then the equation of the cylinder is $y^2 + z^2 = r^2$, and the intersection of the removed square based solid and the plane $z = 0$ is a square region R whose vertices are $(\pm\frac{r}{2}, \pm\frac{r}{2})$. Use the fact that $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ to get

$$V = 8 \int_0^{\frac{r}{2}} \int_0^{\frac{r}{2}} \sqrt{r^2 - y^2}dxdy = 8\left(\frac{r}{2}\right)\left[\frac{y}{2}\sqrt{r^2 - y^2} + \frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_0^{\frac{r}{2}} = r^3\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right).$$