EXAM 2 Solution - Math 156

NAME:

Instruction: Circle your answers and show all your work CLEARLY. Messing around may result in losing credits, since the grader may be forced to pick the worst to grade. Solutions with answer only and without supporting procedures will have little credits.

Departmental Policy: No Calculators Permitted.

Part 1 This part of the exam is a set of short answer questions. You need to fill up the blank with you short answers.

1. (6 %) Determine if the given integral is convergent or divergent.

(a) $\displaystyle\int_{-\infty}^{6} e^{x/6}dx = \lim_{A\to-\infty} 6e - 6e^{A/6} = 6e.$

Answer: Convergent

(b) $\displaystyle\int_0^1 \frac{3}{x^5}dx = \lim_{t\to 0^+}\left[\frac{3}{-4x^4}\right]_t^1$ does not exist.

Answer: Divergent

2. (6 %) Find the area of the region bounded by $y = |x|$, $y = 0$, and $-1 \le x \le 1$.

Answer: 1

PART 2: This portion of the exam will be graded on a partial credit basis. **Answers without supporting work shown on the paper will receive NO credit. Partial credit will be given to solutions that belong to part of a correct solution.**

3. (12 %) Write out the form of the partial fraction decomposition of $\dfrac{2x^3-4x-8}{(x^2-x)(x^2+4)}$ AND determine the coefficients.

Solution: Factor the denominator completely: $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$.

Set up the partial fractions

$$\begin{aligned}\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} &= \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}\\ &= \frac{A(x-1)(x^2+4)+Bx(x^2+4)+(Cx+D)x(x-1)}{(x^2-x)(x^2+4)}.\end{aligned}$$

Find the coefficients. Equal both numerators

$$\begin{aligned}2x^3-4x-8 &= A(x-1)(x^2+4)+Bx(x^2+4)+(Cx+D)x(x-1)\\ &= (A+B+C)x^3+(-A-C+D)x^2+(4A+4B-D)x-4A.\end{aligned}$$

Compare the constant coefficients to get $-8=-4A$, and so $A=2$. Set $x=1$ to get $2-4-8=5B$, and so $B=-2$. As $A+B+C=2$ and $A=2$, $B=-2$, we have $C=2$. Compare the coefficients of x^2 to get $-A-C+D=0$, and then use $A=C=2$ to get $D=4$.

4. (14 %) A region is bounded by curves $y=2-x^2$ and $y=x$. Find **the area** of this region.

Solution: Then solve the system $y=2-x^2$ and $y=x$ to get $2-x^2=x$, and so $x^2+x-2=0$ or $(x+2)(x-1)=0$. Therefore, these two curves intersect at $(-2,-2)$ and $(1,1)$. The region is bounded above by $y=2-x^2$ and below by $y=x$.

The area is

$$A=\int_{-2}^{1}[(2-x^2)-x]dx = \left[\frac{-x^3}{3}-\frac{x^2}{2}+2x\right]_{-2}^{1} = \frac{9}{2}.$$

5. (14 %) A region is bounded by $y^2=3-x$ and $y=x-1$. Find **the area** of this region..

Solution: Rewrite the curves as functions of y: $x=3-y^2$ and $x=y+1$. Then solve the system $x=3-y^2$ and $x=y+1$ to get $3-y^2=y+1$, and so $y^2+y-2=0$ or $(y-1)(y+2)=0$. Therefore, these two curves intersect at $(2,1)$ and $(-1,-2)$. The curve $x=3-y^2$ is the right boundary and $x=y+1$ is the left boundary.

The area is

$$A=\int_{-2}^{1}[(3-y^2)-(y+1)]dy=\int_{-2}^{1}(-y^2-y+2)dy=\left[\frac{-y^3}{3}-\frac{y^2}{2}+2y\right]_{-2}^{1}=\frac{9}{2}.$$

6. (14 %) Find **the volume** of the solid obtained by rotating the region bounded by $y=\sec(x)$, $y=0$, $x=0$, and $x=\frac{\pi}{4}$ about the x-axis.

Solution: The curves bounding the region suggest that $a = 0$ and $b = \frac{\pi}{4}$. We use cross-section method. For any x in $[0, \frac{\pi}{4}]$, the radius of the cross-section circle is $y = \sec(x)$. The answer is

$$V = \pi \int_0^{\pi/4} \sec^2(x)dx = \pi \left[\tan(x)\right]_0^{\pi/4} = \pi \left(\tan(\frac{\pi}{4}) - \tan(0)\right) = \pi(1-0) = \pi.$$

7. (14 %) Find **the volume** of the solid obtained by rotating the region bounded by $y = x^2 + 1$, $y = 0$, and $x = 0$, and $x = 1$ about the y-axis. (Hint: Use the shell technique).

