EXAM 1 Solutions - Math 156

1. (6 %) Find the interval(s) on which the function $F(x) = \int_1^{x^2} e^{\sin(t)}dt$ is increasing.

Solution. Set $u = x^2$. Use chain rule $\frac{dF}{dx} = \frac{dF}{du} \cdot \frac{du}{dx}$ and FTC to compute

$$\frac{dF}{du}e^{\sin(u)} \text{ and } \frac{du}{dx} = 2x. \text{ Thus } \frac{dF}{dx} = e^{\sin(x^2)} \cdot (2x).$$

This asks us to find on which interval(s) we have $F'(x) > 0$. Now we have $\frac{dF}{dx} = e^{\sin(x^2)} \cdot (2x)$. Since $2e^{\sin(x^2)} > 0$, $F'(x) > 0$ takes place exactly on the interval $(0, \infty)$.

2. (6 %) Find an equation of the tangent line of $f = F(x)$ at $x = \frac{\pi}{2}$ by filling up the blank, where

$$F(x) = \int_{-\frac{\pi}{2}}^{x} \sin^3(t)dt.$$

Solution. The y-intersect is $F(\frac{\pi}{2}) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3(t)dt = 0$ as $\sin^3(t)$ is an odd function. The slope is $F'(\frac{\pi}{2})$. By FTC, $F'(x) = \sin^3(x)$. As $\sin(\frac{\pi}{2}) = 1$, we conclude that $F'(\frac{\pi}{2}) = 1$. Put these in the blank to get the answer: $y - 0 = 1(x - \frac{\pi}{2})$.

3. (11 %) Find the average value of the function $y = \sin(\frac{\pi x}{2})$ on the interval $[0, 2]$.

Solution. To compute $\int \sin(\frac{\pi x}{2})dx$, we set $u = \frac{\pi x}{2}$ to get

$$\int \sin(\frac{\pi x}{2})dx = \frac{2}{\pi} \int \sin(u)du = \frac{-2}{\pi} \cos(\frac{\pi x}{2}).$$

Thus the average is (use $\cos(0) = 1$ and $\cos(\pi) = -1$)

$$\frac{1}{2-0} \int_0^2 \sin(\frac{\pi x}{2})dx = \frac{1}{2} \left[\frac{-2}{\pi} \cos(\frac{\pi x}{2})\right]_0^2 = \frac{1}{2} \cdot \frac{-2}{\pi} \cdot (-1-1) = \frac{2}{\pi}.$$

4. (11 %) Use an appropriate substitution to evaluate $\int_0^1 2(1-x^2)^3 x dx$.

Solution. Set $u = u(x) = 1 - x^2$. Then $u(0) = 1$, $u(1) = 0$, and $du = -2xdx$. Thus

$$\int_0^1 2(1-x^2)^3 x dx = -\int_1^0 u^3 du = \left[-\frac{u^4}{4}\right]_1^0 = \frac{1}{4}.$$

5. (11 %) Evaluate $\int \tan^2(x)(1 + \sec^2(x))dx$.

Solution. One solution is to write $\int \tan^2(x)(1+\sec^2(x))dx = \int \tan^2(x)dx + \int \tan^2(x)\sec^2(x)dx$. Now we know how to deal with each integral by recalling $d\tan(x) = \sec^2(x)dx$. For the first,

$\int \tan^2(x)dx = \int(\sec^2(x)-1)dx = \tan(x) - x + C_1$. For the second integral, we set $u = tan(x)$ to get $\int \tan^2(x)\sec^2(x)dx = \int u^2 du = \frac{u^3}{3} = \frac{\tan^3(x)}{3} + C_2$. Thus the answer is (combining $C = C_1 + C_2$)

$$\int \tan^2(x)(1+\sec^2(x))dx = \int \tan^2(x)dx + \int \tan^2(x)\sec^2(x)dx = \tan(x) - x + \frac{\tan^3(x)}{3} + C.$$

6. (11 %) Use integration by pars to evaluate $\int xe^{-x}dx$.

Solution. Set $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ and $v = \int dv = -e^{-x}$. Use integration by pars to get

$$\int xe^{-x}dx = -xe^{-x} - \int(-1)e^{-x}dx = -xe^{-x} - e^{-x} + C.$$

7. (11 %) Use integration by pars to evaluate $\int x\ln(x^2)dx$.

