Graphs (Matroids) with $k \pm \epsilon$-disjoint spanning trees (bases)

Hong-Jian Lai, Ping Li and Yanting Liang

Department of Mathematics

West Virginia University

Morgantown, WV

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- Survey Paper: E. M. Palmer, [On the spanning tree packing number of a graph, a survey, Discrete Math. 230 (2001) 13 - 21].

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- $a_1(G)$:= edge-arboricity, the minimum number of spanning trees whose union equals $E(G)$.
- Theorem (Nash-Williams, [J. London Math. Soc. 39 (1964)]) $a_1(G) \leq k$ iff $\forall X \subseteq E(G)$, $|X| \leq k|V(G[X])| - \omega(G[X])$.

When does $\tau(G) = k$?

- By Nash-Williams and Tutte, for a connected G, $\tau(G) = k$ if and only if both of the following holds:
  (i) $\forall X \subseteq E(G)$, $|E - X| \geq k(\omega(G - X) - 1)$, and
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- What is next?

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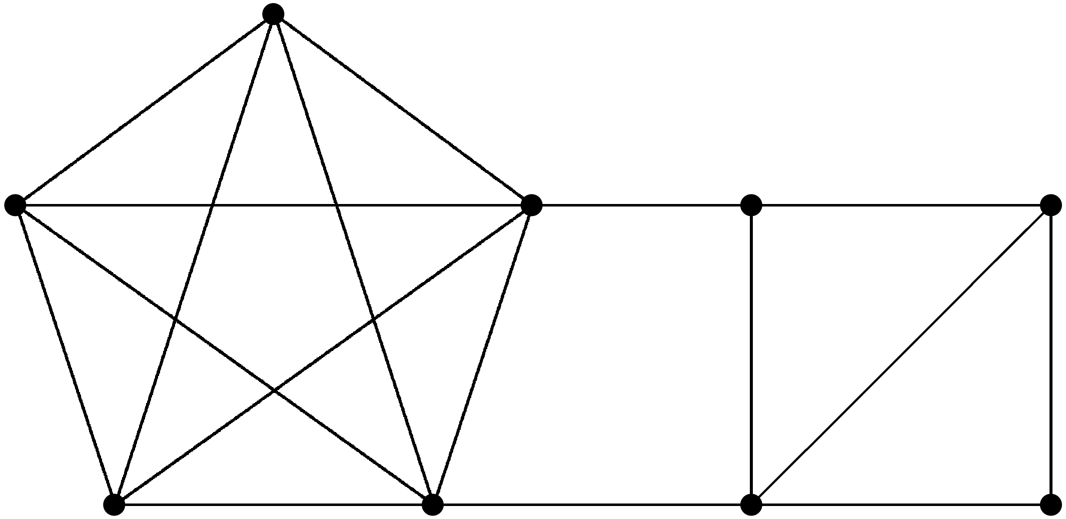
- 

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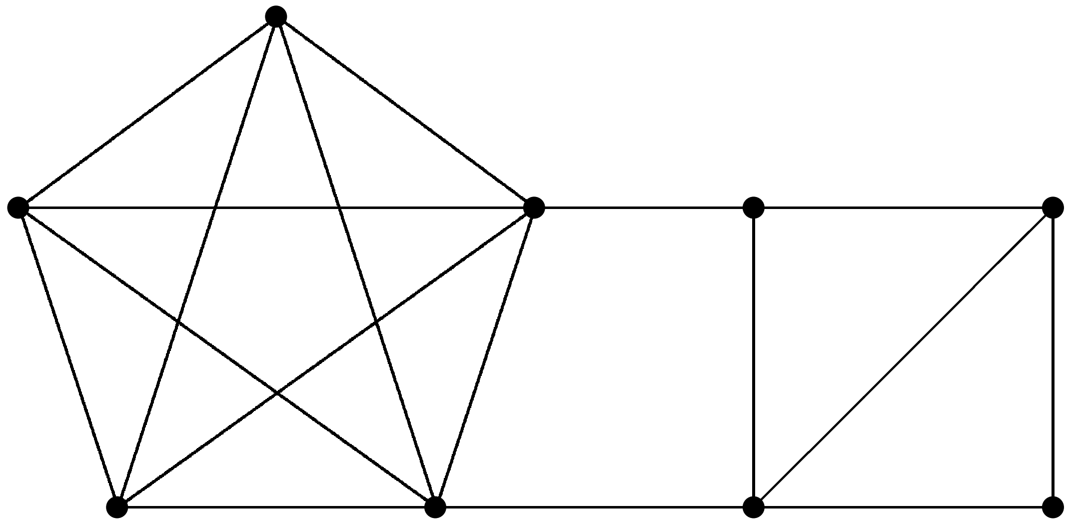
- 
- Every spanning tree must use one of the two edges in the 2-cut. Thus $F(G, 2) = 1$.

