SOLUTIONS TO EXAM 1 (Math 251)

1. Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Compute each of the following.

(1A) (3 %) $2\mathbf{a} - 3\mathbf{b} = (2(2, 3, -4) - 3(1, -1, 2) = (4, 6, -8) - (3, -3, 6) = (1, 9, -14)$.

(1B) (3 %) $(2\mathbf{a}) \cdot (3\mathbf{b}) = (4, 6, -8) \cdot (3, -3, 6) = (4)(3) + (6)(-3) + (-8)(6) = 12 - 18 - 48 = -54$.

(1C) (3 %) $|2\mathbf{a} - 3\mathbf{b}| = |(1, 12, -14)| = \sqrt{1^2 + 9^2 + (-14)^2} = \sqrt{1 + 81 + 196} = \sqrt{278}$.

(1D) (3 %) $\mathbf{a} \times \mathbf{b} = (6 - 4, -(4 - (-4)), -2 - 3) = (2, -8, -5)$.

(1E) (3 %) Find two unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
First compute $|\mathbf{a} \times \mathbf{b}| = \sqrt{2^2 + (-8)^2 + (-5)^2} = \sqrt{4 + 64 + 25} = \sqrt{93}$. Thus the following vectors are the desired ones:

$$\mathbf{u}_1 = \left(\frac{2}{\sqrt{93}}, \frac{-8}{\sqrt{93}}, \frac{-5}{\sqrt{93}} \right) \text{ and } \mathbf{u}_2 = \left(\frac{-2}{\sqrt{93}}, \frac{8}{\sqrt{93}}, \frac{5}{\sqrt{93}} \right).$$

(1F) (3 %) Find $Comp_{\mathbf{a}}\mathbf{b}$.

$$Comp_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{2 - 3 - 8}{\sqrt{4 + 9 + 16}} = \frac{-9}{\sqrt{29}}.$$

(1G) (3 %) Determine the value of x so that the vector $\mathbf{u} = (1, 1, x)$ is perpendicular to $\mathbf{a}$.
This amounts to find x such that $\mathbf{u} \cdot \mathbf{a} = 0$. Thus from $\mathbf{u} \cdot \mathbf{a} = 2 + 3 - 4x = 0$, we have $x = \frac{5}{4}$.

(1H) (3 %) Let $\mathbf{c} = (1, 1, 1)$. Compute $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.
One can use the determinant of a 3 by 3 array as indicated in the text book. A better solution here is to use your **correct** answer in (1D).

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (2, -8, -5) \cdot (1, 1, 1) = 2 - 8 - 5 = -11.$$

2. (10 %) Write an equation of the sphere with the line segment joining (1,2,1) and (3,2,-3) as a diameter.

Solution The center of the sphere is $x_0 = \frac{1+3}{2} = 2, y_0 = \frac{2+2}{2} = 2$ and $z_0 = \frac{-3+1}{2} = -1$. The square of radius is the square of the distance from the point $(1, 2, 1)$ to the center $(2, 2, -1)$, which is equal to $(2 - 1)^2 + (2 - 2)^2 + (-1 - 1)^2 = 5$. Thus the answer is

$$(x - 2)^2 + (y - 2)^2 + (z + 1)^2 = 5.$$

3. (10 %) Find the area of the triangle whose vertices are $P(1, 1, 1)$, $Q(2, 3, 4)$ and $R(3, 4, 5)$.

Solution Set $\mathbf{a} = \overline{PQ} = (1, 2, 3)$, $\mathbf{b} = \overline{QR} = (1, 1, 1)$. Compute $\mathbf{a} \times \mathbf{b} = (3-2, -(3-1), 2-1) = (1, -2, 1)$. Thus the answer is

$$\text{area } = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{1+4+1}}{2} = \frac{\sqrt{6}}{2}.$$

4. (10 %) Given four points $A(0, 0, 0)$, $B(1, 1, 0)$, $C(1, 0, 1)$ and $D(0, 1, 1)$ in the space, find the volume of the parallelepiped with $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ as three adjacent edges.

Solution Let $\mathbf{a} = \overline{AB} = (1, 1, 0)$, $\mathbf{b} = \overline{AC} = (1, 0, 1)$ and $\mathbf{c} = \overline{AD} = (0, 1, 1)$. Compute the mix product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 - 1 - 1 = -2.$$

Thus the answer is $|-2| = 2$.

