Math156 Review for Exam 4

1. What will be covered in this exam: Representing functions as power series, Taylor and Maclaurin series, calculus with parametric curves, calculus with polar coordinates.

2. Exam Rules: This is a closed note and close-text exam. Formula sheet is not allowed. Calculators of any kind will not be allowed. Any electronic devices, including a cell phone, will **not** be allowed.

3. Expectations In this exam, students are expected to do each of the following:
(A) (**representing functions as power series**) to know how to use both algebraic methods and calculus methods (term by term differentiation and integration within the interval of convergence) to present a function as a power series.
(B) (**Taylor and Maclaurin series**) know how to find Taylor and Maclaurin series of a function near a given point, and to use both algebraic methods and calculus methods in computing Taylor and Maclaurin series of a function near a given point.
(C) (**calculus with parametric curves**) to know the skills of converting an (x, y)-coordinated function into parametric functions and vise versa; and to compute the differentiation and integration (including area and arc length computations) of parametric functions.
(D) (**calculus with polar coordinates**) to know the skills of converting an (x, y)-coordinated function into polar coordinated functions and vise versa; and to compute the differentiation and integration (including area and arc length computations) of polar coordinated functions.

4. Summary Sheet (help yourself to make up your own summary materials)

Maclairin Series

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $r = 1$.

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $r = \infty$.

(3) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $r = \infty$.

(4) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $r = \infty$.

(5) $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, $r = 1$.

(6) $(1+x)^k = \sum_{n=0}^{\infty} \left(\begin{array}{c} k \\ n \end{array} \right) x^n$, $r = 1$.

(7) $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, $r = 1$.

Taylor Series The Taylor series of $f(x)$ at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Area in polar coordinates $A = \int_{\alpha}^{\beta} \frac{r^2}{2} d\theta$.

Arc length in polar coordinates $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$.

5. Warming Up Exercises

By working on these and similar exercises, you will become more familiar with the related concepts and skills. It will help to warm you up and prepare well for the coming exam. Please note that there is no implication of any kind that any of these problem will be in the exam.

Representing functions as power series: Find the power series representation for the functions and determine the interval of convergence.

(1) $f(x) = \dfrac{1}{1+x}$. (Answer: $f(x) = \dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$. $I = (-1, 1)$)

(2) $f(x) = \dfrac{3}{1-x^4}$. (Answer: $f(x) = \dfrac{3}{1-x^4} = \sum_{n=0}^{\infty} 3x^{4n}$. $I = (-1, 1)$)

(3) $f(x) = \dfrac{2}{1-x^3}$. (Answer: $f(x) = \dfrac{1}{1-x^3} = \sum_{n=0}^{\infty} 2x^{3n}$. $I = (-1, 1)$)

(4) $f(x) = \dfrac{1}{1+9x^2}$. (Answer: $f(x) = \dfrac{1}{1+9x^2} = \sum_{n=0}^{\infty}(3x)^{2n}$. $I = (-1/3, 1/3)$)

(5) $f(x) = \dfrac{1}{x-5}$. (Answer: $f(x) = \dfrac{1}{x-5} = \dfrac{-1}{5}\dfrac{1}{1-x/5} = \sum_{n=0}^{\infty} \dfrac{-x^n}{5^{n+1}}$. $I = (-5, 5)$)

(6) $f(x) = \dfrac{x}{4x+1}$. (Answer: $f(x) = \dfrac{x}{4x+1} = \dfrac{x}{1-(-4x)} = x\sum_{n=0}^{\infty}(-1)^n(4x)^n = \sum_{n=0}^{\infty}(-4)^n x^{n+1}$. $I = (-1/4, 1/4)$)

(7) $f(x) = \dfrac{x}{9+x^2}$. (Answer: $f(x) = \dfrac{x}{9+x^2} = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{9^{2n}}$. $I = (-3, 3)$)

(8) $f(x) = \dfrac{x^2}{a^3-x^3}$. (Answer: $f(x) = \dfrac{x^2}{a^3-x^3} = \sum_{n=0}^{\infty} \dfrac{x^{3n+2}}{a^{3n+3}}$. $I = (-|a|, |a|)$)

Find the power series representation for the functions and determine the radius of convergence.