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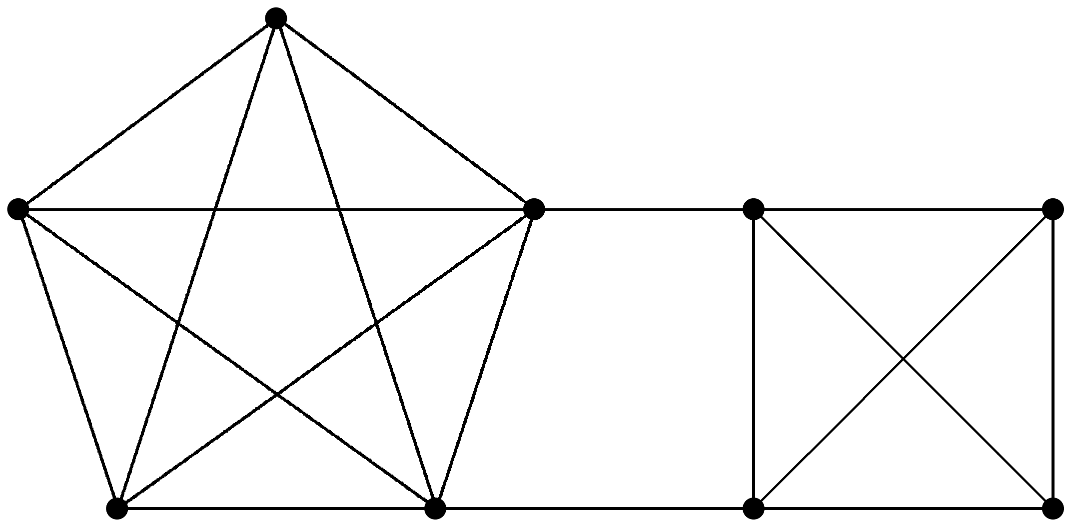

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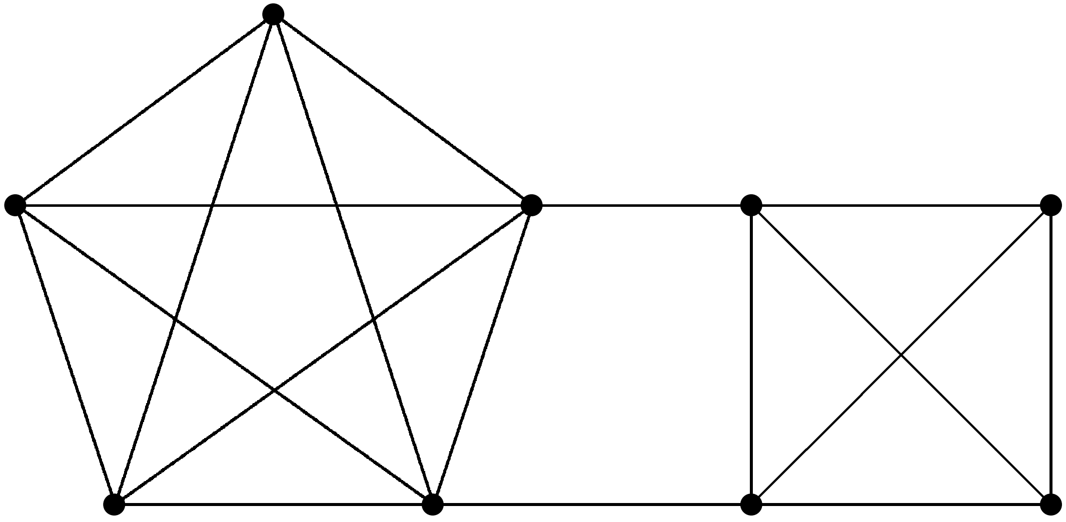
- 
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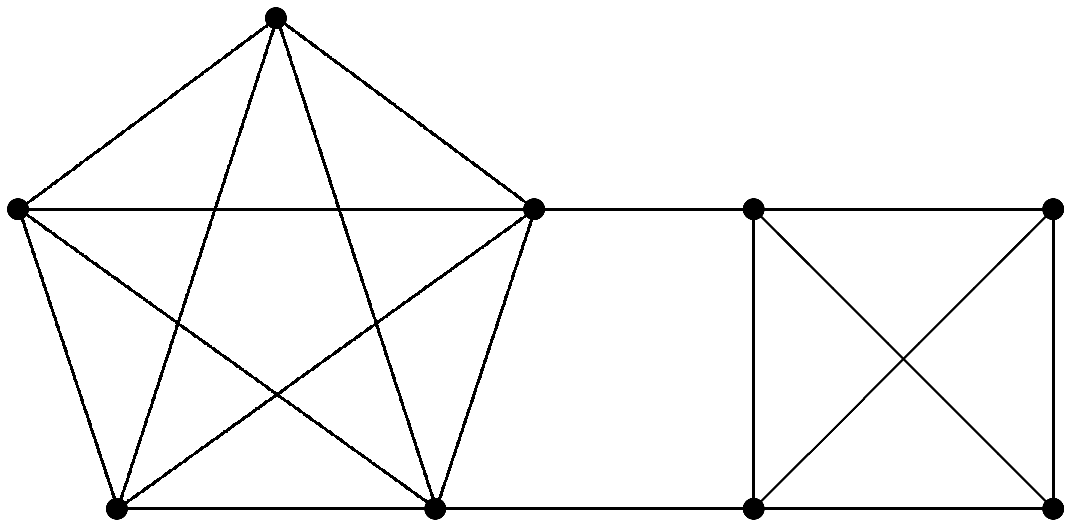
- 

