Math156 Review for Exam 2

1. What will be covered in this exam: improper integrals, using integrals to compute areas, volumes, length of arcs.

2. Exam Rules: This is a closed note and close-text exam. Formula sheet is not allowed. Calculators of any kind will not be allowed. Any electronic devices, including a cell phone, will not be allowed.

3. Expectations In this exam, students are expected:
(A) (**improper integrals**) to understand the mathematical idea of converting an improper integral into a proper integral and computing it with the appropriate limiting process, to be able to identify whether the improper integral is over an infinite interval or has discontinuities, and to use appropriate method accordingly.
(B) (**area computation**) to understand the mathematical idea of converting the area computation of irregular shape regions into the computing of the area of a rectangular region together with the corresponding summation and limiting process, and to use this understanding to correctly setting up the integral (including determining the integration bounds and the integrand).
(C) (**volume computation**) to understand the mathematical idea of converting the volume computation of irregular shape solids into the computing the volume of either the thin pieces of regular shape solids (cross-section/the disk method) or the cylindrical shells with the corresponding summation and limiting process, and to use this understanding to correctly setting up the integral (including determining the integration bounds and the integrand).
(D) (**arc length computation**) to understand the mathematical idea of converting the arc length computation of a curve into the computing the length of a hypotonus of a right triangle with the corresponding summation and limiting process, and to use this understanding to correctly setting up the integral (including determining the integration bounds and the integrand).

4. Related Formulas

4.1. Area Computation

(a) **For vertically simple regions** (those using x-bounds): Area $= \displaystyle\int_a^b (y_{\text{top}}(x) - y_{\text{bottom}}(x))dx$.

(b) **For horizontally simple regions** (those using y-bounds): Area $= \displaystyle\int_c^d (x_{\text{right}}(y) - x_{\text{left}}(y))dy$.

4.2. Volume Computation of Solids Generated by Rotating a Region About an Axis

(a) **Rotation about an axis parallel to x-axis:**
The method of cross sections: (General description) $a \le x \le b$, $r =$ distance from axis of rotation to the point on curve forming the cross-section (radius of the rotated circle), $dx =$ thickness of washer. (Geometry: Area of a disc with radius $r = \pi r^2$).

Bounds: When the axis or rotation is parallel to x-axis, the integration bounds should be the x-bounds. They are the smallest x coordinate a and largest x-coordinate b of points in the region.

Integrand: For a fixed x, the cross-section at x is either a washer (a bigger circle minus a smaller circle) or a circle (can view it as the smaller circle radius being 0). Use the πr^2 to compute the area of the cross-section $A(x)$.
Integral:

$$\text{Volume} = \int_a^b \text{area of cross-section at } x \cdot \text{thickness of the washer/circle} = \int_a^b \pi(r^2_{\text{larger}} - r^2_{\text{smaller}})dx.$$

The method of cylindrical shells: (General description) $c \le y \le d$, $r =$ distance from axis of rotation to the point on curve (radius of cylinder base), $x_{\text{right}}(y) - x_{\text{left}}(y) =$ height of the cylinder at y, dy=thickness of cylindrical shell. (Geometry: Area of the lateral face of a topless and bottomless cylinder with radius r and height $h = 2\pi rh$).

Bounds: When the axis or rotation is parallel to x-axis, the integration bounds should be the y-bounds. They are the smallest y coordinate c and largest y-coordinate d of points in the region.

Integrand: For a fixed y, the shell at y has thickness dy, radius $r(y)$ = distance from y to the y-coordinate of the axis of rotation, and height $h(y)$ = the right x-coordinate at y minus the left x-coordinate at y. The volume of this shell is $2\pi r(y)h(y)dy$.
Integral:

$$\text{Volume} = \int_c^d 2\pi \text{distance from } y \text{ to axis} \cdot \text{ height of shell at } y \cdot \text{ thickness of shell} = \int_c^d 2\pi r(y)(x_{\text{right}}(y) - x_{\text{left}}(y))dy.$$

(b) **Rotation about an axis parallel to y-axis:**
The method of cross sections: (General description) $c \le y \le d$, r = distance from axis of rotation to the point on curve forming the cross-section (radius of the rotated circle), dy = thickness of washer. (Geometry: Area of a disc with radius $r = \pi r^2$).

Bounds: When the axis or rotation is parallel to y-axis, the integration bounds should be the y-bounds. They are the smallest y coordinate c and largest y-coordinate d of points in the region.

