Math 156 Fall 2015 Quiz 4

Name:

Important Instruction: You must use the shell technique to do Problem 1. You have the freedom to choose either the cross-section method or the shell technique in Problems 2 and 3.

1: Use the shell technique to find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = x^3, y = 8$ and $x = 0$.

2: Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \frac{x^2}{4}$, $y = 5 - x^2$.

3: Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \sqrt{x-1}, y = 0$ and $x = 5$. (For this problem, set up the integral only. **Do not evaluate the integral**).

1: (Section 7.3, Problem 11) Use the shell to find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = x^3, y = 8$ and $x = 0$.

Solution, Cross-section Method: (This method is included for your information and comparison).
Bounds: Axis of rotation is x-axis, and so we use x-bounds with $a = 0$. Solve $y = x^3$ and $y = 8$ for x to get $b = 2$.
Integrand: For any x with $0 \le x \le 2$, the cross-section has two circles. The bigger radius is $r_1 = 8$ and the smaller is $r_2 = x^3$. Therefore, the cross-section area is $A(x) = \pi(8^2 - (x^3)^2) = \pi(64 - x^6)$.
Integral: The volume is

$$\pi \int_0^2 (64 - x^6)dx = \pi \left[64x - \frac{x^7}{7}\right]_0^2 = \pi\left(128 - \frac{128}{7}\right) = \frac{768\pi}{7}.$$

Solution, Shell technique:
Bounds: Axis of rotation is x-axis, and so we use y-bounds with $c = 0$ and $d = 8$.
Integrand: For any y with $0 \le y \le 8$, radius of the shell $r = y$ and the height of the shell $h = \sqrt[3]{y}$.
Integral: The volume is

$$2\pi \int_0^8 y \cdot y^{\frac{1}{3}} dy = 2\pi \int y^{\frac{4}{3}} dy = 2\pi \left[\frac{3y^{\frac{7}{3}}}{7}\right]_0^8 = 2\pi \cdot \frac{3 \cdot 2^7}{7} = \frac{768\pi}{7}.$$

2: (Section 7.2, Problem 6) Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \frac{x^2}{4}$, $y = 5 - x^2$.

Solution, Cross-section Method: The shape of the graph indicated that it is better to use cross-section than the shell, as the height of the shell will have different expressions with different values of y.
Bounds: Axis of rotation is x-axis, and so we use x-bounds. Solve the system $y = \frac{x^2}{4}$ and $y = 5 - x^2$ for x. Then $\frac{x^2}{4} = 5 - x^2$, or $x^2 = 20 - 4x^2$, resulting $5x^2 = 20$ or $x^2 = 4$. Therefore $a = -2$ and $b = 2$.
Integrand: For any x with $-2 \le x \le 2$, the cross-section has two circles. The bigger radius is $r_1 = 5 - x^2$ and the smaller is $r_2 = \frac{x^2}{4}$. Therefore, the cross-section area is $A(x) = \pi((5 - x^2)^2 - (\frac{x^2}{4})^2) = \pi(25 - 10x^2 + x^4 - \frac{x^4}{16}) = \pi(25 - 10x^2 + \frac{15x^4}{16})$.
Integral: The volume is

$$\begin{aligned} \pi \int_{-2}^2 \left(25 - 10x^2 + \frac{15x^4}{16}\right) dx &= \pi \left[25x - \frac{10x^3}{3} + \frac{15x^5}{5 \cdot 16}\right]_{-2}^2 \\ = 2\pi\left(50 - \frac{80}{3} + 6\right) &= 2\pi\left(\frac{150 - 80}{3} + 6\right) = 2\pi\left(\frac{70}{3} + 6\right) = 2\pi\left(\frac{86}{3}\right) = \frac{172\pi}{3}. \end{aligned}$$

Some of us are puzzled where to stop. Lots of the arithmetic errors have been observed in combining the fractions. **A suggestion:** in future quizzes and exams, we should have at least clearly write down expressions like $2\pi\left(56 - \frac{80}{3}\right)$ as our answer with or without continuing doing the additional arithmetic steps.

3: (Section 7.3, Problem 19) Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = \sqrt{x-1}, y = 0$ and $x = 5$. (For this problem, set up the integral only. **Do not evaluate the integral**).

