Determine the relationship of a line and a plane (in parametric equations)

Example: Given a line L and a plane $\mathcal{P}$:

$$\begin{aligned} L: &\quad x = 7 - 6r, y = 3 + 3r, z = 28 + 3r \\ \mathcal{P}: &\quad x = 7s + 3t, y = 4s - 2t, z = -5s + 6t. \end{aligned}$$

Determine if L and $\mathcal{P}$ intersect or are parallel.

Solution: The normal vector of $\mathcal{P}$ is $\mathbf{n} = (7, 4, -5) \times (3, -2, 6) = (24 - 10, -(42 + 15), -14 - 12) = (14, -57, -26)$, and the line L is parallel to $\mathbf{v} = (-6, 3, 3)$. As the dot product $\mathbf{n} \cdot \mathbf{v} = (-6)(14) + (-57)(3) + (-26)(3) \neq 0$, L and $\mathcal{P}$ are not parallel to each other.

We need to see where the two objects intersect. Substitute the parametric equations of L into the equations of $\mathcal{P}$ to get:

$$\begin{cases} 7 - 6r = 7s + 3t \\ 3 + 3r = 4s + 2t \\ 28 + 3r = -5s + 6t \end{cases} \Longrightarrow \begin{cases} 7s + 3t + 6r = 7 \\ 4s + 2t - 3r = 3 \\ -5s + 6t - 3r = 28 \end{cases} \Longrightarrow t = 4, s = -1, \text{ and } r = \frac{1}{3}.$$

Thus the point of intersection is

$$x = 7 - \frac{6}{3} = 5, y = 3 + \frac{3}{3} = 4, z = 28 + \frac{3}{3} = 29.$$

Determine the relationship of two planes (in parametric equations)

Example: Given two planes $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\begin{aligned} \mathcal{P}_1: &\quad x = 7 - s + t, y = 3 + s, z = 9 + 2s - t \\ \mathcal{P}_2: &\quad x = 5 + 2t, y = 3 + 2s + t, z = 4 - s - t. \end{aligned}$$

Determine the relationship between them.

Solution: The planes have normal vectors $\mathbf{a} = (-1, 1, 2) \times (1, 0, -1) = (-1 - 0, -(1 - 2), 0 - 1) = (-1, 1, -1)$, and $\mathbf{b} = (0, 2, -1) \times (2, 1, -1) = (-2 - (-1), -(0 + 2), 0 - 4) = (-1, -2, -4)$, respectively. As $\mathbf{a}$ cannot be a scalar product of $\mathbf{b}$ (try to solve $c\mathbf{a} = \mathbf{b}$ to convince yourself), the two planes are not parallel, and so they are not equal either.

The consine of the angle θ between the two planes can be found as

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{(-1)(-1) + (1)(-2) + (-1)(-4)}{\sqrt{3} \cdot \sqrt{21}} = \frac{3}{3\sqrt{7}} = \frac{1}{\sqrt{7}}, \text{ and so } \theta = \cos^{-1}\frac{1}{\sqrt{7}}.$$