Compute unit tangent and unit normal vectors, tangential and normal components (for 2D vectors)

Example: Find the unit tangent and unit normal vectors, tangential and normal components of the curve $x = t - \sin t, y = 1 - \cos t$ at $t = \frac{\pi}{2}$.

Solution: The **position vector** is $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$.
(Step 1) Compute the **velosity vector** $\mathbf{v}(t) = \mathbf{r}'(t) = (1 - \cos t, \sin t)$, and the **speed** $|\mathbf{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$.
(Step 2) Compute the **unit tangent vector**:

$$\mathbf{T}(t) = \frac{1}{|\mathbf{v}|\mathbf{v}} = \left(\frac{1 - \cos t}{\sqrt{2 - 2\cos t}}, \frac{\sin t}{\sqrt{2 - 2\cos t}}\right).$$

When $t = \frac{\pi}{2}$, $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$. Thus $|\mathbf{v}| = \sqrt{2}$, ans so $\mathbf{T}(\frac{\pi}{2}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
(Step 3) Compute the **acceleration vector** and the **tangential component**:

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{v}'(t) = (\sin t, \cos t). \\ a_T &= \frac{d|\mathbf{v}|}{dt} = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{\sin t - \sin t \cos t + \sin t \cos t}{\sqrt{2 - 2\cos t}} = \frac{\sin t}{\sqrt{2 - 2\cos t}}. \end{aligned}$$

When $t = \frac{\pi}{2}$, $\mathbf{a} = (1, 0)$ and $a_T = \frac{1}{\sqrt{2}}$.
(Step 4) Compute, at $t = \frac{\pi}{2}$, (view the vectors as 3D vectors) $\mathbf{v} \times \mathbf{a} = (1, 1, 0) \times (1, 0, 0) = (0, 0, -1)$. Then use it to compute the **curvature**

$$\kappa(t) = \frac{1}{|\mathbf{v}|}\frac{\mathbf{T}}{dt} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \ \kappa(\frac{\pi}{2}) = \frac{1}{(\sqrt{2})^3},$$

and the **normal component** at $t = \frac{\pi}{2}$,

$$a_N = \kappa \mathbf{v}^2 = \frac{1}{(\sqrt{2})^3}(\sqrt{2})^2 = \frac{1}{\sqrt{2}}.$$

(Step 5) Compute the **unit normal vector** at $t = \frac{\pi}{2}$,

$$\mathbf{N} = \frac{1}{a_N}(\mathbf{a} - a_T\mathbf{T}) = \sqrt{2}\left(1, 0\right) - \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\sqrt{2} - 1, -1\right).$$