Integrand: For a fixed y, the cross-section at y is either a washer (a bigger circle minus a smaller circle) or a circle (can view it as the smaller circle radius being 0). Use the πr^2 to compute the area of the cross-section $A(x)$.
Integral:

$$\text{Volume} = \int_c^d \text{ area of cross-section at } x \cdot \text{thickness of the washer/circle} = \int_c^d \pi(r_{\text{larger}}^2 - r_{\text{smaller}}^2)dy.$$

The method of cylindrical shells: (General description) $a \le x \le b$ r = distance from axis of rotation to the point on curve (radius of cylinder base), $y_{\text{larger}}(x) - y_{\text{smaller}}(x)$ = height of the cylinder at x, dx=thickness of cylindrical shell. (Geometry: Area of the lateral face of a topless and bottomless cylinder with radius r and height $h = 2\pi rh$).

Bounds: When the axis or rotation is parallel to y-axis, the integration bounds should be the x-bounds. They are the smallest x coordinate a and largest x-coordinate b of points in the region.

Integrand: For a fixed x, the shell at x has thickness dx, radius $r(x)$ = distance from x to the x-coordinate of the axis of rotation, and height $h(x)$ = the top y-coordinate at x minus the bottom y-coordinate at x. The volume of this shell is $2\pi r(x)h(x)dx$.
Integral:

$$\text{Volume} = \int_a^b 2\pi \text{distance from } x \text{ to axis} \cdot \text{ height of shell at } x \cdot \text{ thickness of shell} = \int_a^b 2\pi r(x)(y_{\text{larger}}(x) - y_{\text{smaller}}(x))dx.$$

4.3. Arc Length Computation

(a) Length of curve $y = y(x)$ with $a \le x \le b$ is $L = \displaystyle\int_a^b \sqrt{1 + (y'(x))^2}dx$.

(b) Length of curve $x = x(y)$ with $c \le y \le d$ is $L = \displaystyle\int_c^d \sqrt{1 + (x'(y))^2}dy$.

5. Warming Up Exercises

By working on these and similar exercises, you will become more familiar with the related concepts and skills. It will help to warm you up and prepare well for the coming exam. Please note that there is no implication of any kind that any of these problem will be in the exam.

Improper Integrals For each of the following improper integrals, determine if it is convergent. If it is convergent, try to find its value.

(1) $\int_e^\infty \frac{1}{x\ln x}dx$. (Answer: Use $u = \ln(x)$ in the integral. $\int_e^\infty \frac{1}{x\ln x}dx = \lim_{A\to\infty} \ln(\ln(A))$ is divergent.)

(2) $\int_0^2 \frac{x}{\sqrt{4-x^2}}dx$. (Answer: use $u = 4 - x^2$, $\int_0^2 \frac{x}{\sqrt{4-x^2}}dx = \lim_{d\to 2^-} -\sqrt{4-d^2} + 2 = 2$.)

(3) $\int_{-4}^0 \frac{1}{\sqrt{x+4}}dx$. (Ans $= 4$, $u = x + 4$).

(4) $\int_0^\infty xe^{-x}dx$. (Ans $= 1$, by parts).

(5) $\int_0^\infty xe^{-x^2}dx$. (Ans $= \frac{1}{2}$, $u = -x^2$.)

(6) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$. (Ans $= \pi$. use $\lim_{A\to\infty} \tan^{-1}(A) = \frac{\pi}{2}$ and $\lim_{A\to-\infty} \tan^{-1}(A) = -\frac{\pi}{2}$.)

(7) $\int_{-\infty}^\infty \frac{x^2dx}{9+x^6}$. (Ans $= \frac{\pi}{9}$, use $u = x^3$ and $\lim_{A\to\infty} \tan^{-1}(A) = \frac{\pi}{2}$.)

Area Computations Find the area of the region bounded by give curves in each of the following problems.