Solution: The curves bounding the region suggest that $a = 0$ and $b = 1$ for us to use the shell technique. For any x in $[0, 2]$, the cylinder generated by the vertical line at x in the region has radius $r = x$ and height $h = x^2 + 1$ (the y-coordinate on the curve bounded above the region). The integral is:

$$V = 2\pi \int_0^1 x(x^2+1)dx = 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 = \frac{3\pi}{2}.$$

8. (10 %) Setup an integral that computes **the volume** of the solid obtained by rotating the region bounded by $y = x^2 + 1$, $y = 0$, and $x = 0$, and $x = 1$ about the line $x = 3$. **Do not evaluate the integral**.

Solution: The curves bounding the region suggest that $a = 0$ and $b = 1$ for us to use the shell technique. For any x in $[0, 2]$, the cylinder generated by the vertical line at x in the region has radius $r = 3 - x$ and height $h = x^2 + 1$ (the y-coordinate on the curve bounded above the region). The integral is:

$$V = 2\pi \int_0^1 (3-x)(x^2+1)dx.$$

9. (10 %) Setup an integral that computes **the volume** of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4x - x^2$ about the line $y = 6$. **Do not evaluate the integral**.

Solution: Solve the system $y = x^2$, $y = 4x - x^2$ to get $x^2 = 4x - x^2$ and so $x^2 - 2x = 0$, or $x(x-2) = 0$. Therefore, the curves intersect at $(0, 0)$ and $(2, 4)$. The region is bounded above by $y = 4x - x^2$ and below by $y = x^2$. We use cross-section. For any x in $[0, 2]$, the cross-section area is $A(x) = \pi[(6-x^2)^2 - (6-4x+x^2)^2]$. The integral is:

$$V = \pi \int_0^2 [(6-x^2)^2 - (6-4x+x^2)^2]dx.$$

Grade Distribution of Exam 2:

Meaning of the scores: The highest score of exam 2 is 100/100.
at least 90 = Very good, familiar with the related materials and skillful, with minimal computational errors. Keep on!

80-89 = good, familiar with most of the related materials, with a few computations errors. Make an effort to do better.
70-79 = OK, not so familiar with the related materials, with relatively more computational errors. We have room to improve. (For this quiz, not familiar with differentiation).
60-69 = Passing, We are ont the borderline of failing. It indicates that we are less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.
at most 59 = We failed. We need to catch it up. It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

Scores	90-99	80-89	70-79	60-69	≤ 59
Frequency	12	4	7	4	8
Percentage	34.3	11.4	20.0	11.4	22.8

Discussions and Comments

1 Problems: Algebra deficiencies and differentiation deficiencies appear to be the major factor causing most of errors in this test. Many errors are due to the lack of knowledge on straight lines, parabola and negative exponents, and the erroneous use of differentiation/integration formulas.

Understanding negative exponents. Here are some errors observed in solutions of Problem 1.

Erroneous way	Correct way:
$\lim_{A\to\infty} e^{-A/6} = \infty$	$\lim_{A\to\infty} e^{-A/6} = 0$
$\lim_{t\to 0^+} x^{-4} = 0$	$\lim_{t\to 0^+} x^{-4} = \infty$ (does not exists).

What is factor completely? In solution of Problem 4, many use $x^2 - x$ as a denominator without recognizing $x^2 - x = x(x-1)$. Consequently, the solutions are basically all wrong.

Use quadratic formula. In determining the integration bounds, we often need to solve a quadratic equation. In problem 5, quite a few of us did the following: Set $-y^2+3 = y+1$. Simplify it to get $-y^2+2 = y$. Then $y = 1$ and $y = -1$. I have no clue how the final answers were obtained. However, I would offer a simple and never wrong approach: use of quadratic formula. Rewrite $-y^2+2 = y$ to $y^2+y-2 = 0$, then $y = (-1\pm\sqrt{1-4(1)(-2)})/2$ to get $y = -2$ and $y = 1$.

2 Problems: Setting up integrals. Most of us did very well in setting up integrals. Some of us still have problems in choosing dx or dy, in determining bounds, and in finding the integrands. The choice of dx or dy was in my email to all before the exam. Once we use dx, w should use the x-bounds; and once we decide to use dy, we should use y-bounds. (Some of us use dx but y-bounds in several problems.) If we use the disc (cross-section) method, the we are computing area of a disc, which is πr^2. We then need to figure out what is the radius (in the event of a washer, the radii). If we use the shell method, the we are computing area of the lateral surface of a shell, which is $2\pi rh$. We then need to figure out what is the radius and what is the eight of the shell. Most of the errors are caused for not doing such computing.