- $|V(G)| = 8, |E(G)| = 18, k = 2$ and $|E(G)| - 2(|V(G)| - 1) = 4$.
- $E_2(G) = E(K_5)$, (by inspection, or proof postponed).

Matroids as a Generalization of Graphs

- A matroid M consists of a finite set $E = E(M)$ and a collection $\mathcal{I}(M)$ of independent subsets of E, satisfying these axioms:
(I1) $\emptyset$ is independent.
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- Rank of a subset X: $r(X) =$ cardinality of a maximal independent subset in X.

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- Rank $r_{M(G)}(X) = |V(X)| - \omega(X)$.

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- Theorem (Edmonds, [J. Res. Nat. Bur. Standards Sect. B **69B**, (1965), 73-77]) Let M be a matroid with $r(M) > 0$. Each of the following holds.
  (i) $\tau(M) \geq k$ if and only if $\forall X \subseteq E(M)$, $|E(M) - X| \geq k(r(M) - r(X))$.
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- Suppose that $\tau(M) < k$. A $(\tau \geq k)$-extension of M is a matroid M' that contains M as a restriction with $\tau(M') \geq k$. What is the minimum $|E(M')| - |E(M)|$ among all $(\tau \geq k)$-extension of M?

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- $d(X) \geq 1$, equality holds iff X is independent ($G[X]$ is a forest).

Matroid and Graph Contractions

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$$r_{M(G/X)}(Y) = |V(X \cup Y)| - \omega(X \cup Y) - |V(X)| + \omega(X).$$

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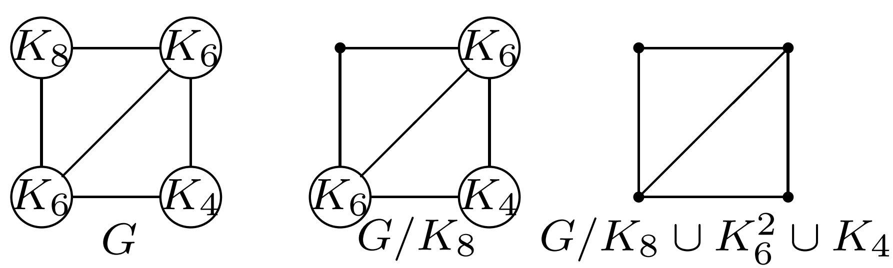

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- Useful Facts ([Discrete Appl. Math. 40 (1992) 285-302]) Each of the following holds.
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- Theorem (Edmonds, fractional form, [DAM (1992)]) For integers $p \geq q > 0$,
  (i) $\tau(M) \geq \frac{p}{q}$ iff M has p bases such that every element of M is in at most q of them.
  (ii) $\gamma \leq \frac{p}{q}$ iff M has p bases such that every element of M is in at least q of them.

