EXAM 2 - Math 251

1. (10 %) Let $\rho = 2\sin\phi$ denote an equation in spherical coordinates.

(1A) Convert it to cylindrical coordinates.

(1B) Convert it to rectangular coordinates.

Solution Note that $r = \rho\sin\phi$. Then multiply ρ to both sides to get $\rho^2 = 2\rho\sin\phi$.

(1A) For cylindrical coordinates, note that $\rho^2 = r^2 + z^2$. Thus the answer is

$$r^2 + z^2 = 2r.$$

(1B) Apply $r^2 = x^2 + y^2$ to get the answer for rectangular coordinates

$$x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2} \text{ or } (x^2 + y^2 + z^2)^2 = 4x^2 + y^2.$$

2. (10 %) Evaluate the following limits

(2A) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 2 + y^2}{x^2 - 2 + y^2}$.

Solution $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 2 + y^2}{x^2 - 2 + y^2} = \frac{0^2 + 2 + 0^2}{0^2 - 2 + 0^2} = -1.$

(2B) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$.

Solution Let the limit be taken along the x-axis that is, set $y = 0$, we have $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^2 + 0^2}{x^2 - 0^2} = 1$; and let the limit be taken along the y-axis that is, set $x = 0$, we have $\displaystyle\lim_{(0,y)\to(0,0)} \frac{0^2 + y^2}{0^2 - y^2} = -1$. Therefore, the limit does not exist.

3. (10 %) Find an equation of the tangent plane at the point (1,-1,-1) to the surface $z = xy$.

Solution Compute $z_x = y$ and $z_y = x$. Therefore, a normal vector for the tangent plane at $(1, -1, -1)$ is $\mathbf{n} = (-1, 1, -1)$, and so the equation is

$$-(x - 1) + (y + 1) - (z + 1) = 0.$$

4. (15 %)

(4A) Compute all the first order partial derivatives of $f(x, y, z) = (x^2 + y^3 + z^4)e^{xyz}$.

Solution Use product rule for each of the partial derivatives:

$$\begin{aligned} f_x &= 2xe^{xyz} + (x^2 + y^3 + z^4)(yz)e^{xyz} \\ f_y &= 3y^2e^{xyz} + (x^2 + y^3 + z^4)(xz)e^{xyz} \\ f_x &= 4z^3e^{xyz} + (x^2 + y^3 + z^4)(xy)e^{xyz} \end{aligned}$$

(4B) Verify that $z_{xy} = z_{yx}$, where $z = x^2 e^{y^2}$.

Solution First compute $z_x = 2xe^{y^2}$ and $z_y = 2yx^2 e^{y^2}$. Then compute $z_{xy} = 4xye^{y^2}$ and $z_{yx} = 4yxe^{y^2}$, and so $z_{xy} = z_{yx}$.

5. (10 % for correct procedure and 5% for accuracy of solution) Find the highest and the lowest point of the surface given by

$$z = f(x, y) = x^2 + 2xy + 3y^2$$

over a square region with vertices $(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)$.

Solution We first compute $z_x = 2x + 2y$ and $z_y = 2x + 6y$. Setting $z_x = 0$ and $z_y = 0$ to get the only critical point $(0, 0)$. Note that $f(0, 0) = 0$.

Consider each of the boundaries. Let L_1 denote the boundary $\{(x, 1) : -1 \le x \le 1\}$. Then $f(x, 1) = x^2 + 2x + 3 = (x + 1)^2 + 2$. Therefor, apply Calculus I or High school algebra to get maximum $f(1, 1) = 6$ and minimum $f(-1, 1) = 2$.

Let L_2 denote the boundary $\{(x, -1) : -1 \le x \le 1\}$. Then $f(x, -1) = x^2 - 2x + 3 = (x - 1)^2 + 2$. Therefor, apply Calculus I or High school algebra to get maximum $f(-1, -1) = 6$ and minimum $f(1, -1) = 2$.

Let L_3 denote the boundary $\{(1, y) : -1 \le y \le 1\}$. Then $f(1, y) = 1 + 2y + 3y^2$. Therefor, apply Calculus I to get maximum $f(1, 1) = 6$ and minimum $f(-1, 1) = 2$.