(1) $f(x) = \ln(5-x)$. (Answer: $f(x) = \ln 5 - x = -\displaystyle\int \frac{dx}{5-x} = -\frac{1}{5}\sum_{n=0}^{\infty}\int\left(\frac{x}{5}\right)^n dx = C - \sum_{n=1}^{\infty}\frac{x^n}{n5^n} = \ln(5) - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$. $R = 5$)

(2) $f(x) = \dfrac{x^2}{(1-2x)^2}$. (Answer: $\dfrac{2}{(1-2x)^2} = \displaystyle\int \frac{dx}{1-2x} = \sum_{n=0}^{\infty}\int(2x)^n dx = \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n$. $f(x) = \dfrac{x^2}{2}\cdot\dfrac{2}{(1-2x)^2} = \displaystyle\sum_{n=0}^{\infty} 2^n(n+1)x^n$. $R = 1/2$)

(3) $f(x) = \dfrac{x^3}{(x-2)^2}$. (Answer: $\dfrac{1}{2-x} = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$. $f(x) = \dfrac{x^3}{(x-2)^2} = x^3\dfrac{d}{dx}\dfrac{1}{2-x} = x^3\displaystyle\sum_{n=1}^{\infty}\frac{nx^{n-1}}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{nx^{n+2}}{2^{n+1}}$, $R = 2$)

(4) $f(x) = \tan^{-1}(x/3)$. (Answer: $f'(x) = \dfrac{1}{3(1+(x/3)^2)} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}$. $R = 3$))

(5) $f(x) = \dfrac{1}{x-5}$. (Answer: $f(x) = \dfrac{1}{x-5} = \dfrac{-1}{5}\dfrac{1}{1-x/5} = \sum_{n=0}^{\infty} \dfrac{-x^n}{5^{n+1}}$. $I = (-5, 5)$)

(6) $f(x) = \dfrac{x}{4x+1}$. (Answer: $f(x) = \dfrac{x}{4x+1} = \dfrac{x}{1-(-4x)} = x\sum_{n=0}^{\infty}(-1)^n(4x)^n = \sum_{n=0}^{\infty}(-4)^n x^{n+1}$. $I = (-1/4, 1/4)$)

(7) $f(x) = \dfrac{x}{9+x^2}$. (Answer: $f(x) = \dfrac{x}{9+x^2} = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{9^{2n}}$. $I = (-3, 3)$)

(8) $f(x) = \dfrac{x^2}{a^3 - x^3}$. (Answer: $f(x) = \dfrac{x^2}{a^3 - x^3} = \sum_{n=0}^{\infty}(\dfrac{x^{3n+2}}{a^{3n+3}}$. $I = (-|a|, |a|)$)

Find the Maclaurin series for each function f given below, and determine its radius of convergence.

(1) $f(x) = \tan^{-1}(x^2)$. (Ans: $f(x) = \tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+2}}{2n+1}$, $r = 1$.)

(2) $f(x) = \dfrac{x^2}{1+x}$. (Ans: $f(x) = \dfrac{x^2}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^{n+2}$, $r = 1$.)

(3) $f(x) = \ln(1-x)$. (Ans: $f(x) = -\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$, $r = 1$.)

(4) $f(x) = xe^{2x}$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{2^n x^{n+1}}{n!}$, $r = \infty$.)

(5) $f(x) = \sin(x^4)$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$, $r = \infty$.)

(6) $f(x) = \dfrac{1}{\sqrt[4]{16-x}}$. (Ans: $f(x) = \dfrac{1}{2} + \sum_{n=1}^{\infty} \dfrac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1} n!} x^n$, $r = 16$.)

