Find and classify critical points

Useful facts: The discriminant $\Delta = f_{xx}f_{yy} - f_{xy}^2$ at a critical point $P(x_0, y_0)$ plays the following role:

1. If $\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).

2. If $\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).

3. If $\Delta(x_0, y_0) < 0$, then f has neither a local minimum nor a local maximum at (x_0, y_0) (saddle point at P).

Example (1) : Find and classify the critical points of $f(x, y) = x^2+4xy+2y^2+4x-8y+3$.

Solution: Compute $f_x = 2x + 4y + 4$ and $f_y = 4x + 4y - 8$. Solve $f_x = 0$ and $f_y = 0$ to get the only critical point $(6, -4)$. Note that $f(6, -4) = 31$.

Compute $f_{xx} = 2, f_{xy} = 4$ and $f_{yy} = 4$, and so $\Delta = (2)(4) - 4^2 < 0$ at any point. Therefore at the critical point $(6, -4, 31)$, the surface has a saddle point.

Example (2) : Find and classify the critical points of $f(x, y) = x^2 - 2xy + y^3 - y$.

Solution: Compute $f_x = 2x - 2y$ and $f_y = 3y^2 - 2x - 1$. Solve $f_x = 0$ and $f_y = 0$ to get $x = y$ and $(3y + 1)(y - 1) = 3y^2 - 2y - 1 = 0$. Thus we obtain two critical points $(1, 1)$ and $(-1/3, -1/3)$. Note that $f(1, 1) = -1$, $f(-1/3, -1/3) = 5/27$.

Compute $f_{xx} = 2, f_{xy} = -4$ and $f_{yy} = 6y$. At $(1, 1)$, $\Delta(1, 1) = 12 - (-4)^2 < 0$, and at $(-1/3, -1/3, 5/27)$, $\Delta(-1/3, -1/3) =< 0$. Therefore at these two points $(1, 1, -1)$ and $(-1/3, -1/3, 5/27)$, the surface has saddle points.

Example (3) : Find and classify the critical points of $f(x, y) = 3xy - x^3 - y^3$.

Solution: Compute $f_x = 3y - 3x^2$ and $f_y = 3x - 3y^2$. Solve $f_x = 0$ and $f_y = 0$ and note that $x \geq 0$ and $y \geq 0$ to get two critical points $(0, 0)$ and $(1, 1)$. Note that $f(0, 0) = 0$, $f(1, 1) = 1$.

Compute $f_{xx} = -6x, f_{xy} = 3$ and $f_{yy} = -6y$. At $(0, 0, 0)$, $\Delta(0, 0) = 0 - 3^2 < 0$, and at $(1, 1, 1)$, $\Delta(1, 1) > 0$, and $f_{xx}(1, 1) < 0$. Therefore at $(0, 0, 0)$, the surface has a saddle points, and at $(1, 1, 1)$, the surface has a local maximum.

Example (4) : Find and classify the critical points of $f(x, y) = x^3 + y^3 + 3xy + 3$.

Solution: Compute $f_x = 3x^2 + 3y$ and $f_y = 3y^2 + 3x$. Solve $f_x = 3x^2 + 3y = 0$ and $f_y = 3y^2 + 3x = 0$. Note that both $x \leq 0$ and $y \leq 0$. Combine the two equations to get

$x(x^3+1) = 0$ and so $x = 0$ or $x = -1$. As $y \leq 0$, we substitute $x = 0$ and $x = -1$ respectively to get two critical points $(0, 0)$ and $(-1, -1)$. Note that $f(0, 0) = 3$ and $f(-1, -1) = 4$.

Compute $f_{xx} = 6x$, $f_{xy} = 3$ and $f_{yy} = 6y$. As $\Delta(0, 0) = -9 < 0$, the surface has a saddle point at $(0, 0, 3)$. As $\Delta(-1, -1) = 27 > 0$ and $f_{xx}(-1, -1) = -6 < 0$, the surface has a local maximum at $(-1, -1, 4)$.

Example (5) : Find and classify the critical points of $f(x, y) = 2xye^{-x^2-y^2}$.

Solution: Compute $f_x = 2ye^{-x^2-y^2} - 4x^2ye^{-x^2-y^2} = 2(1 - 2x^2)ye^{-x^2-y^2}$ and $f_y = 2(1 - 2y^2)xe^{-x^2-y^2}$. Solve $f_x = 0$ and $f_y = 0$ to get solutions $(0, 0)$, $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Note that $f(0, 0) = 0$, $f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = e^{-1}$ and $f(-1/\sqrt{2}, 1/\sqrt{2}) = f(1/\sqrt{2}, -1/\sqrt{2}) = -e^{-1}$.

Compute $f_{xx} = 2(-4xye^{-x^2-y^2} - 2x(1 - 2x^2)ye^{-x^2-y^2}) = -4xy(3 - x^2)e^{-x^2-y^2}$, $f_{xy} = 2(1 - 2x^2)(1 - 2y^2)e^{-x^2-y^2}$ and $f_{yy} = -4xy(3 - y^2)e^{-x^2-y^2}$, and so $\Delta(0, 0) = -4 < 0$ (saddle point),; $\Delta(1/\sqrt{2}, 1/\sqrt{2}) = \Delta(-1/\sqrt{2}, -1/\sqrt{2}) = 49 > 0$, and $f_{xx}(1/\sqrt{2}, 1/\sqrt{2}) = f_{xx}(-1/\sqrt{2}, -1/\sqrt{2}) < 0$ (local maximum); $\Delta(-1/\sqrt{2}, 1/\sqrt{2}) = \Delta(1/\sqrt{2}, -1/\sqrt{2}) = 49 > 0$, and $f_{xx}(-1/\sqrt{2}, 1/\sqrt{2}) = f_{xx}(1/\sqrt{2}, -1/\sqrt{2}) > 0$ (local minimum).