(1) $y = x, y = x^2$. (Ans $= \frac{1}{6}$, integral $\int_0^1 (x - x^2)dx$.)
(2) $y = x^2, y = 4x - x^2$. (Ans $= \frac{8}{3}$, integral $\int_0^2 (4x - 2x^2)dx$.)
(3) $x = 2y^2, x = 4 + y^2$. (Ans $= \frac{32}{3}$, integral $\int_{-2}^2 (4 - y^2)dy$.)
(4) $y = \sin(x), y = \frac{2x}{\pi}, x \geq 0$. (Ans $= 1 - \frac{\pi}{4}$, integral $\int_0^{\pi/2} (\sin(x) - \frac{2x}{\pi})dx$.)
(5) $y = |x|, y = x^2 - 2$. (Ans $= \frac{20}{3}$, integral $\int_{-2}^2 (|x| - x^2 + 2)dx = 2\int_0^2 (x - x^2 + 2)dx$.)
(6) $y = x^2 - x - 6, y = 0$. (Ans $= \frac{125}{6}$, integral $\int_{-2}^3 (x - x^2 + 6)dx$.)
(7) $y = 20 - x^2, y = x^2 - 12$. (Ans $= \frac{512}{3}$, integral $\int_{-4}^4 (32 - 2x^2)dx$.)
(8) $y = e^x - 1, y = x^2 - x$. (Ans $= e - \frac{11}{6}$, integral $\int_0^1 ((e^x - 1) - (x^2 - x))dx$.)
(9) $y + x = 0, x = y^2 + 3y$. (Ans $= \frac{32}{3}$, integral $\int_{-4}^0 (-y^2 - 4y)dy$.)

Volume Computations In each of the following problems, find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(1) $y = \frac{1}{x}, x = 1, x = 2, y = 0$, about $y = 0$. (Ans $= \frac{\pi}{2}$, cross-section, integral $\pi\int_1^2 (\frac{1}{x})^2dx$.)
(2) $y = x^3, y = x, x \geq 0$, about $y = 0$. (Ans $= \frac{4\pi}{21}$, cross-section, integral $\pi\int_0^1 (x^2 - x^6)dx$.)
(3) $y^2 = x, 2y = x$, about $x = 0$. (Ans $= \frac{64\pi}{15}$, cross-section, integral $\pi\int_0^2 (4y^2 - y^4)dy$.)
(4) $y^2 = x, y = x^2$, about $x = -1$. (Ans $= \frac{29\pi}{30}$, cross-section, integral $\pi\int_0^1 [(\sqrt{y} + 1)^2 - (y^2 + 1)^2]dy$.)
(5) $y = x, y = \sqrt{x}$, about $y = 1$. (Ans $= \frac{\pi}{6}$, cross-section, integral $\pi\int_0^1 [(1 - x)^2 - (1 - \sqrt{x})^2]dx$.)
(6) $y = \frac{1}{x}, x = 1, x = 2, y = 0$, about $x = 0$. (Ans $= 2\pi$, shell, integral $2\pi\int_1^2 x(\frac{1}{x})dx$.)
(7) $y = x^2, x = -1, x = -2, y = 0$, about $x = 0$. (Ans $= \frac{15\pi}{2}$, shell, integral $2\pi\int_{-2}^{-1} (-x) \cdot x^2dx$.)
(8) $y = x^2, x = 1, y = 0$, about $x = 0$. (Ans $= \frac{\pi}{2}$, shell, integral $2\pi\int_0^1 x \cdot x^2dx$.)
(9) $x = \sqrt{y}, x = 0, y = 1$, about $y = 0$. (Ans $= \frac{4\pi}{5}$, shell, integral $2\pi\int_0^1 y\sqrt{y}dy$.)
(10) $y = x^2, x = y^2$, about $y = -1$. (Ans $= \frac{29\pi}{30}$, shell, integral $2\pi\int_0^1 (y + 1)(\sqrt{y} - y^2)dy$.)
(11) $y = x^2, x = -1, x = -2, y = 0$, about $x = -1$. (Ans $= \frac{17\pi}{6}$, shell, integral $2\pi\int_{-2}^{-1} (-1 - x) \cdot x^2dx$.)

For each of the following problems, set up an integral that computes the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Do not evaluate the integrals (I include some answers anyway).

(1) $y = 2x, y = x^2$, about $y = 0$. (Ans $= \frac{64\pi}{15}$, cross-section, integral $\pi\int_0^2 (4x^2 - x^4)dx$.)
(2) $x = 1 + y^2, y = x - 3$, about $x = 0$. (Ans $= \frac{117\pi}{5}$, cross-section, integral $\pi\int_{-1}^2 [(y + 3)^2 - (1 + y^2)^2)dy$.)
(3) $x = 0, x = 9 - y^2$, about $x = -1$. (Ans $= \frac{1656\pi}{5}$, cross-section, integral $\pi\int_{-3}^3 [(10 - y^2)^2 - 1)dy$.)
(4) $y = x^2 + 1, y = 9 - x^2$, about $y = -1$. (Ans $= 256\pi$, cross-section, integral $\pi\int_{-2}^2 [(10 - x^2)^2 - (x^2 + 2)^2)dx$.)