(7) $f(x) = e^{-x/2}$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{2^n n!}$, $r = \infty$.)

(8) $f(x) = \cos(\pi x)$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$, $r = \infty$.)

(9) $f(x) = x \tan^{-1} x$. (Ans: $f(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+2}}{2n+1}$, $r = 1$.)

(10) $f(x) = x^2 e^{-x}$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+2}}{n!}$, $r = \infty$.)

(11) $f(x) = x\cos(2x)$. (Ans: $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n} x^{2n+1}}{(2n)!}$, $r = \infty$.)

(12) $f(x) = \dfrac{x^3}{\sqrt{2+x}}$. (Ans: $f(x) = \dfrac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^{2n+1/2} n!} x^n$, $r = 2$.)

Evaluate the integrals as an infinite series, and determine the radius of convergence.

(1) $\int x\cos(x^3)dx$. (Ans: $\int x\cos(x^3)dx = \sum_{n=0}^{\infty} \int (-1)^n \dfrac{x^{6n+1}}{(2n)!} dx = C + \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{6n+2}}{(6n+2)(2n)!}$, $r = \infty$.)

(2) $\int \dfrac{\sin(x)}{x} dx$. (Ans: $\int \dfrac{\sin(x)}{x} dx = \sum_{n=0}^{\infty} \int (-1)^n \dfrac{x^{2n}}{(2n+1)!} dx = C + \sum_{n=1}^{\infty}(-1)^n \dfrac{x^{2n+1}}{(2n+1)(2n+1)!}$, $r = \infty$.)

(3) $\int \dfrac{e^x - 1}{x} dx$. (Ans: $\int \dfrac{e^x - 1}{x} dx = \sum_{n=0}^{\infty} \int \dfrac{x^{n-1}}{n!} dx = C + \sum_{n=1}^{\infty} \dfrac{x^n}{nn!}$, $r = \infty$.)

Use Series to compute the limits.

(1) $\lim_{x\to 0} \dfrac{x - \tan^{-1}(x)}{x^3}$. (Ans: $\lim_{x\to 0} \dfrac{x - \tan^{-1}(x)}{x^3} = \lim_{x\to 0} \dfrac{1}{x^3} \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1} = \dfrac{1}{3}$.)

(2) $\lim_{x\to 0} \dfrac{1 - \cos(x)}{1 - x - e^x}$. (Ans: $\lim_{x\to 0} \dfrac{1 - \cos(x)}{1 - x - e^x} = \lim_{x\to 0} \dfrac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n+1}}{\sum_{n=2}^{\infty} \frac{-x^n}{n!}} = -1$.)

(3) $\lim_{x\to 0} \dfrac{\sin(x) - x - \frac{x^3}{6}}{x^5}$. (Ans: $\lim_{x\to 0} \dfrac{\sin(x) - x - \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \sum_{n=2}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} x^5 = \dfrac{1}{5!} = \dfrac{1}{120}$.)

(4) $\lim_{x\to 0} \dfrac{\tan(x) - x}{x^3}$. (Ans: $\lim_{x\to 0} \dfrac{\tan(x) - x}{x^3} = \lim_{x\to 0} \dfrac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots - x}{x^3} = \dfrac{1}{3}$.)

Find the Taylor series for each function f below centered at the given value a, and determine its radius of convergence.

(1) $f(x) = x^3 + 2x + 1$, $a = 1$. (Ans: $4 + 5(x-1) + 3(x-1)^2 + (x-1)^3$, $r = \infty$.)
(2) $f(x) = x^3$, $a = -1$. (Ans: $-1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$, $r = \infty$.)
(3) $f(x) = e^x$, $a = 3$. (Ans: $f(x) = e^3 \cdot e^{x-3} = \sum_{n=0}^{\infty} \frac{e^3(x-3)^n}{n!}$, $r = \infty$.)
(4) $f(x) = \ln(x)$, $a = 2$. (Ans: $f(x) = \ln(2+(x-2)) = \ln 2 + \ln(1 - \frac{x-2}{2}) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n2^n}$, $r = 2$.)