Characterizations

- Theorem (Catlin, Grossman, Hobbs & HJL, [Discrete Appl. Math. 40 (1992) 285-302]) The following are equivalent.
  (i) $\eta(M) = d(M)$.
  (ii) $\gamma(M) = d(M)$.
  (iii) $\eta(M) = \gamma(M)$.
  (iv) $\eta(M) = \frac{p}{q}$, M has p bases such that each element is in exactly q of them.
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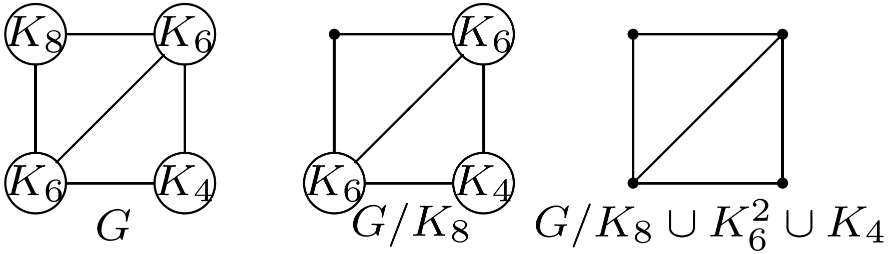

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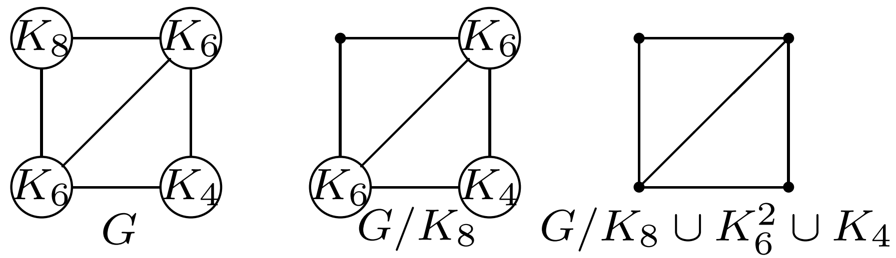

- Each of K_8, K_6, K_4 is η-maximal.

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- (i) There exist an integer $m > 0$, and an m-tuple $(l_1, l_2, ..., l_m)$ of rational numbers such that

$$\eta(M) = l_1 < l_2 < ... < l_m = \gamma(G),$$

and a sequence of subsets

$$J_m \subset ... \subset J_2 \subset J_1 = E(M);$$

such that for each i with $1 \leq i \leq m$, $M|J_i$ is an η-maximal restriction of M with $\eta(M|J_i) = l_i$.

A Decomposition Theorem (ii)

- (ii) The integer m, the sequences of fractions $\eta(M) = l_1 \leq l_2 \leq ... \leq l_m = \gamma(M)$ and subsets $J_m \subset ... \subset J_2 \subset J_1 = E(M)$ are uniquely determined by M.

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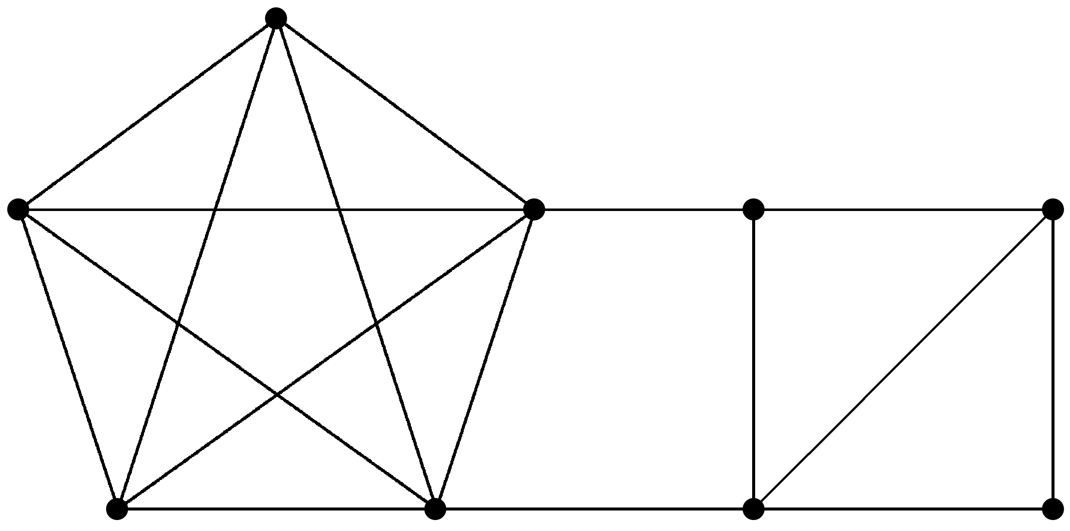
- 