Let L_4 denote the boundary $\{(-1, y) : -1 \le y \le 1\}$. Then $f(-1, y) = 1 - 2y + 3y^2$. Therefor, apply Calculus I to get maximum $f(-1, -1) = 6$ and minimum $f(-1, 1) = 2$.

Summing up, the highest points on the surface are $(1, 1, 6)$ and $-1, -1, 6)$; and the lowest points is $(0, 0, 0)$.

6. (10 %) Find every point on the surface $z = 3x^2 + 12x + 4y^3 - 12y + 1$ at which the tangent plane is horizontal.

Solution Compute $z_x = 6x + 12$ and $z_y = 12y^2 - 12$. Therefore, the critical points are $(-2, 1)$ and $(-2, -1)$. Compute $f(-2, 1) = -19$ and $f(-2, -1) = -3$. Hence at $(-1, 1, -19)$ and at $(-1, -1, -3)$, the surface has horizontal tangent planes.

7. (10 %) Find the dimension of the open-topped (rectangular) box with volume 500 in^3 that has minimum total surface area.

Solution Let x, y, z denote the dimension of the box. Then $xyz = 500$ or $z = \frac{500}{xy}$. The total surface area is formulated as

$$f(x,y) = xy + 2xz + 2yz = xy + 2(x+y)\frac{500}{xy} = xy + \frac{1000}{x} + \frac{1000}{y}.$$

Compute the partial derivatives to get

$$f_x = y - \frac{1000}{x^2},\ f_y = x - \frac{1000}{y}.$$

Solve the system of $f_x = 0$ and $f_y = 0$ to get $x = y = 10$, and so $z = \frac{500}{xy} = 5$.

8. (20 %)Let $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t)$ be a space curve (viewed as a position vector of a moving particle). Compute each of the following.

(8A) The velocity, the speed and the unit tangent vector.
(8B) The acceleration.
(8C) The curvature at the point when $t = 0$.
(8D) The unit normal vector at the point when $t = 0$.

Solution (8A) The velocity $\mathbf{v} = (e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t)$ and (use $\sin^2 t + \cos^2 t = 1$)

$$v = \sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2 + e^{2t}} = e^t\sqrt{3}.$$

The unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{e^t\sqrt{3}}(e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t) = \frac{1}{\sqrt{3}}(\cos t - \sin t, \sin t + \cos t, 1).$$

(8B) Differentiate $\mathbf{v}$ to get $\mathbf{a}$.

$$\mathbf{a}(t) = (e^t(\cos t - \sin t) + e^t(-\sin t - \cos t), e^t(\sin t + \cos t) + e^t(\cos t - \sin t), e^t) = e^t(-2\sin t, 2\cos t, 1).$$

(8C) At $t = 0$, $\mathbf{v} = (1,1,1)$, $v = \sqrt{3}$, and $\mathbf{a} = (0,2,1)$. Compute $\mathbf{v} \times \mathbf{a} = (1,1,1) \times (0,2,1) = (-1,-1,2)$, and so $|\mathbf{v} \times \mathbf{a}| = \sqrt{1+1+4} = \sqrt{6}$. Then use it to compute the **curvature**

$$\kappa(0) = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{6}}{(\sqrt{3})^3} = \frac{\sqrt{2}}{3}.$$

(8D) At $t = 0$, $\mathbf{T}$, a_N, and a_T are, respectively,

$$\mathbf{T} = \frac{1}{\sqrt{3}}(1,1,1), a_N = \kappa \mathbf{v}^2 = \frac{\sqrt{2}}{3}(3) = \sqrt{2},\ a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{2+1}{\sqrt{3}} = \sqrt{3}.$$

Thus

$$\mathbf{N} = \frac{1}{a_N}(\mathbf{a} - a_T\mathbf{T}) = \frac{1}{\sqrt{2}}\left((0,2,1) - \sqrt{3}\frac{1}{\sqrt{3}}(1,1,1)\right) = \frac{1}{\sqrt{2}}(-1,1,0).$$