Parametric Curves and Calculus with Parametric Curves

Eliminate the parameter to find a Cartesian equation.

(1) $x = 1 + 3t$, $y = 2 - t^2$. (Ans. $y = \frac{-1}{9}(x-1)^2 + 2$.)
(2) $x = t^2$, $y = t^3$. (Ans. $x = y^{3/2}$.)

Differentiation and Integration with Parametric Curves.

(1) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = t^3 - 12t, y = t^2 - 1$. (Ans: $\frac{dy}{dx} = \frac{2t}{3t^2 - 12}$, $\frac{d^2y}{dx^2} = \frac{d(y'_x}{dx} = \frac{-2(t^2+4)}{9(t^2-4)^3}$.)

(2) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x = t + \ln(t), y = t - \ln(t)$. (Ans: $\frac{dy}{dx} = \frac{t-1}{t+1}$, $\frac{d^2y}{dx^2} = \frac{d(y'_x}{dx} = \frac{2t}{(t+1)^3}$.)

(3) Find an equation of the line tangent to the curve $x = 2t^2 + 1$, $y = t^3/3 - t$ at $t = 3$. (Ans: $y - 6 = \frac{2}{3}(x - 19)$.)

(4) Find an equation of the line tangent to the curve $x = \cos(\theta) + \sin(2\theta)$, $y = \sin(\theta) + \cos(2\theta)$ at $\theta = 0$. (Ans: $y - 1 = \frac{1}{2}(x-1)$.)

(5) Find the points on the curve $x = 2t^3 + 3t^2 - 12t$, $y = 2t^3 + 3t^2 + 1$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(x, y) = (20, -3)$, vertical at $(-7, 6)$.)

(6) Find the points on the curve $x = \cos(3\theta)$, $y = \sin(2\theta)$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(x, y) = (0, \pm 2)$, vertical at $(\pm 1, 0)$ and $(\pm 1, \pm\sqrt{3})$.)

(7) Find the points on the curve $x = 10 - t^2$, $y = t^3 - 12t$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(x, y) = (20, -3)$, vertical at $(-7, 6)$.)

(8) Find the points on the curve $x = \cos(3\theta)$, $y = \sin(2\theta)$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(x, y) = (6, \pm 16)$, vertical at $(10, 0)$.)

(9) Find equations of the tangents to the curve $x = 3t^2 + 1$ and $y = 2t^3 + 1$ that pass through the point $(4, 3)$. (Ans: $y - 3 = x - 4$, and $y - (-15) = -2(x - 13)$.)

(10) At what points on the curve $x = t^3 + 4t$ and $y = 6t^2$ is the tangent line parallel to the line $x = -7t$ and $y = 12t - 5$? (Ans: $(x, y) = (-5, 6)$ or $(\frac{-208}{27}, \frac{32}{3})$)

(11) Find the area bounded by the curve $x = t - 1/t$ and $y = t + 1/t$ and the line $y = 2.5$. (Ans: Set $t + 1/t = y = 2.5$ to get $t = 1.5$ and $t = 2$. $A = \int_{1.5}^{2} (2.5 - y)dx(t) = \int_{\frac{1}{2}}^{2} \left(\frac{5}{2} - t - \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt = \frac{15}{4} - 4\ln 2$.)

(12) Find the area bounded by the curve $x = \cos(t)$ and $y = e^t$, $0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$. (Ans: $A = \int_0^1 (y-1)dx = \int_{\pi/2}^{0} (e^t - 1)(-\sin(t))dt = \frac{e^{\pi/2} - 1}{2}$.)