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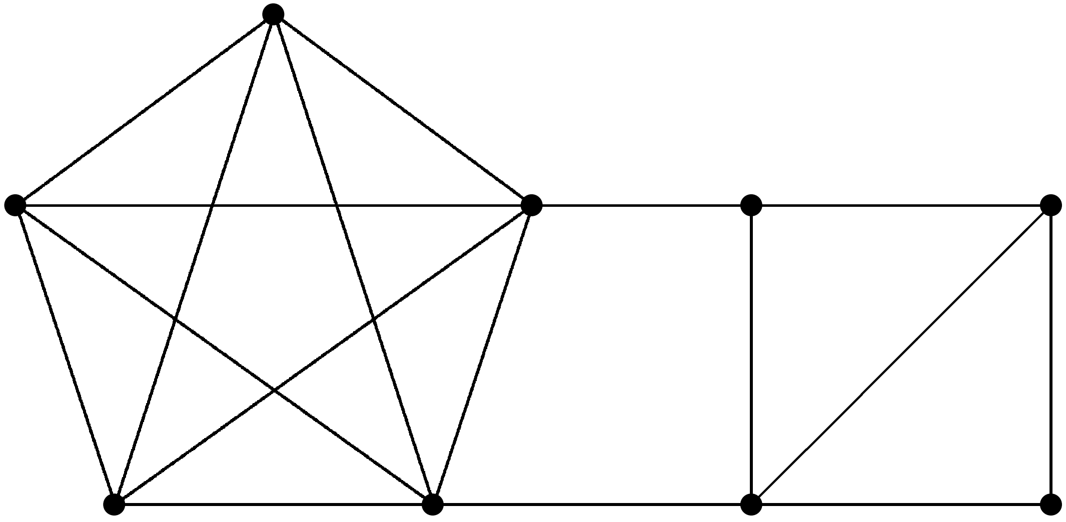
- 
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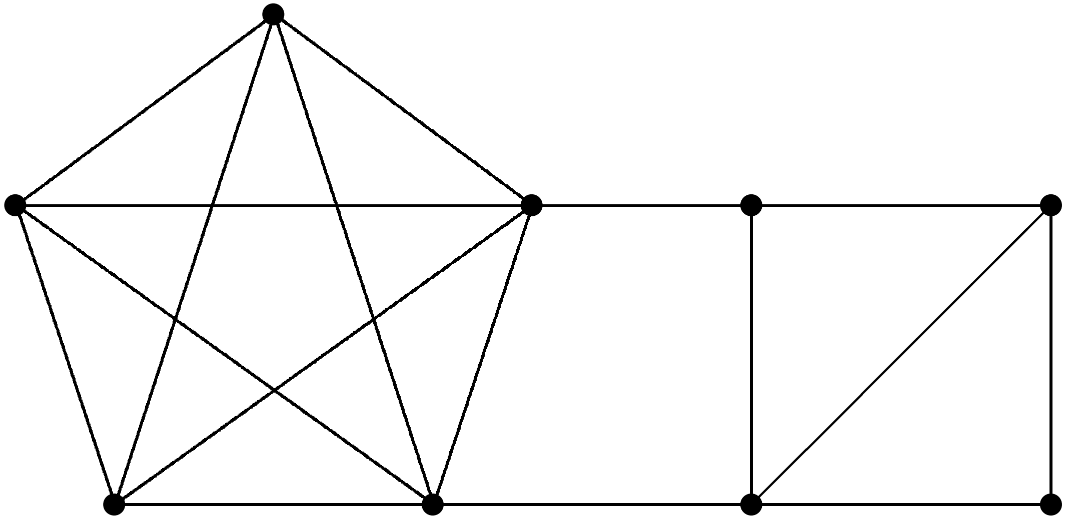
- 
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- $i_2 = \frac{5}{2}$, $i_1 = \frac{7}{4}$.

