## Compute derivatives of implicit functions

Facts: An equation $F(x, y)=0$ involving variables $x$ and $y$ ( may define $y$ as a function $y=y(x)$. To compute $y^{\prime}=\frac{d y}{d x}$, one can apply the following procedure.
(Step 1) View $y=y(x)$ and differentiate both sides of the equation $F(x, y)=0$ with respect to $x$. This will yield a new equation involving $x, y$ and $y^{\prime}$.
(Step 2) Solve the resulting equation from (Step 1) for $y^{\prime}$.

Example 1 Given $x^{4}+x^{2} y^{2}+y^{4}=48$, find $\frac{d y}{d x}$.
Solution: View $y=y(x)$ and differentiate both sides of the equation $x^{4}+x^{2} y^{2}+y^{4}=48$ to get

$$
4 x^{3}+2 x y^{2}+2 x^{2} y y^{\prime}+4 y^{3} y^{\prime}=0 .
$$

To solve this new equation for $y^{\prime}$, we first combine those terms involving $y^{\prime}$,

$$
\left(2 x^{2} y+4 y^{3}\right) y^{\prime}=-4 x^{3}-2 x y^{2},
$$

and then solve for $y^{\prime}$ :

$$
y^{\prime}=\frac{-4 x^{3}-2 x y^{2}}{2 x^{2} y+4 y^{3}} .
$$

Example 2 Find an equation of line tangent to the curve $x y^{2}+x^{2} y=2$ at the point (1, -2).

Solution: The slope $m$ of this line, is $\frac{d y}{d x}$ at $(1,-2)$, and so we need to find $y^{\prime}$ first. Apply implicit differentiation. We differentiate both sides of the equation $x y^{2}+x^{2} y=2$ with respect to $x$ (view $y=y(x)$ in the process) to get

$$
y^{2}+2 x y y^{\prime}+2 x y+x^{2} y^{\prime}=0 .
$$

Then we solve for $y^{\prime}$. First we have $\left(2 x y+x^{2}\right) y^{\prime}=-y^{2}-2 x y$, and then

$$
y^{\prime}=\frac{-y^{2}-2 x y}{2 x y+x^{2}} .
$$

At $(1,-2)$, we substitute $x=1$ and $y=-2$ in $y^{\prime}$ to get the slope $m=\frac{-(-2)^{2}-2(1)(-2)}{2(1)(-2)+1^{2}}=0$, and so the tangent line is $y=-2$.

Example 3 Find all the points on the graph of $x^{2}+y^{2}=4 x+4 y$ at which the tangent line is horizontal.

Solution: First find $y^{\prime}$. We differentiate both sides of the equation $x^{2}+y^{2}=4 x+4 y$ with respect to $x$ (view $y=y(x)$ in the process) to get

$$
2 x+2 y y^{\prime}=4+4 y^{\prime}
$$

Then we solve for $y^{\prime}$. First we have $(2 y-4) y^{\prime}=4-2 x$, and then

$$
y^{\prime}=\frac{2-x}{y-2}
$$

Note that when $x=2$, the equation $x^{2}+y^{2}=4 x+4 y$ becomes $4+y^{2}=8+4 y$, or $y^{2}-4 y=4$. Solve this equation we get $y=2+\sqrt{8}$ and $y=2-\sqrt{8}$. Therefore, at $(2,2-\sqrt{8})$ and $(2,2+\sqrt{8})$, the curve has horizontal tangent lines.