(13) Find the length of the curve $x = \frac{t}{1+t}$ and $y = \ln(1+t)$, $0 \le t \le 2$. (Ans: $L = \int_0^2 \frac{\sqrt{t^2 + 2t + 2}}{(1+t)^2} dt =$

$\int_1^3 \frac{u^2+1}{u^2} du$ with $u = t+1$, Thus $L = -\frac{\sqrt{10}}{3} + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})$.)

(14) Find the length of the curve $x = e^t + e^{-t}$ and $y = 5 - 2t$, $0 \le t \le 3$. (Ans: $L = \int_0^3 (e^t + e^{-t})dt = e^3 - e^{-3}$.)

Polar Coordinates and Calculus with Polar Coordinates

Switching between polar coordinates and Cartesian coordinates.

(1) Find Cartesian equation for $r = -3\sin(\theta)$. (Ans: $x^2 + y^2 = 3x$.)

(2) Find Cartesian equation for $r = 2\sin(\theta) + 2\cos(\theta)$. (Ans: $x^2 + y^2 = 2x + 2y$.)

(3) Find Cartesian equation for $r = \csc(\theta)$. (Ans: $y = 1$.)

(4) Find Cartesian equation for $r = \tan(\theta)\sec(\theta)$. (Ans: $x^2 = y$.)

(5) Find polar equation for $x = -y^2$. (Ans: $r = -\cot(\theta)\csc(\theta)$.)

(6) Find Cartesian equation for $x + y = 9$. (Ans: $r = \frac{9}{\cos(\theta)+\sin(\theta)}$.)

(7) Find polar equation for $x^2 + y^2 = 2cx$. (Ans: $r = 2c\cos(\theta)$.)

(8) Find Cartesian equation for $x^2 - y^2 = 2$. (Ans: $r^2 = \sec(2\theta)$. Note that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$).

Differentiation and Integration with Polar Coordinates.

(1) Find the slope of tangent line to $r = 2\sin(\theta)$ at $\theta = \frac{\pi}{6}$. (Ans: $\frac{dy}{dx} = \tan(2\theta)$, slope $= \tan(\frac{\pi}{3}) = \sqrt{3}$.)

(2) Find the slope of tangent line to $r = 2 - \sin(\theta)$ at $\theta = \frac{\pi}{3}$. (Ans: $\frac{dy}{dx} = \frac{2\cos(\theta) - \sin(2\theta)}{-2\sin(theta) - \cos(2\theta)}$, slope $= \frac{2-\sqrt{3}}{1-2\sqrt{3}}$.)

(3) Find the slope of tangent line to $r = 1/\theta$ at $\theta = \pi$. (Ans: $\frac{dy}{dx} = \frac{-\sin(\theta) + \theta\cos(\theta)}{-\cos(theta) - \theta\sin(\theta)}$, slope $= -\pi$.)

(4) Find the slope of tangent line to $r = \sin(3\theta)$ at $\theta = \pi/6$. (Ans: $\frac{dy}{dx} = \frac{3\cos(3\theta))\sin(\theta) + \sin(3\theta)\cos(\theta)}{3\cos(3\theta))\cos(\theta) - \sin(3\theta)\sin(\theta)}$, slope $= -\sqrt{3}$.)

(5) Find the points on $r = 3\cos(\theta)$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(\frac{3}{\sqrt{2}}, \frac{\pi}{4}), (-\frac{3}{\sqrt{2}}, \frac{3\pi}{4})$, vertical at $(3, 0), (0, \pi/2)$.)

(6) Find the points on $r = e^\theta$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(e^{\pi(n-1)/4}, \pi(n-\frac{1}{4}))$, vertical at $(e^{\pi(n+1)/4}, \pi(n+\frac{1}{4}))$.)

(7) Find the points on $r = 1 + \cos(\theta)$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(\frac{3}{2}, \frac{\pi}{3}), (0, \pi), (\frac{3}{2}, \frac{5\pi}{3})$, vertical at $(2, 0), (\frac{1}{2}, \frac{2\pi}{3}), (\frac{1}{2}, \frac{4\pi}{3})$.)