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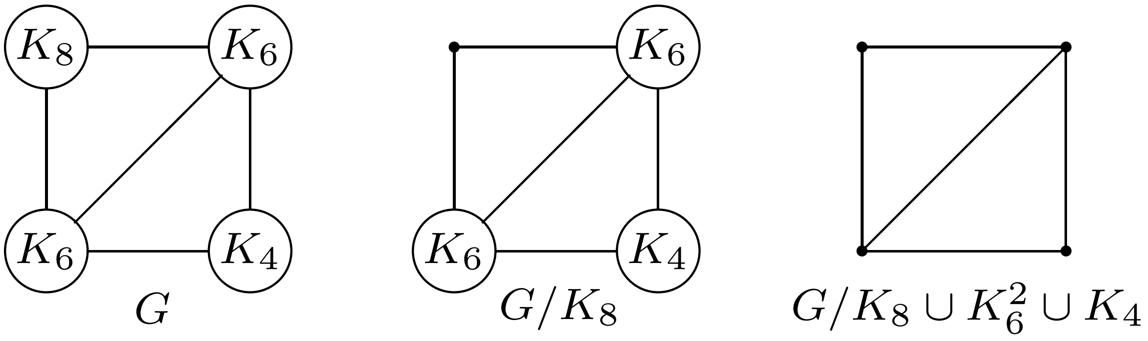
- 

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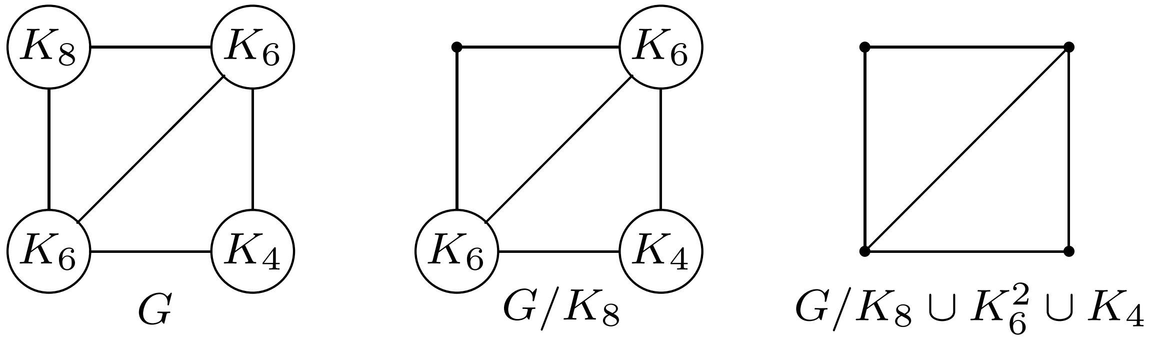
- 

- $m = 4$, $J_4 = E(K_8)$, $J_3 = E(K_6) \cup E(K_6) \cup J_4$, $J_2 = J_3 \cup E(K_4)$.

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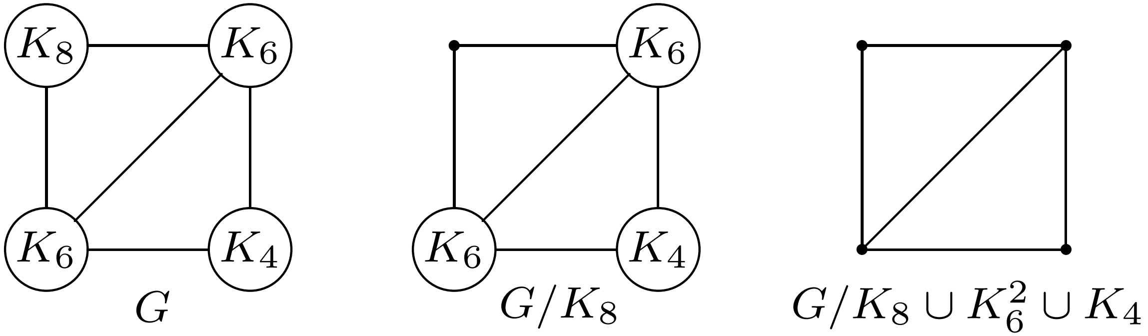
- 

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- $i_4 = 4$, $i_3 = 3$, $i_2 = 2$ and $i_1 = \frac{5}{3}$.

Characterization of Excessive Elements

- If $k > 0$ is an integer such that $k < ß_m = \gamma(M)$, then in the η-spectrum, there exists a smallest i_{j_0} such that $i_{j_0} > k$. J_{j_0} is the η**-maximal subset at level** k of M.

