Compute partial derivatives

Example (1) : Compute the first order partial derivatives of $f(x,y) = \dfrac{xy}{x^2+y^2}$.

Solution: View y as a constant to compute f_x.

$$f_x = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - yx^2}{(x^2+y^2)^2}.$$

View x as a constant to compute f_y. (You may also observe the symmetry between x and y),

$$f_y = \frac{x(x^2+y^2) - 2y(xy)}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}.$$

Example (2) : Compute the first order partial derivatives of $f(x,y) = x^y$.

Solution: Fancy rules are not needed. View y as a constant to compute $f_x = yx^{y-1}$; and view x as a constant to compute $f_y = x^y \ln x$.

Example (3) : Verify that $z_{xy} = z_{yx}$ if $z = 2x^3 + 5x^2y - 6y^2 + xy^4$.

Solution: View y as a constant to compute $z_x = 6x^2 + 10xy + y^4$; then view x as a constant to compute $z_{xy} = (z_x)_y = 10x + 4y^3$.

Similarly, view x as a constant to compute $z_y = 5x^2 - 12y + 4xy^3$; then view y as a constant to compute $z_{yx} = (z_y)_x = 10x + 4y^3$. It is now clear that $z_{xy} = z_{yx}$.

Find an equation of the tangent plane

Useful Fact: A normal vector of the plane tangent to a surface $z = f(x,y)$ at a point is $(z_x, z_y, -1)$ at P.

Example (1) : Find an equation of a plane tangent to $z = \sin \dfrac{\pi xy}{2}$ at $P(3,5,-1)$.

Solution: Compute the partial derivatives

$$z_x = \frac{\pi y}{2} \cos \frac{\pi xy}{2}, \text{ and } z_y = \frac{\pi x}{2} \cos \frac{\pi xy}{2}.$$

Note that $\cos \frac{15\pi}{2} = 0$. Thus a normal vector of the plane is $\mathbf{n} = (0,0,-1)$, and so an equation of the plane is $z = -1$.

Example (2) : Find an equation of a plane tangent to $z = \sqrt{x^2+y^2}$ at $P(3,-4,5)$.

Solution: Compute the partial derivatives

$$z_x = \frac{x}{\sqrt{x^2+y^2}}, \text{ and } z_y = \frac{y}{\sqrt{x^2+y^2}}.$$

Thus a normal vector of the plane is $\mathbf{n} = (\frac{3}{5}, -\frac{4}{5}, -1)$, and so an equation of the plane is

$$3(x-3) - 4(y+4) - 5(z-5) = 0.$$

Example (3) : Show that there at which the plane tangent to $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

Solution: Compute $z_x = 2x + 2y - 6$ and $z_y = 2x + 4y + 8$. A horizontal plane must have a normal vector parallel to $\mathbf{k}$, and so the point at which the tangent plane is horizontal must satisfy $z_x = 0$ and $z_y = 0$. Solve this system

$$\begin{cases} 2x + 2y - 6 = 0 \\ 2x + 4y + 8 = 0 \end{cases}$$

to get the answer $x = 10, y = -7$. Note that $z(10,-7) = -58$. Therefore, at $(10,-7,-58)$, the tangent plane is horizontal, and this is the only point with such a property.