(8) Find the points on $r^2 = e\sin(2\theta)$ where the tangent line is horizontal or vertical. (Ans: Horizontal at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}), (\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3})(0, 0)$, vertical at $(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}), (\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6})(0, 0)$.)

(9) Find the area of the region bounded by $r = \sqrt{\theta}$, $0 \le \theta \le \frac{\pi}{4}$. (Ans: $A = \int_0^{\pi/4} \frac{\theta}{2} d\theta = \frac{\pi^2}{64}$.)

(10) Find the area of the region bounded by $r = e^{\theta/2}$, $\pi \le \theta \le 2\pi$. (Ans: $A = \int_\pi^{2\pi} \frac{e^\theta}{2} d\theta = \frac{e^{2\pi} - e^\pi}{2}$.)

(11) Find the area of the region bounded by $r = \sin(\theta)$, $\pi/3 \le \theta \le 2\pi/3$. (Ans: $A = \int_{\pi/3}^{2\pi/3} \frac{\sin^2(\theta)}{2} d\theta =$

$\dfrac{\pi}{12} + \dfrac{\sqrt{3}}{8}$.)

(12) Find the area of the region bounded by $r = \sqrt{\sin(\theta)}$, $0 \le \theta \le \pi$. (Ans: $A = \displaystyle\int_0^{\pi} \frac{\sin(\theta)}{2} d\theta = 1$.)

(13) Find the area of the region inside $r = 4\sin(\theta)$ and outside $r = 2$. (Ans: $4\sin(\theta) = 2$ to get bounds. $A = 2\displaystyle\int_{\pi/6}^{\pi/2} \frac{(4\sin(\theta))^2 - 2^2}{2} d\theta = \frac{4}{3}\pi + 2\sqrt{3}$.)

(14) Find the area of the region inside $r = 1 - \sin(\theta)$ and outside $r = 1$. (Ans: $1 - \sin(\theta) = 1$ to get bounds. $A = \displaystyle\int_{\pi}^{2\pi} \frac{(1-\sin(\theta))^2 - 1^2}{2} d\theta = \frac{\pi}{4} + 2$.)

(15) Find the area of the region inside $r = 3\cos(\theta)$ and outside $r = 1 + \cos(\theta)$. (Ans: $3\cos(\theta) = 1 + \cos(\theta)$ to get bounds. $A = 2\displaystyle\int_0^{\pi/3} \frac{(3\cos(\theta))^2 - (1-\cos(\theta))^2}{2} d\theta = \frac{9\pi}{2}$.)

(14) Find the area of the region inside $r = 2 + \sin(\theta)$ and outside $r = 3\sin(\theta)$. (Ans: $2 + \sin(\theta) = 3\sin(\theta)$ to get bounds. $A = \displaystyle\int_0^{2\pi} \frac{(2+\sin(\theta))^2 - (3\sin(\theta))^2}{2} d\theta = \frac{\pi - 2}{8}$.)

(15) Find the area of the region enclosed by one loop of $r = \sin(2\theta)$. (Ans: $A = \displaystyle\int_0^{\pi/2} \frac{\sin^2(2\theta)}{2} d\theta = \frac{\pi}{8}$.)

(16) Find the area of the region enclosed by one loop of $r = 4\sin(3\theta)$. (Ans: $A = \displaystyle\int_0^{\pi/3} \frac{4\sin^2(3\theta)}{2} d\theta = \frac{4\pi}{3}$.)

(17) Find the area of the region enclosed by one loop of $r = 1+2\sin(\theta)$ (inner loop). (Ans: $A = 2\displaystyle\int_{7\pi/6}^{3\pi/2} \frac{(1+2\sin(\theta))^2}{2} d\theta = \pi - \frac{3\sqrt{3}}{2}$.)