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- Theorem Let $k \geq 2$ be an integer. Let M be a graph with $\tau(M) \geq k$. Then each of the following holds.
  (i) $E_k(M) = E(M)$ if and only if $\eta(M) > k$.
  (ii) In general, if $\eta(M) = k$ and if $m > 1$, then $E_k(G) = J_2$ equals to the η-maximal subset at level k of M.

The Cycle Matroid Case

- Theorem Let $k \geq 2$ be an integer, and G be a connected graph with $\tau(G) \geq k$. Let $\eta(M) = l_1 \leq l_2 \leq ... \leq l_m = \gamma(M)$ and $J_m \subset ... \subset J_2 \subset J_1 = E(M)$ denote the η-spectrum and η-decomposition of $M(G)$, respectively. Then each of the following holds.
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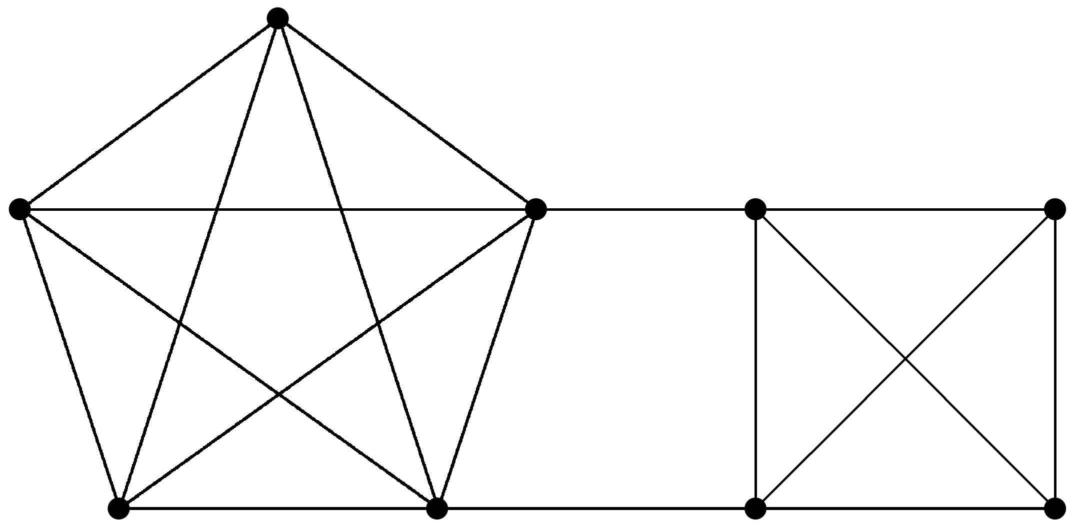

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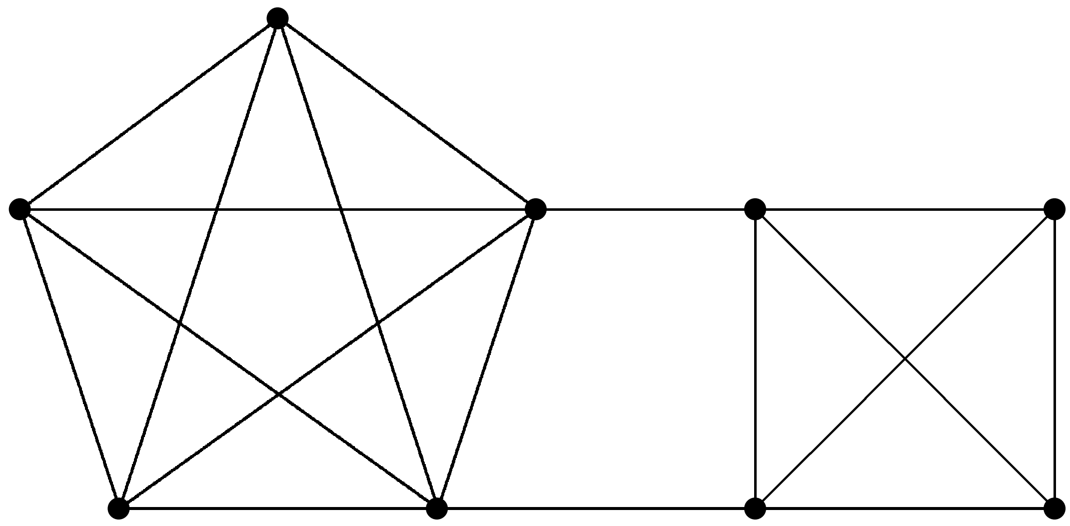
- 