(5) $y = \cos(x), y = 0, x = \dfrac{3\pi}{2}, x = \dfrac{5\pi}{2}$, about $x = 0$. (shell, integral $2\pi \int_{5\pi/2}^{3\pi/2} x\cos(x)dx$.)
(6) $y = x^3, y = x^2$, about $y = 1$. (cross-section, integral $\pi \int_0^1 [(1-x^3)^2 - (1-x^2)^2]dx = \pi \int_0^1 (x^6 - x^4 - 2x^3 + 2x^2)dx$.)
(7) $y = x^3, y = 8, x = 0$, about $x = 2$. (shell, integral $2\pi \int_0^2 (8-x^3)(2-x)dx$.)

Arc Length Computations Find the length of the curve.

(1) $y = 1 + 6x^{\frac{3}{2}}$, $0 \le x \le 1$. (Ans= $\frac{2}{243}(82\sqrt{82}-1)$, integral $\int_0^1 \sqrt{1+81x}dx$.)
(2) $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3}$, $0 \le x \le 1$. (Ans= $\frac{1261}{240}$, integral $\int_0^1 \sqrt{1 + (\frac{5x^4}{6} - \frac{3x^{-4}}{10})^2}dx = \int_0^1 (\frac{5x^4}{6} + \frac{3x^{-4}}{10})dx$.)
(3) $y = \ln(\cos(x))$, $0 \le x \le \dfrac{\pi}{3}$. (Ans= $\ln(2+\sqrt{3})$, integral $\int_0^{\pi/3} \sqrt{\sec^2(x)}dx = \int_0^{\pi/3} \sec(x)dx$.)
(4) $y = \dfrac{1}{6}(x^2+4)^{\frac{3}{2}}$, $0 \le x \le 3$. (Ans= $\frac{15}{2}$, integral $\int_0^3 \sqrt{1 + (\frac{x\sqrt{x^2+4}}{2})^2}dx = \int_0^3 (\frac{x^2}{2} + 1)dx$.)
(5) $y = 2\ln(\sin \dfrac{x}{2})$, $\dfrac{\pi}{3} \le x \le \pi$. (Ans= $-2\ln(2-\sqrt{3})$. This is a positive number, as $0 < 2 - \sqrt{3} < 1$. Integral $\int_{\pi/3}^{\pi} \sqrt{1 + \cot^2(x/2)}dx = \int_{\pi/3}^{\pi} \csc(x/2)dx$.)
(6) $y^2 = 4(x+4)^3$, $0 \le x \le 2$ and $y > 0$. (Ans= $\frac{2}{27}(55\sqrt{55} - 37\sqrt{37})$, use $y = 2(x+4)^{3/2}$, integral $\int_0^2 \sqrt{1+9(x+4)}dx$.)
(7) $y = \dfrac{x^2}{2} - \dfrac{\ln(x)}{4}$, $2 \le x \le 4$. (Ans= $6 + \frac{\ln(2)}{4}$, integral $\int_2^4 \sqrt{1 + (x - \frac{1}{4x})^2}dx = \int_2^4 \sqrt{(x + \frac{1}{4x})^2}dx$.)
(8) $x = \dfrac{\sqrt{y}(y-3)}{3}$, $1 \le y \le 9$. (Ans= $\frac{32}{3}$, integral $\int_1^9 \sqrt{1 + (\frac{\sqrt{y}}{2} - \frac{1}{2\sqrt{y}})^2}dy = \int_1^9 \sqrt{(\frac{\sqrt{y}}{2} + \frac{1}{2\sqrt{y}})^2}dy$.)
(9) $y = \ln(\sec(x))$, $0 \le x \le \pi/4$. (Ans= $\ln(\sqrt{2}+1)$, integral $\int_0^{\pi/4} \sqrt{1+\tan^2(x)}dx$.)

For each of the following problems, set up an integral that computes the length of the curve. Do not evaluate the integrals.

(1) $y = \cos(x)$, $0 \le x \le 2\pi$. (Ans: $\int_0^{2\pi} \sqrt{1+\sin^2(x)}dx$.)
(2) $x = y + y^3$, $1 \le y \le 4$. (Ans: $\int_1^4 \sqrt{1 + (1+3y^2)^2}dy = \int_1^4 \sqrt{2 + 6y^2 + 9y^4}dy$.)