5. (16 %) Write an equation of the plane $\mathcal{P}$ through the point $P(1, 1, 1)$ and parallel to the plane $3x + 4y = z + 10$; also write the equations (both in parametric form and in symmetric form) of the line L that passes through $P(1, 1, 1)$ and is perpendicular to the plane $\mathcal{P}$.

Solution Note that a normal vector of the plane $3x + 4y = z + 10$ is $\mathbf{n} = (3, 4, -1)$. As the two planes are parallel, $\mathbf{n}$ is perpendicular to both of them and so $\mathbf{n}$ can serve as a normal vector of $\mathcal{P}$. Noting also that $P(1, 1, 1)$ is a point on the plane, we obtain an equation of $\mathcal{P}$ as

$$3(x - 1) + 4(y - 1) - (z - 1) = 0.$$

Now observe that L is parallel to $\mathbf{n}$, and so the equations of L are

$$\begin{cases} x = 1 + 3t \\ y = 1 + 4t \\ z = 1 - t \end{cases} \quad \text{and} \quad \frac{x-1}{3} = \frac{y-1}{4} = \frac{z-1}{-1}.$$

6. (10 %) Find an equation of a plane $\mathcal{P}$ that passes through the point $O(0, 0, 0)$ and contains the line of intersection of the planes $x - y + z = 1$ and $x + y - z = 1$.

Solution We need first to find the intersection line L. Note that $\mathbf{n}_1 = (1, -1, 1)$ and $\mathbf{n}_2 = (1, 1, -1)$ are normal vectors of the two planes $x - y + z = 1$ and $x + y - z = 1$, respectively, the line L is parallel to the vector $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (1 - 1, -(-1 - 1), 1 - (-1)) = (0, 2, 2)$. Solve the system of equations

$$\begin{cases} x - y + z = 1 \\ x + y - z = 1 \end{cases}$$

by setting $z = 0$, we can find that $P(1,0,0)$ is a point on L. Thus the equations of L is $x = 1, y = 2t, z = 2t$.

Note that $P(1,0,0)$ is also a point on the plane $\mathcal{P}$. Let $\mathbf{a} = \overline{OP} = (1,0,0)$. Then both $\mathbf{a}$ and $\mathbf{v}$ are parallel to $\mathcal{P}$. Thus a normal vector of $\mathcal{P}$ can be $\mathbf{n}' = \mathbf{a} \times \mathbf{n} = (0,-2,2)$. Hence the equation of the plane $\mathcal{P}$ is

$$-2(y-0) + 2(z-0) = 0, \text{ or simply } y = z.$$

7. (10 %) Given the position vector $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} - 4t\mathbf{k}$, find the velocity and acceleration vectors, and the speed at time t.

Solution

$$\begin{aligned}
\text{velocity} &= (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} - 4\mathbf{k} \\
\text{acceleration} &= (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} \\
\text{speed} &= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = \sqrt{25} = 5
\end{aligned}$$

8. (10 %) Given the acceleration vector $\mathbf{a}(t) = (9\sin 3t)\mathbf{i} + (9\cos 3t)\mathbf{j} + 4\mathbf{k}$, initial velocity $\mathbf{v}_0 = \mathbf{v}(0) = 2\mathbf{i} - 7\mathbf{j}$, and initial position $\mathbf{r}_0 = \mathbf{r}(0) = 3\mathbf{i} + 4\mathbf{j}$, find the position vector $\mathbf{r}(t)$ at time t.

Solution We first find the velocity

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(t)dt + \mathbf{v}_0 = \left(\int_0^t 9\sin 3t dt + 2, \int_0^t 9\cos 3t dt - 7, \int_0^t 4dt\right) = (-3\cos 3t + 2, 3\sin 3t - 7, 4t).$$

Then we find the position vector

$$\begin{aligned}
\mathbf{r}(t) &= \int_0^t \mathbf{v}(t)dt + \mathbf{r}_0 = \left(\int_0^t (-3\cos 3t + 2)dt + 3, \int_0^t (3\sin 3t - 7)dt + 4, \int_0^t 4tdt\right) \\
&= \left(-\sin 3t + 2t + 3, -\cos 3t - 7t + 4, 2t^2\right).
\end{aligned}$$