Compute partial derivatives with Chain Rule

Formulae: These are the most frequently used ones:

1. If $w = f(x, y)$ and $x = x(t)$ and $y = y(t)$ such that f, x, y are all differentiable. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}. \tag{1}$$

2. If $w = f(x_1, x_2, \cdots, x_m)$ and for each i, $(1 \le i \le n)$, $x_i = x_i(t_1, t_2, \cdots, t_n)$ such that $f, x_1, \cdots, x_m$ are all differentiable. Then

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_i}. \tag{2}$$

Example (1) : Given $w = \ln(u + v + z)$, with $u = \cos^2 t$, $v = \sin^2 t$ and $z = t^2$, find dw/dt both by using the chain rule and by expressing w explicitly as a function of t before differentiating.

Solution: First we apply Chain Rule (1):

$$\frac{dw}{dt} = \frac{\partial w}{\partial u}\frac{du}{dt} + \frac{\partial w}{\partial v}\frac{dv}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{u+v+z}(-2\cos t\sin t + 2\sin t\cos t + 2t) = \frac{2t}{u+v+z}.$$

Next, we express w as a function of t by substituting u, v and z into w to get $w = \ln(\cos^2 t + \sin^2 t + t^2) = \ln(1 + t^2)$. Therefore, $dw/dt = 2t/(1 + t^2)$.

Example (2) : Given $w = yz + zx + xy$, $x = s^2 - t^2$, $y = s^2 + t^2$ and $z = s^2t^2$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

Solution: This is a partial derivative problem and so we apply Chain Rule (2).

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = (z + y)(2s) + (z + x)2s + (x + y)2st^2 \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = -(z + y)(2t) + (z + x)2t + (x + y)2s^2t
\end{aligned}$$

Example (3) : Given $p = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, write the chain rule formulas giving the partial derivatives of the dependent variable p with respect to each independent variable.

Solution: This is a partial derivative problem and so we apply Chain Rule (2).

$$\begin{aligned}
\frac{\partial p}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\
\frac{\partial p}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}
\end{aligned}$$

Example (4) : Given $x^3 + y^3 + z^3 = xyz$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as functions of x, y and z.

Solution: As z is an implicit function of x and y, implicit differentiation must be used. Just view $z = z(x, y)$ everywhere z occurs when we differentiate both sides of the equation.
(Step 1) View $z = z(x, y)$ and differentiate both sides of the equation with respect to x to get

$$3x^2 + 3z^2\frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x}.$$

(Step 2) Solve for $\frac{\partial z}{\partial x}$ in the resulting equation above.

$$z_x = \frac{\partial z}{\partial x} = \frac{yz - 3x^2}{3z^2 - xy}.$$

Do the same for y.
(Step 1) View $z = z(x, y)$ and differentiate both sides of the equation with respect to y to get

$$3y^2 + 3z^2\frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}.$$

(Step 2) Solve for $\frac{\partial z}{\partial y}$ in the resulting equation above.

$$z_y = \frac{\partial z}{\partial y} = \frac{xz - 3y^2}{3z^2 - xy}.$$

Example (5) : Given $x^3 + y^3 + z^3 = xyz$ as the equation of a surface, find an equation of the plane tangent to this surface at the point $P(1, -1, -1)$.

Solution 1: View $z = z(x, y)$, and so the vector $\mathbf{n} = (z_x, z_y, -1)$ at P will be a normal vector of the tangent plane. From Example (4) above, we have, at $P(1, -1, -1)$,

$$\begin{aligned} z_x &= \left.\frac{yz - 3x^2}{3z^2 - xy}\right|_{(x,y,z)=(1,-1,-1)} = \frac{(-1)(-1) - 3}{3(-1)^2 - 1(-1)} = \frac{-2}{4} = -\frac{1}{2}, \\ z_y &= \left.\frac{xz - 3y^2}{3z^2 - xy}\right|_{(x,y,z)=(1,-1,-1)} = \frac{1(-1) - 3}{3(-1)^2 - 1(-1)} = \frac{-4}{4} = -1. \end{aligned}$$

Hence the equation of the tangent plane is

$$-\frac{1}{2}(x - 1) - (y + 1) - (z + 1) = 0, \text{ or } (x - 1) + 2(y + 1) + 2(z + 1) = 0.$$

Solution 2: One can also set $F(x, y, z) = x^3 + y^3 + z^3 - xyz$ and view the equation of the surface is $F(x, y, z) = 0$. In this case, the vector $\mathbf{u} = (F_x, F_y, F_z)$ at $P(1, -1, -1)$ can be a normal vector of the tangent plane. We compute the partial derivatives:

$$F_x = \frac{\partial F}{\partial x} = 3x^2 - yz,\ F_y = \frac{\partial F}{\partial y} = 3y^2 - xz,\ F_z = \frac{\partial F}{\partial z} = 3z^2 - xy.$$

Therefore, $\mathbf{u} = (2, 4, 4)$ and so an equation of the tangent plane is

$$2(x-1) + 4(y+1) + 4(z+1) = 0, \text{ or } (x-1) + 2(y+1) + 2(z+1) = 0.$$

Example (6) : Suppose that $w = f(u)$ and that $u = x + y$. Show that $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$.

Solution: Notice that $\frac{\partial u}{\partial x} = 1 = \frac{\partial u}{\partial y}$. By Chain Rule (2),

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y}.$$

Example (7) : Suppose that $w = f(x, y)$ and that $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

Solution: Apply Chain Rule (2) to compute the first order of partial derivatives

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = w_x \cos\theta + w_y \sin\theta \\
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -w_x r\sin\theta + w_y r\cos\theta.
\end{aligned}$$

Then, we apply Chain Rule (2) again to compute the second order of partial derivatives (making use of $w_{xy} = w_{yx}$),

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2} &= w_{xx}\cos^2\theta + w_{xy}\cos\theta\sin\theta + w_{yx}\sin\theta\cos\theta + w_{yy}\sin^2\theta \qquad (3)\\
&= w_{xx}\cos^2\theta + (w_{xy} + w_{yx})\cos\theta\sin\theta + w_{yy}\sin^2\theta,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 w}{\partial \theta^2} &= w_{xx}r^2\sin^2\theta - w_{xy}r^2\sin\theta\cos\theta - w_x r\cos\theta + w_{yy}r^2\cos^2\theta + w_{yx}r^2\cos\theta(-\sin\theta) - w_y r\sin\theta \\
&= w_{xx}r^2\sin^2\theta + w_{yy}r^2\cos^2\theta - (w_{xy} + w_{yx})r^2\sin\theta\cos\theta - w_x r\cos\theta - w_y r\sin\theta.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} &= w_{xx}\sin^2\theta + w_{yy}\cos^2\theta - (w_{xy} + w_{yx})\sin\theta\cos\theta - \frac{w_x\cos\theta + w_y\sin\theta}{r} \qquad (4)\\
\frac{1}{r}\frac{\partial w}{\partial r} &= \frac{w_x\cos\theta + w_y\sin\theta}{r}. \qquad (5)
\end{aligned}$$

Now add equations (3), (4) and (5) side by side and apply $\sin^2\theta + \cos^2\theta = 1$ to get the conclusion.