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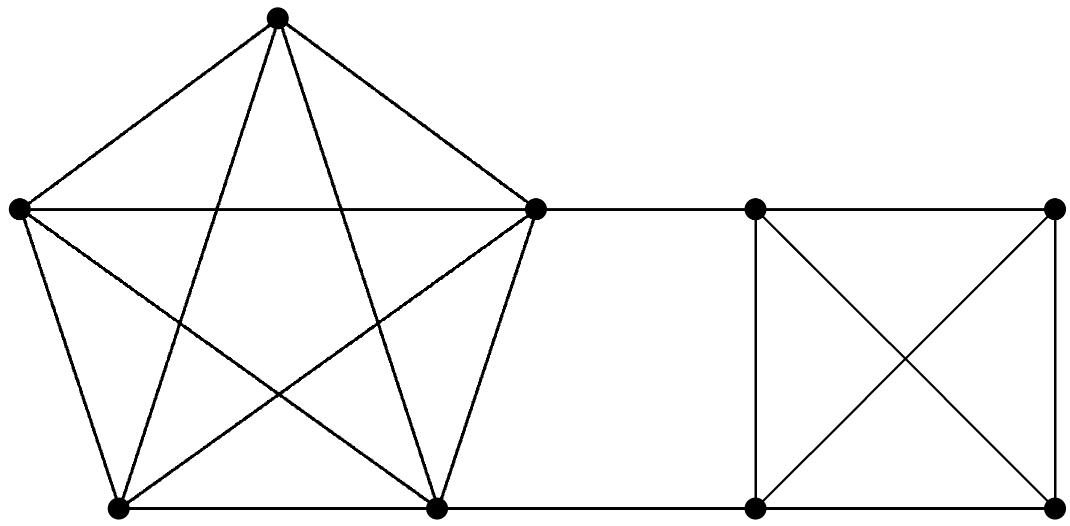
- 
- $m = 2$, $E_2(G) = J_2 = E(K_5)$, $J_1 = E(G)$.
- $i_2 = \frac{5}{2}$, $i_1 = 2$.

What Must Be Added to Have k Disjoint Bases?

- Let G be a graph, and let $F(G,k)$ denote the minimum number of additional edges that must be added to G to result in a graph G' with $\tau(G') \geq k$.

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- If $\tau(M) \geq k$, then $M' = M$.

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- $F(M,k) = \min\{|E(M')| - |E(M)| : M'$ is a $(\tau \geq k)$-extension of $M\}$.

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- Theorem Let M be a matroid and let $k > 0$ be an integer. Each of the following holds.
  (i) $\eta(M) \geq k$ if and only if $F(M,k) = 0$.
  (ii) If $\gamma(M) \leq k$, then

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- Theorem Let G be a graph. If (edge-arboricity) $a_1(G) \le k$, then $F(G,k) = k(|V(G)| - \omega(G)) - |E(G)|$.

Application: The Graph Case

- Theorem (Haas, Theorem 1 of [Ars Combinatoria, 63 (2002), 129-137]) The following are equivalent for a graph G, integers $k > 0$ and $l > 0$.
  (i) $E(G)| = k(|V(G)| - 1) - l$ and for subgraphs H of G with at least 2 vertices, $|E(H)| \leq k(|V(H)| - 1)$.
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- Proof: Either (ii) or $|E(H)| \leq k(|V(H)| - 1)$ in (i) implies $\gamma(M(G)) \leq k$. Hence by our theorem, $l = F(G, k) = k(|V(G)| - 1) - |E(G)|$.

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- Theorem For integer $k > 0$, let M be a matroid with $\tau(M) \le k$ and let $i(k)$ denote the smallest i_j in $\eta(M) = l_1 \le l_2 \le ... \le l_m = \gamma(M)$ such that $i(k) \ge k$. Then
  (i) $F(M, k) = k(r(M) - r(J_{i(k)}) - |E(M) - J_{i(k)}|$.
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- Theorem (D. Liu, H.-J. Lai and Z.-H. Chen, Theorems 3.4 and 3.10 of [Ars Combinatoria, 93 (2009), 113-127]) Let G be a connected graph with $\tau(M(G)) \leq k$ and let $i(k)$ denote the smallest i_j in the spectrum of $M(G)$ such that $i(k) \geq k$. Then
  (i) $F(G, k) =$
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(ii) $F(G, k) = \max_{Y \subseteq E(G)} \{k[\omega(G - Y) - 1] - |Y|\}$.
- Proof: Apply the theorem to cycle matroids.

Thank you!