MATH 251 - Worksheet 9

1. Find the the mass and centroid of the first-octant region that is interior to $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ (with density $\delta \equiv 1$).

Solution Project the solid on the yz-plane. Then on the yz-plane, the region R is bounded by $y = 0$, $z = 0$ and $y^2 + z^2 = 1$. Thus

$$\begin{aligned}
m &= \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} dxdydz = \int_0^1 (1 - z^2)dz = \frac{2}{3} \\
\overline{x} &= \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} xdxdydz = \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1 - z^2}{2} dydz \\
&= \frac{3}{2} \int_0^1 \frac{(1 - z^2)^{\frac{3}{2}}}{2} dz \\
&= \frac{3}{4} \left[\frac{y}{8}(5 - 2y^2)\sqrt{1 - y^2} + \frac{3}{8} \sin^{-1} y \right]_0^1 = \frac{9\pi}{64} \\
\overline{y} &= \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} ydydxdz == \frac{9\pi}{64} \\
\overline{z} &= \frac{3}{2} \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} zdydxdz = \frac{3}{2} \int_0^1 (1 - z^2)zdz = \frac{3}{8}.
\end{aligned}$$

2. Find the volume of the region that lies inside both $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 - 2x = 0$.

Solution One can compute the volume of the first quadrant of the solid (thus the solid lies between $z = \sqrt{4 - (x^2 + y^2)}$, and $z = 0$). The projection on the xy-plane is a region bounded by $(x - 1)^2 + y^2 = 1$ and $x = 0$, or $r = 2\cos\theta$ with $0 \le \theta \le \pi/2$ in polar. Therefore, the volume is, in cylindrical coordinates,

$$\begin{aligned}
\text{V} &= 4 \int_0^{\pi/2} \int_0^{2\cos\theta} \int_0^{\sqrt{4-r^2}} rdzdrd\theta = 4 \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{4 - r^2}drd\theta \\
&= 4 \int_0^{\pi/2} \left[\frac{-1}{3}(4 - r^2)^{\frac{3}{2}} \right]_0^{2\cos\theta} d\theta = \frac{32}{3} \int_0^{\pi/2} (1 - \sin^3\theta)d\theta \\
&= \frac{32}{3} \int_0^{\pi/2} (1 - (1 - \cos^2\theta)\sin\theta)d\theta = \frac{32}{3} \left[\theta + \cos\theta - \frac{\cos^3\theta}{3} \right]_0^{\pi/2} = \frac{16}{9}(3\pi - 4).
\end{aligned}$$

3. Find the volume of the region bounded by $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$.

Solution Project the intersection of the two surfaces down to the xy-plane (by getting rid of z in the system of equations $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$), we get a region R on the xy-plane bounded by $x^2 + y^2 = 4$. Use cylindrical coordinates, we have

$$\text{V} = \int_{-\pi}^{\pi} \int_0^2 \int_{x^2+2y^2}^{12-2x^2-y^2} rdzdrd\theta$$

$$= \int_{-\pi}^{\pi}\int_{0}^{2} r(12-3r^2)drd\theta = \int_{-\pi}^{\pi} 12d\theta = 24\pi.$$

4. (Use spherical coordinates) Find the volume of the region bounded by the plane $z = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution As at lateral the boundary of the cone, $\tan\phi = 1$, $\phi \le \frac{\pi}{4}$. Thus $-\pi \le \theta \le \pi$, $0 \le \phi \le \pi/4$ and $0 \le \rho \le \rho\sin\phi$.

$$\begin{aligned} \mathrm{V} &= \int_{-\pi}^{\pi}\int_{0}^{\pi/4}\int_{0}^{\sec\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{1}{3}\int_{-\pi}^{\pi}\int_{0}^{\pi/4} \sec^3\phi \sin\phi d\rho d\phi d\theta \\ &= \frac{1}{3}\int_{-\pi}^{\pi}\int_{0}^{\pi/4} \sec^2\phi \tan\phi d\phi d\theta \qquad\qquad \text{set } u = \tan\phi \\ &= \frac{1}{3}\int_{-\pi}^{\pi} d\theta \int_{0}^{1} udu = \frac{\pi}{3}. \end{aligned}$$

5. (Use cylindrical coordinates) Find the volume of the region bounded by the plane $z = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution For each fixed θ with $-\pi \le \theta \le \pi$, the cross section is a triangle on rz-plane bounded by $r = z$, $r = 0$ and $z = 1$. Thus

$$\begin{aligned} \mathrm{V} &= \int_{-\pi}^{\pi}\int_{0}^{1}\int_{0}^{z} rdrdzd\theta \\ &= \int_{-\pi}^{\pi}\int_{0}^{1} \frac{z^2}{2} dzd\theta = \frac{2\pi}{6} = \frac{\pi}{3}. \end{aligned}$$