Show a parametric curve is lying on a surface

Example: Show that the graph of the curve $x = \sin t, y = \cos t, z = \cos 8t$ lies on te vertical circular cylinder $x^2 + y^2 = 1$.

Solution: Substitute the parametric equations of the curve into the left hand side of $x^2 + y^2$, and apply a trigonometry identity to get

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1,$$

and so every point of the parametric curve satisfies the equation of the cylinder, and so the curve lies on the cylinder.

Evaluate vector functions

Example Given $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, find the values of $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ at $t = \frac{\pi}{4}$.

Solution: Compute $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. Thus

$$\mathbf{r}(\frac{\pi}{4} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}, \text{ and } \mathbf{r}'(\frac{\pi}{4} = \frac{-\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}.$$

Find velocity and acceleration vectors

Example Given the position vector $\mathbf{r}(t) = 12t\mathbf{i} + (5\sin 2t)\mathbf{j} - (5\cos 2t)\mathbf{k}$, find its velocity and acceleration vectors and its speed at t.

Solution: They are

$$\begin{aligned}
\text{velocity } \mathbf{v}(t) &= \mathbf{r}'(t) = 12\mathbf{i} + (10\cos 2t)\mathbf{j} + (10\sin 2t)\mathbf{k} \\
\text{acceleration } \mathbf{a}(t) &= \mathbf{v}'(t) = -(20\sin 2t)\mathbf{j} + (20\cos 2t)\mathbf{k} \\
\text{speed } |\mathbf{v}(t)| &= \sqrt{12^2 + (10\cos 2t)^2 + (10\sin 2t)^2} = \sqrt{144 + 100} = \sqrt{244}.
\end{aligned}$$

Compute the integrals of vector functions

Example Compute

$$\int_1^e \left(\frac{1}{t}\mathbf{i} - \mathbf{j}\right) dt.$$

Solution: The answer is

$$\int_1^e \left(\frac{1}{t}\mathbf{i} - \mathbf{j}\right) dt = [\ln |t|\mathbf{i} - t\mathbf{j}]_1^e = \mathbf{i} - (e-1)\mathbf{j}.$$