EXAM 4 - Math 251

1. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that is bounded by $z = x^2 + y^2, x + y = 1, x = 0, y = 0$ and $z = 0$.

Solution: The top surface is $z = x^2 + y^2$ and the bottom is $z = 0$, over the region R on the xy-plane bounded by $x + y = 1, x = 0, y = 0$. Therefore, using rectangular coordinates, the integral is

$$\text{Vol} = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dzdydx.$$

2. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that lies inside both $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.

Solution: Let us compute only the upper half solid. The top surface is $z = \sqrt{4 - (x^2 + y^2)}$ and the bottom is $z = 0$, over the region R on the xy-plane bounded by $x^2 + y^2 = 1$. Therefore, using cylindrical coordinates, the integral is

$$\text{Vol} = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} rdzdrd\theta.$$

3. (10 %) Find the integral (clearly indicates what coordinate system is used, and the bounds of the integration) that computes the volume of of a solid that lies inside both $x^2 + y^2 + z^2 = 16$ and $z = \sqrt{x^2 + y^2}$.

Solution: The top surface is $z = \sqrt{16 - (x^2 + y^2)}$ and the bottom is $z = \sqrt{x^2 + y^2}$. Therefore, it is more convenient to use spherical coordinates. The largest ϕ is when $z = \sqrt{x^2 + y^2} = r$, which means $\tan\phi = \frac{r}{z} = 1$, and so $\phi = \pi/4$. In this case $-\pi \leq \theta \leq \pi$, $0 \leq \phi \leq \pi/4$ and $0 \leq \rho \leq 4$. Therefore, the integral is

$$\text{Vol} = \int_{-\pi}^{\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin\phi drhod\phi d\theta.$$

One can also use cylindrical coordinates. For each fixed θ in the interval $[-\pi, \pi]$, the cross section can be viewed as a region R in the rz-plane, where R is bounded by $r = 0$ on the left, $z = r$ below and $z = \sqrt{16 - r^2}$ above. The maximum value of r can be found by solving $z = r$ and $z = \sqrt{16 - r^2}$ for r. As $r^2 = z^2 = 16 - r^2$, we have $r = \sqrt{8}$ (note that $r > 0$ and so we throw away $r = -\sqrt{8}$ in the solution). Therefore,

$$\text{Vol} = \int_{-\pi}^{\pi} \int_0^{\sqrt{8}} \int_r^{\sqrt{16-r^2}} rdzdrd\theta.$$

4. (10 %) Compute the following integral by using the spherical coordinates.

$$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2} dzdydx.$$

Solution: The upper surface of the integration solid is $z = \sqrt{4-x^2-y^2}$ and the bottom is $z = 0$. The projection of this solid on the xy-plane is the region enclosed by $x^2 + y^2 = 4$. In spherical coordinates, for each fixed θ_0 in $[-\pi, \pi]$, the cross section (the intersection of the plane $\theta = \theta_0$ and the solid) will be a quarter of the circle $\rho^2 = 4$ (with θ being a constant). Therefore, $0 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/2$. Thus the integral is

$$\begin{aligned} I &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2} dzdydx \\ &= \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^5 \cos^2\phi \sin\phi drhod\phi d\theta \\ &= \int_{-\pi}^{\pi} \int_{0}^{\pi/2} \frac{32}{3} \cos^2\phi \sin\phi d\phi d\theta \\ &= \frac{32}{3} \int_{-\pi}^{\pi} \left[-\frac{\cos^3\phi}{3} \right]_0^{\pi/2} d\theta = \frac{64\pi}{9}. \end{aligned}$$

5. (10 %) Given a vector field $\mathbf{F} = (3x, -2y, -4z)$, compute div$\mathbf{F}$ and curl$\mathbf{F}$.

Solution:

$$\begin{aligned} \text{div}\mathbf{F} &= 3 + (-2) + (-4) = -3. \\ \text{curl}\mathbf{F} &= (0-0, 0-0, 0-0) = (0,0,0) \end{aligned}$$

6. (15 %) Given a function $f(x, y) = xy$, and a curve $C : x = 3t, y = t^4$ with $0 \leq t \leq 1$, find $\int_C f ds$ and $\int_C f dx$.

Solution: $x' = 3$ and $y' = 4t^3$. Thus $ds = \sqrt{9 + 16t^6} dt$.

$$\begin{aligned} \int_C f ds &= \int_0^1 3t^5 \sqrt{9+16t^6} dt \qquad \text{set } u = 9 + 16t^6 \\ &= \frac{1}{32} \int_9^{25} u^{1/2} du = \frac{1}{32} \left[\frac{2u^{3/2}}{3} \right]_9^{25} = \frac{49}{24} \\ \int_C f dx &= \int_0^1 9t^5 dt = \frac{9}{6} = \frac{3}{2}. \end{aligned}$$

7. (12 %) Given a vector field $\mathbf{F} = (2xy^2 + 3x^2, 2x^2y + 4y^3)$, do the following.

(7A) Verify that this is a conservative field.

(7B) Find a potential of $\mathbf{F}$.

Solution: (7A) Here $P = 2xy^2+3x^2$ and $Q = 2x^2y+4y^3$. As $P_y = 4xy = Q_x$, this is a conservative field.

(7B) Compute $f(x,y) = \int Pdx = \int(2xy^2+3x^2)dx = x^2y^2+x^3+c(y)$. Then $2x^2y+4y^3 = Q = f_y = 2x^2y+c'(y)$, and so $c'(y) = 4y^3$. Therefore $c(y) = \int 4y^3dy = y^4$ and so $f(x,y) = x^2y^2+x^3+y^4$.

8. (12 %) Do both of the following.

(8A) Verify that $\mathbf{F} = (\cos y, -x\sin y)$ is a conservative field.

(8B) Compute the integral $\int_C \cos y dx - x\sin y dy$ for a curve C from $(0,0)$ to $(2,\pi)$.

Solution: (8A) Here $P = \cos y$ and $Q = -x\sin y$. As $P_y = -\sin y = Q_x$, this is a conservative field.

(8B) One solution is to find a potential function $f = x\cos y$, using a method similar to (7B). Therefore,

$$\int_C \cos y dx - x\sin y dy = f(2,\pi) - (0,0) = 2(-1) - 0 = -2.$$

Another solution is to choose a specific path such as $C_1 : 0 \le x \le 2$ with $y = 0$ followed by $C_2 : 0 \le y \le \pi$ with $x = 2$. In this case, $dy = 0$ in C_1 and $dx = 0$ in C_2. As $\cos 0 = 1$ and $\int \sin y dy = -\cos y + C$, we have

$$\int_C \cos y dx - x\sin y dy = \int_{C_1} \cos 0 dx - \int_{C_2} 2\sin y dy = \int_0^2 dx - 2\int_0^\pi \sin y dy = -2.$$

9. (10 %) Find a potential function for the conservative field $\mathbf{F} = (yz, xz+y, xy+1)$.

Solution: Let C denote the straight line from $(0,0,0)$ to (x_0,y_0,z_0). Then $C : x = x_0t, y = y_0t, z = z_0t$ with $0 \le t \le 1$. Let f be a potential function of $\mathbf{F}$ such that $f(0,0,0) = 0$. Then

$$f(x_0,y_0,z_0) = \int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^1 (3x_0y_0z_0t^2 + y_0^2t + z_0)dt = x_0y_0z_0 + \frac{y_0^2}{2} + z_0.$$

Thus $f(x,y,z) = xyz + \frac{y^2}{2} + z$.