MATH 251 - Worksheet 10

NAME:

I.D.:

Instruction: Circle your answers and show all your work CLEARLY. Solutions with answer only and without supporting procedures will have little credit. Leave your answer in fractions.

1. Compute the divergence and the curl of the vector field $\mathbf{F} = (xy^2, yz^2, zx^2)$.

Solution:

$$\begin{aligned} \text{div}\mathbf{F} &= y^2 + z^2 + x^2. \\ \text{curl}\mathbf{F} &= (0 - 2yz, -(2xz - 0), 0 - 2xy) = (-2yz, -2xz, -2xy). \end{aligned}$$

2. Let $\mathbf{r} = (x, y, z)$. Compute both $\nabla \cdot \mathbf{r}$ and $\nabla \times \mathbf{r}$.

Solution:

$$\begin{aligned} \text{div}\mathbf{F} &= 1 + 1 + 1 = 3. \\ \text{curl}\mathbf{F} &= (0 - 0, -(0 - 0), 0 - 0) = (0, 0, 0). \end{aligned}$$

3. Given $f(x, y) = x + y$, and the curve $C : x = e^t + 1$, $y = e^t - 1$, $0 \le t \le \ln 2$. compute

$$\int_C f ds, \int_C f dx, \text{ and } \int_C f dy.$$

Solution:

$$\begin{aligned} \int_C f ds &= \int_0^{\ln 2} [(e^t + 1) + (e^t - 1)]\sqrt{2}e^t dt = 2\sqrt{2}\int_0^{\ln 2} e^{2t} dt = \left[\frac{2\sqrt{2}e^{2t}}{2}\right]_0^{\ln 2} = 3\sqrt{2}. \\ \int_C f dx &= \int_0^{\ln 2} (2e^t)e^t dt = 2\int_0^{\ln 2} e^{2t} dt = \left[\frac{2e^{2t}}{2}\right]_0^{\ln 2} = 3. \\ \int_C f dy &= \int_0^{\ln 2} (2e^t)e^t dt = 2\int_0^{\ln 2} e^{2t} dt = \left[\frac{2e^{2t}}{2}\right]_0^{\ln 2} = 3. \end{aligned}$$

4. Compute $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where $\mathbf{F} = (yz^2, xz^2, 2xyz)$ and C is the path from $(-1, 2, -2)$ to $(1, 5, 2)$ that consists of three segments parallel to the z-axis, x-axis and y-axis, in that order.

Solution: Let $C_1 : -2 \leq z \leq 2$ with $x = -1$ and $y = 2$, $C_2 : -1 \leq x \leq 1$ with $y = 2$ and $z = 2$ and $C_3 : 2 \leq y \leq 5$ with $x = 1$ and $z = 2$. Then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_{-2}^{2} 2(-1)(2)z dz + \int_{-1}^{1} (2)(2^2) dx + \int_{2}^{5} (1)(2^2) dy = 0 + 16 + 12 = 28.$$

5. Compute $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where $\mathbf{F} = (yz^2, xz^2, 2xyz)$ and C is a path from $(-1, 2, -2)$ to $(1, 5, 2)$ that we do not know its equations, but we do know that $\mathbf{F}$ has a potential function.

Solution: One can guess a potential function $f(x, y, z) = xyz^2$ and verify that $\nabla f = \mathbf{F}$. Thus

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = f(1, 5, 2) - f(-1, 2, -2) = 20 - (-8) = 28.$$

6. Compute the line integral

$$\int_{(0,0)}^{(1,-1)} 2xe^y dx + x^2 e^y dy.$$

Solution: Let $P = 2xe^y$ and $Q = x^2 e^y$. Then$P_y = P = Q_x$. Therefore, this is a conservative field and so the integration in independent of path. Choose $C_1 : 0 \leq x \leq 1$ with $y = 0$ and C_2 is the reverse path of $-1 \leq y \leq 0$ with $x = 1$. Then

$$\int_{(0,0)}^{(1,-1)} 2xe^y dx + x^2 e^y dy = \int_0^1 2x dx - \int_{-1}^{0} e^y dy = 1 - (e^0 - e^{-1}) = e^{-1}.$$

Another way is to find a potential function $f(x, y) = x^2 e^y$ of the related vector field, and so the integral is $f(1, -1) - f(0, 0) = e^{-1}$.