MATH 251 - QUIZ 5

1. Given $w = \sqrt{u^2+v^2+z^2}$ and $u = 3e^t \sin s$, $v = 3e^t \cos s$ and $z = 4e^t$, find $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$.

Solution Apply chain rule to get

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{2u}{2\sqrt{u^2+v^2+z^2}}3e^t\cos s + \frac{2v}{2\sqrt{u^2+v^2+z^2}}3e^t(-\sin s)\\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\
&= \frac{2u}{2\sqrt{u^2+v^2+z^2}}3e^t\sin s + \frac{2v}{2\sqrt{u^2+v^2+z^2}}3e^t\cos s + \frac{2z}{2\sqrt{u^2+v^2+z^2}}4e^t
\end{aligned}$$

Note that $\sqrt{u^2+v^2+z^2} = \sqrt{9e^{2t}\sin^2 t + 9e^{2t}\cos^2 t + 16e^{2t}} = \sqrt{25e^{2t}} = 5e^t$. We can substitute u, v, z in their expressions of t to get $w_s = 0$ and $w_t = 5e^t$.

2. Assume that $z = z(x, y)$ satisfy the equation $x^5 + xy^2z + yz^3 = 3$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution Apply implicit differentiation. For z_x, we have these steps.
(Step 1) Partially differentiate both sides with respect to x,

$$5x^4 + y^2z + xy^2z_x + 3yz^2z_x = 0.$$

(Step 2) Solve the above equation for z_x.

$$(xy^2 + 3yz^2)z_x = -(5x^4 + y^2z), \text{ and so } z_x = -\frac{5x^4 + y^2z}{xy^2 + 3yz^2},$$

For z_y, we have these steps.
(Step 1) Partially differentiate both sides with respect to y,

$$2xyz + xy^2z_y + z^3 + 3yz^2z_y = 0.$$

(Step 2) Solve the above equation for z_y.

$$(xy^2 + 3yz^2)z_y = -(2xyz + z^3), \text{ and so } z_y = -\frac{2xyz + z^3}{xy^2 + 3yz^2},$$

3. (Continuation of Problem 2) Given a surface with the equation $x^5 + xy^2z + yz^3 = 3$, find an equation of the plane tangent to this surface at the point $P(1, 1, 1)$.

Solution A normal vector of the plane is $\mathbf{n} = (z_x, z_y, -1)$ at $P(1, 1, 1)$. Therefore $\mathbf{n} = -(6/4, 3/4, 1)$ and so the following is an equation of the plane:

$$\frac{6}{4}(x-1) + \frac{3}{4}(y-1) + (z-1) = 0 \text{ or } 6(x-1) + 3(y-1) + 4(z-1) = 0.$$