Solution, Cross-section Method:
Bounds: Axis of rotation is x-axis, and so we use x-bounds. The intersection of $y = \sqrt{x-1}$ and $y = 0$ is at $(1, 0)$, and so $a = 1$ and $b = 5$.
Integrand: For any x with $1 \le x \le 5$, the cross-section has a circle with radius $r = \sqrt{x-1}$.
Integral: The volume is (evaluation is not needed).

$$\pi \int_1^5 (x-1)dx = \pi \left[\frac{x^2}{2} - x\right]_1^5 = \pi\left(\frac{25}{2} - 5 - \frac{1}{2} + 1\right) = 8\pi.$$

Solution, Shell technique:
Bounds: Axis of rotation is x-axis, and so we use y-bounds with $c = 0$. Solve $y = \sqrt{x-1}$ and $x = 5$ for y to get $d = 2$.
Integrand: For any y with $0 \le y \le 2$, radius of the shell $r = y$ and the height of the shell $h = 5 - (1 + y^2) = 4 - y^2$.
Integral: The volume is (evaluation is not needed).

$$2\pi \int_0^2 y(4-y^2)dy = 2\pi \int_0^2 (4y - y^3)dy = 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 2\pi(8-4) = 8\pi.$$

Grade distribution of this quiz (including the worksheet extra credit):

Meaning of the scores:
$\geq 9, 10$ = **Very good**, familiar with the related materials and skillful, with minimal computational errors. Keep on!
8 = **OK**, not so familiar with the related materials, with relatively more computational errors. We have room to improve.
7 = **Passing**, less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.
6 = **Borderline**. We need to catch it up. If you have trouble doing your homework, it might be time to visit your instructor to get help. Do not wait to let the trouble accumulate.
Below 6 = This might be a dangerous warning signal. **We are failing!** It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

Scores	≥ 10	9	8	7	6	≤ 5
Frequency	2	9	6	4	10	3
Percentage	5.9	26.5	17.7	11.7	29.4	8.8

Discussions and Comments

The main purpose of this quiz is 2-fold: to see if students master the art of cress-section method and the shell method in finding volumes of solids; to see how students are doing homework exercises, as all these quiz problems, as indicated below, are from exercises.

It is phenomenal that many students are very good in setting up the integrals. (Most of the students with grade 7 or higher have at least 2 of the 3 integrals setting correctly.) Many of the errors are in integration, geometry and algebra. (I will not summarize algebraic errors as they vary from person to person.)

(1) **integration errors:** Quite a few students are still unfamiliar with the integration formulas in Section 5.2. The following have been found in our solutions.

Erroneous way: $\int y \cdot y^{1/3} dy = \frac{y^2}{2} \cdot \frac{y^{4/3}}{4/3}$ Correct way: $\int y \cdot y^{1/3} dy = \int y^{4/3} dy = \frac{3y^{7/3}}{7} + C.$

(2) **integration bounds errors:** Many errors are in using dx bounds for dy integrals or vice versa. Students are strongly encouraged to read the formulas in 4.2 of the review guide Math156 Review for Exam 2 displayed in the Math 156 website.

(3) **geometry errors: (area formula)** As we have discussed in class: in this exam, it is very important to know what we are computing, as the integrands we set up in our volume computing are either the cross section area (in rotating solids, area of a circle) times the thickness, or the lateral surface area of a shell times the thickness. The correct way to compute the area of a circle of radius r is πr^2; and the correct way to compute the area of the lateral surface area of a shell with radius r and height h is $2\pi rh$. Most of the geometry errors observed in this quiz are using πr^2 to compute the lateral surface area of a shell, and use $2\pi r$ to compute the area of a circle.

(4) **geometry errors: (length computing)** Another very common problem found in the quiz is that many are having trouble computing the length of a line segment. Here is a simple way to remember it correctly. For a horizontal line segment, the length is the right end x-coordinate minus the left end x-coordinate. For a vertical line segment, the length is the top end y-coordinate minus the bottom end y-coordinate.

The only way to avoid these errors in the future is practicing more until the correct methods are firmly in our minds.