# Math 156 Fall 2015 Quiz 9

# Name:

**Instruction.** Need to show your work to get your answer. Solutions without supporting work, even with correct answer, will have at most half the credit.

**1**: Find the Maclauring series of $f(x) = \sin(-3x)$.

**2**: Find the Maclauring series of $f(x) = \ln(2 - x)$.

**3**: Find the sum of $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + ... + (-1)^n \frac{e^n}{n!} + ....$

**4**: Find the first 4 terms of the Maclauring series of $f(x) = \frac{1}{\sqrt[3]{1+x}}$.

**Solutions**

**1**: Find the Maclauring series of $f(x) = \sin(-3x)$.

**Solution.** Since the Maclauring series of $\sin(u)$ is

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad (-\infty, \infty),$$

Replace $x$ in the formula above by $(-3x)$ to get the answer:

$$\sin(-3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-3x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{2n+1}}{(2n+1)!} x^{2n+1}.$$

**2**: Find the Maclaurin series of $f(x) = \ln(2-x)$.

**Solution 1.** (Use Maclaurin series of $\ln(1-x)$ and substitution. )
If we have in mind the formula (lectured in class)

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad [-1, 1).$$

Write $2 - x = 2(1 - \frac{x}{2})$. Thus $\ln(2-x) = \ln(2(1-\frac{x}{2})) = \ln(2) + \ln(1-\frac{x}{2})$. Replace $x$ in the formula above by $\frac{x}{2}$ to get the answer:

$$\ln(2-x) = \ln(2) + \ln(1 - \frac{x}{2}) = \ln(2) + -\sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{n+1}}{n+1} = \ln(2) + -\sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}(n+1)}, \quad [2, -2).$$

**Solution 2.** (Use differentiation and integration)

**Step 1.** Compute $f'(x) = \frac{-1}{2-x} = \frac{-1}{2} \cdot \frac{1}{1-\frac{x}{2}}$.

**Step 2.** Find the power series of $f'(x)$.

$$f'(x) = \frac{-1}{2-x} = \frac{-1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{-1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} x^n.$$

**Step 3.** Find the power series of $f(x)$, (inside the interval of convergence)

$$f(x) = \int f'(x)dx = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} \int x^n dx = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}(n+1)} x^{n+1} + C.$$

As $f(0) = \ln(2) = C$,

$$f(x) = \ln(2) + \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}(n+1)} x^{n+1}.$$

**Step 4.** Find the radius of convergence. (Use ratio test) Here $|u_n(x)| = \frac{1}{2^{n+1}(n+1)} |x^{n+1}|$ and so we need

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \frac{1}{2^{(n+1)+1}((n+1)+1)} |x^{(n+1)+1}| \cdot \frac{2^{n+1}(n+1)}{|x^{n+1}|} = \lim_{n\to\infty} \frac{|x|}{2} \cdot \frac{n+1}{n+2} = \frac{|x|}{2} < 1.$$

Thus $|x| < 2$. It follows that $|x| < 2$, and so the radius of convergence is $R = 2$.

**3**: Find the sum of $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + ... + (-1)^n \frac{e^n}{n!} + ....$

**Solution.** The series we want the sum is $\sum_{n=0}^{\infty} (-1)^n \frac{e^n}{n!}$. This reminds us of the Maclauring series of $e^x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty).$$

Compare the two, we find the answer:

$$\sum_{n=0}^{\infty} (-1)^n \frac{e^n}{n!} = \sum_{n=0}^{\infty} \frac{(-e)^n}{n!} = e^{-e}.$$

**4**: Find the first 4 terms of the Maclauring series of $f(x) = \frac{1}{\sqrt[3]{1+x}}$.

**Solution 1.** (Use Maclaurin series of $(1 + x)^{-1/3}$. ) Write down the Maclaurin series formula of $(1 + x)^k$ (lectured in class)

$$(1 + x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)(k-2)...(k-n+1)}{n!} x^n, \quad \text{radius of convergence} = 1.$$

Now $k = -\frac{1}{3}$. We can write down the first 4 terms as follows

$$\begin{array}{ll} n = 0 & a_0 = 1 \\ n = 1 & a_1 = -\frac{1}{3} \\ n = 2 & \frac{-\frac{1}{3}(-\frac{1}{3}-1)}{2!} = \frac{-\frac{1}{3}(-\frac{4}{3})}{2!} = \frac{4}{2 \cdot 3^2} = \frac{2}{9}. \\ n = 3 & \frac{-\frac{1}{3}(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{3!} = \frac{-\frac{1}{3}(-\frac{4}{3})(-\frac{7}{3})}{3!} = \frac{4 \cdot 7}{3! \cdot 3^3} = -\frac{14}{81}. \end{array}$$

**Answer:**

$$\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{1}{3}x + \frac{4}{9}x^2 - \frac{14}{81}x^3 + ...$$

**Solution 2.** (Use the definition routine)
**Step 1.** Compute the derivatives. Let $f(x) = (1 + x)^{-1/3}$.

$$\begin{aligned} f'(x) &= -\frac{1}{3}(1+x)^{\frac{-1}{3}-1} = -\frac{1}{3}(1+x)^{\frac{-4}{3}}. \\ f''(x) &= \frac{-1}{3} \cdot \frac{-4}{3}(1+x)^{\frac{-4}{3}-1} = \frac{4}{9}(1+x)^{\frac{-7}{3}}. \\ f'''(x) &= \frac{4}{9} \cdot \frac{-7}{3}(1+x)^{\frac{-7}{3}-1} = \frac{-28}{27}(1+x)^{\frac{-10}{3}}. \end{aligned}$$

**Step 2.** Use $a_n = \frac{f^{(n)}(0)}{n!}$ to find the coefficients.

$$\begin{aligned}
a_0 &= \frac{f(0)}{0!} = 1 \\
a_1 &= \frac{f'(0)}{1!} = -\frac{1}{3}. \\
a_2 &= \frac{f''(0)}{2!} = \frac{4}{9} \cdot \frac{1}{2} = \frac{2}{9}. \\
a_3 &= \frac{f'''(0)}{3!} = \frac{-28}{27} \cdot \frac{1}{6} = \frac{-14}{81}.
\end{aligned}$$

**Step 3.** Answer:

$$\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{1}{3}x + \frac{4}{9}x^2 - \frac{14}{81}x^3 + ...$$