**Math 156 Fall 2013 Quiz 12**

**Name:**

**Name:** Show your work (full credit will only be given for correct answer with sufficient supporting work), and do problems on both sides.

**1**: Find an equation of the tangent line to the curve $x = 2t^2 + 1, y = \frac{t^3}{3} - t$ at the point $t = 3$.

**2**: Find $\frac{d^2y}{dx^2}$ for the curve $x = t^3 - 12t, y = t^2 - 1$. For which values of $t$ is the curve concave upward?

**3**: Find the points on the curve $x = 2t^3 + 3t^2 - 12t, y = 2t^3 + 3t^2 + 1$ where the tangent is horizontal or vertical.

**4**: Use the parametric equations of an ellipse $x = 3\cos\theta, y = 4\sin\theta$, $0 \leq \theta \leq 2\pi$, to find the area that it encloses. (Hint: Find the area of the first quadrant and then use symmetry. )

**Solutions**
**1**: Find an equation of the tangent line to the curve $x = 2t^2 + 1, y = \frac{t^3}{3} - t$ at the point $t = 3$.

**Solution.** The tangent line has equation

$$y - \underline{y_0} = \underline{m}(x - \underline{x_0}),$$

where $(x_0, y_0)$ is the point on the curve when $t = 3$, and $m$ is the slope of tangent at that point.
**Computing $(x_0, y_0)$.** When $t = 3$, $x_0 = 2(3^2) + 1 = 19$ and $y_0 = \frac{3^3}{3} - 3 = 6$.
**Computing the slope $m$.** This is $\frac{dy}{dx}$ at that point. We compute

$$\frac{dy}{dt} = t^2 - 1, \ \frac{dx}{dt} = 4t. \text{ and so } \frac{dy}{dx} = \frac{t^2 - 1}{4t}.$$

When $t = 3$, $m = \frac{3^2-1}{4 \cdot 3} = \frac{2}{3}$. Here is the answer:

$$y - 6 = \frac{2}{3}(x - 19).$$

**2**: Find $\frac{d^2y}{dx^2}$ for the curve $x = t^3 - 12t, y = t^2 - 1$. For which values of $t$ is the curve concave upward?

**Solution.** To find $\frac{d^2y}{dx^2}$, we need to find $\frac{dy}{dx}$ first.
**Computing $\frac{dy}{dx}$.** We compute

$$y' = \frac{dy}{dt} = 2t, \ \frac{dx}{dt} = t^2 - 12. \text{ and so } \frac{dy}{dx} = \frac{2t}{3t^2 - 12}.$$

**Computing $\frac{d^2y}{dx^2}$.** We compute (use quotient rule)

$$\frac{dy'}{dt} = \frac{(3t^2 - 12) \cdot 2 - 2t(6t)}{(3t^2 - 12)^2} = \frac{-6t^2 - 24}{(3t^2 - 12)^2}, \ \frac{dx}{dt} = t^2 - 12,$$

Thus

$$y'' = \frac{d^2y}{dx^2} = \frac{-6t^2 - 24}{(3t^2 - 12)^3} = \frac{-6(t^2 + 4)}{(3t^2 - 12)^2 \cdot 3(t + 2)(t - 2)}.$$

**Concavity discussion.** From Calculus I, we need to find $t$ so that $y'' > 0$. Since $6(t^2+4)$ and the square $(3t^2 - 12)^2$ are always positive, the sign of $y''$ is determined by the factors $-3(t+2)(t-2)$. The values $t = -2$ and $t = 2$ partition the reals into three intervals: $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$. Testing (or discussing) indicates that when $-2 < t < 2$, $y'' > 0$ (and so the curve is concave upwards.

**3**: Find the points on the curve $x = 2t^3 + 3t^2 - 12t, y = 2t^3 + 3t^2 + 1$ where the tangent is horizontal or vertical.

**Solution.** We first find $\frac{dy}{dx}$ and then study for what values of $t$ we will have $\frac{dy}{dx} = 0$ for horizontal tangent lines, or $|y'|$ tends to infinity (for vertical tangent lines).
**Computing $\frac{dy}{dx}$.** We compute

$$y' = \frac{dy}{dt} = 6t(t + 1), \ \frac{dx}{dt} = 6t^2 + 6t - 12 = 6(t + 2)(t - 1). \text{ and so } \frac{dy}{dx} = \frac{6t(t + 1)}{6(t + 2)(t - 1)}.$$

**For what value of $t$ we will have $y' = 0$?** Set $\frac{dy}{at} = 6t(t+1) = 0$. We have $t = 0$ (and so $(x, y) = (0, 1)$) or $t = -1$ (and so $(x, y) = (13, 2)$). Thus at the points $(0, 1)$ and $(13, 2)$, the tangent lines of the curve are horizontal.

**For what value of $t$ we will have $|y'|$ goes to infinity?** Set $\frac{dx}{at} = 6(t+2)(t-1) = 0$. We have $t = -2$ (and so $(x, y) = (20, -3)$) or $t = 1$ (and so $(x, y) = (-7, 6)$). Thus at the points $(20, -3)$ and $(-7, 6)$, the tangent lines of the curve are vertical.

**4**: Use the parametric equations of an ellipse $x = 3\cos\theta, y = 4\sin\theta$, $0 \le \theta \le 2\pi$, to find the area that it encloses.

**Solution.** By symmetry, we can compute area in the first quadrant and multiply the answer by 4. Hence

$$\begin{aligned} A &= 4\int_0^3 y dx = 4\int_{\pi/2}^0 4\sin(\theta)(-3\sin(\theta))d\theta \\ &= 4(3)(4)\int_0^{\pi/2} \sin^2(\theta)d\theta = 48\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}d\theta \\ &= 24\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = 12\pi. \end{aligned}$$