Solution. Set $u = \ln(x^2)$ and $dv = xdx$. Then $du = \frac{2x}{x^2}dx$ and $v = x^2/2$. Use integration by pars to get

$$\int \ln(1+x^2)dx = \frac{x^2\ln(x^2)}{2} - \int xdx = \frac{x^2\ln(x^2)}{2} - \frac{x^2}{2} + C.$$

8. (11 %) Use trigonometry substitution to evaluate $\int \frac{x}{\sqrt{4-x^2}}dx$

Solution. Set $x = 2\sin(\theta)$. Then $dx = 2\cos(\theta)d\theta$, and $\sqrt{4-x^2} = \sqrt{4-4\sin^2(\theta)} = 2\cos(\theta)$. Thus

$$\begin{aligned}\int \frac{x^3}{\sqrt{4-x^2}}dx &= \int \frac{2\sin(\theta)}{2\cos(\theta)}2\cos(\theta)d\theta \\ &= \int 2\sin^3(\theta)d\theta = -2\cos(\theta) = -2\frac{\sqrt{4-x^2}}{2} + C = -\sqrt{4-x^2} + C\end{aligned}$$

9. (11 %) Evaluate $\int \frac{dx}{(1+x^2)^{3/2}}$.

Solution. Set $x = \tan(\theta)$. Then $dx = \sec^2(\theta)d\theta$ and $1 + x^2 = 1 + \tan^2(\theta) = \sec^2(\theta)$. Thus

$$\begin{aligned}\int \frac{dx}{(1+x^2)^{3/2}} &= \int \frac{\sec^2(\theta)d\theta}{\sec^3(\theta)} = \int \frac{d\theta}{\sec(\theta)} = \int \cos(\theta)d\theta \\ &= \sin(\theta) + C = \frac{x}{\sqrt{1+x^2}} + C\end{aligned}$$

10. (11 %) Use partial fractions to evaluate $\int \frac{2x+1}{x^2-3x+2}dx$.

Solution. As the fractional integrand is already reduced, we start factoring the denominator completely: $x^2 - 3x + 2 = (x-1)(x-2)$. Write down the partial fractions:

$$\frac{2x+1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}.$$

Now determine A and B (use any method it would work. We use eliminations) Set $x = 1$ in the numerators both sides to get $2+1 = A(1-2)$, and so $A = -3$. Set $x = 2$ in the numerators both sides to get $4+1 = B(2-1)$, and so $B = 5$. How

$$\int \frac{2x+1}{x^2-3x+2}dx = \int \frac{-3}{x-1}dx + \int \frac{5}{x-2}dx = -3\ln|x-1| + 5\ln|x-2| + C.$$

Grade Distribution: There were 37 students taking the test. Please find the grade distribution data in the table below. Two (2) additional credit points have been given to those students who could correctly find the interval on which the function is increasing in Problem 1. The highest score is 101/100 and the lowest score is 14/100 in this test. Three students scores 99/100 or higher.

Recommendations Students with scores at least 79%, keep up the steams! Students with lower grades, do not give up. If your algebraic skill pulled you back, reinforce it. Setting your goal to "knowing how to do" is insufficient for us to get a good grade. Please set your goal to "knowing how to do and be familiar with the skills". More practice is always a way to lead you to success in Calculus.

Grades	90% & up	79% -89 %	69% - 79%	59% - 69%	Below 58%
Frequency	7	7	7	6	10
Percentile	18.9%	18.9%	18.9%	16.3%	27.0%

Common Errors: We list the common errors occurred in this test below, hoping that students will not repeat them in the future. This also provide with students in this section a direction to enhance our algebraic skills and Calculus I skills when working on Calculus problems.

There have been lots of the algebraic errors found in the solutions. As we will put a focus on calculus common errors, we will not list all the high school algebraic errors, which have been marked in the grading of your solutions.

1. Application of Fundamental Theorem of Calculus. We need to differentiate $F(x)$, which is a function of x. One common error in the test is to output a function of t, which indicate the student did not understand what needs to be done in order to solve the problem.

2. Many errors were found as follows:

$$\int \sin(\frac{\pi x}{2})dx = -\cos(\frac{\pi x}{2}). \quad \text{Why this is wrong?}$$

Also, many students consider $\cos(\pi) = 0$ and or $\cos(0) = 0$, which are trigonometry errors.

3. In the solution of Problem 6, many students do not know the difference between $x - e^{-x}$ and $x(-e^{-x})$. Please use your calculator and substitute $x = 1$ to convince yourself that

these are different expression. Some argued with me that their former instructors did the same. I think wrong things should be put to an end and we will not allow using $x - e^{-x}$ for $x(-e^{-x})$ (or something in the similar nature) in all quizzes and exams in this course. Please pay attention this this. I do not want you to lose point because of such errors.

4. Many have various kinds differentiation errors, and these have been marked in the solutions. In the solution of Problem 7, this error has been very common in this test:

$$\text{Set } u = \ln(x^2). \text{ Then } du = \frac{1}{x^2}dx. \quad \text{Why this is wrong?}$$

5. In the Solution of problem 9, some solutions correctly set the substitution $x = \tan(\theta)$ but with a final answer $\sin(\theta)$. We should understand that θ is a tool for us to correctly find the answer and itself is NOT the answer. Our answer should be a function in x.