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Complete family reduction and spanning connectivity in line graphs

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ABSTRACT

Complete families of connected graphs, introduced by Catlin in the 1980s, have been known useful in the study of certain graphical properties that are closed under taking contractions. We show that given any complete family $\mathcal{C}$ of connected graphs such that $\mathcal{C}$ contains graphs with sufficiently many edge-disjoint spanning trees, for any real number a and b with $0 < a < 1$, there exists a finite obstacle family $\mathcal{F} = \mathcal{F}(a, b, \mathcal{C})$ such that for any simple graph G on n vertices satisfying the Ore-type degree condition

$$\min\{d_G(u) + d_G(v) : u, v \in V(G) \text{ and } uv \notin E(G)\} \geq an + b,$$

either $G \in \mathcal{C}$ or G can be contracted to a member in $\mathcal{F}$. This result is applied to the study of spanning connectivity of line graphs. The spanning connectivity is the largest integer s such that for any k with $0 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning subgraph H consisting of k internally disjoint (u, v)-paths. Z. Ryjáček and P. Vrána in [J. Graph Theory, 66 (2011) 152-173] prove that a fascinating conjecture of Thomassen on hamiltonian line graphs is equivalent to that every essentially 4-edge-connected graph has a 2-spanning-connected line graph. We prove that for any essentially 3-edge-connected graph G and any positive integer s, if G satisfies an Ore degree condition lower bounded by an arbitrary linear function in the number of vertices, then $L(G)$ is s-spanning-connected with only finitely many contraction obstacles. When $s = 3$, we determine a finite graph family $\mathcal{J}'(n)$ such that for every simple graph G on $n \geq 156$ vertices with $\kappa(L(G)) \geq 3$ and satisfying $d(u) + d(v) \geq \frac{2(n-6)}{5}$ for any pair of nonadjacent vertices u and v, we have either $\kappa^*(L(G)) \geq 3$ or G is contractible to a member in $\mathcal{J}'(n)$.

1. The problem

Graphs under considerations in this paper are loopless but with possible multiple edges, with undefined notation and terminologies following those in Bondy and Murty [2]. Unless otherwise stated, a graph is assumed to be non-null (with at least one vertex). We shall use $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ to denote the minimum degree, the maximum degree, the

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connectivity and the edge connectivity of a graph G, respectively. For vertices $u, v \in V(G)$ in a graph G, a path (or a trail, respectively) of G with termini u and v will be referred to as an (u, v)-path (or an (u, v)-trail, respectively).

1.1. Complete graph families with strengthened Ore type degree conditions

For a graph G and an edge subset $W \subseteq E(G)$, the **contraction** G/W is the graph formed from G by contracting edges in W with resulting loops deleted. We define $G/\emptyset = G$ and use G/e for $G/\{e\}$. When H is a subgraph of G, then we often use G/H for $G/E(H)$. If H is connected, then the vertex in G/H onto which H is contracted is denoted by v_H, and $PI_G(v_H) = G[V(H)]$ is the **preimage** of v_H in G. (Thus even the connected subgraph H may not be an induced subgraph of G, $PI_G(v_H)$ is an induced subgraph of G spanned by H.) Following [4,7], complete families of connected graphs are defined as follows.

Definition 1.1. ([4,7]) A family $\mathcal{C}$ of nonempty connected graphs is a **complete family** if each of the following holds.
(C1) $K_1 \in \mathcal{C}$.
(C2) If $G \in \mathcal{C}$ and if $e \in E(G)$, then $G/e \in \mathcal{C}$.
(C3) If H is subgraph of s graph G and if $H, G/H \in \mathcal{C}$, then $G \in \mathcal{C}$.

By definition, every spanning tree of the singleton graph K_1 has an empty edge set, and so for any integer $s \geq 0$, K_1 has at least s edge-disjoint spanning trees. Consequently, we in the discussion will use the convention to view $\kappa'(K_1) \geq s$. There are quite a few commonly studied graphical properties which would define complete graph families, as shown in the example below, each of which can be routinely verified.

Example 1.2. Let $s \geq 0$ be an integer.
(i) (Proposition 2.3 of [17]) Let $\mathcal{T}_s = \{G : |V(G)| \geq 1$ and G has at least s edge-disjoint spanning trees$\}$. Then $\mathcal{T}_s$ is a complete family.
(ii) (Example 4 of [15]) Let $\mathcal{E}(s) = \{G : \kappa'(G) \geq s\}$. Then $\mathcal{E}(s)$ is a complete family.
(iii) The family of all connected planar graphs is a complete family.

There have been many studies using degree conditions to investigate certain graphical properties. This motivates our current research. One of the goal of the current research is to seek a unified approach using degree conditions to study graphical properties that are related to complete families. For a graph G, define

$$m_2(G) = \min\{\max\{d_G(u), d_G(v)\} : u, v \in V(G) \text{ and } uv \notin E(G)\}. \tag{1}$$

Thus if for any $u, v \in V(G)$ with $uv \notin E(G)$, we have $d_G(u) + d_G(v) \geq f(n)$, then we must have $m_2(G) \geq f(n)/2$. In this sense, $m_2(G)$ can be viewed as a strengthened Ore type degree condition of a graph G. For any real numbers a and b, and for a simple graph G of order n, we define

$$Z_{a,b}(G) = \{v \in V(G) : d_G(v) < an + b\}. \tag{2}$$

Let $\mathcal{T}_s$ denote the graph family as defined in Example 1.2. One of our main results is the following, which indicates that for any given complete family $\mathcal{C}$, any simple graph G with $m_2(G)$ lower bounded by any linear function of the order of G must be in $\mathcal{C}$ with finitely many contraction obstacles.

Theorem 1.3. *Let s be a positive integer, a and b be real numbers with $0 < a < 1$, $\mathcal{C}$ be a complete family such that $\mathcal{T}_{s+1} \subseteq \mathcal{C}$, and let G be a simple graph of order n satisfying*

$$m_2(G) \geq an + b. \tag{3}$$

If for some integer M_0, we have $|Z_{a,b}(G)| \leq M_0$, then there exists a finite graph family $\mathcal{F}$ such that exactly one of the following holds.
(i) $G \in \mathcal{C}$.
(ii) G is contractible to a member in $\mathcal{F}$.

We shall show that this result implies several former results using degree conditions. In particular, we also present an application of Theorem 1.3 to some studies related to Thomassen's fascinating conjecture that every 4-connected line graph is hamiltonian.

1.2. Thomassen's conjecture on hamiltonian line graphs and spanning connectivity

Let $s \geq 0$ be an integer. If a subgraph H of G consists of s internally disjoint (u, v)-paths (or edge-disjoint (u, v)-trails, respectively) $\{Q_1, Q_2, ..., Q_s\}$, then H is called an $(s; u, v)$**-path-system** (or an $(s; u, v)$**-trail-system**, respectively) of G.

We also use $\{Q_1, Q_2, ..., Q_s\}$ to denote the collection of the corresponding paths (or trails) while using H to denote the subgraph. If $V(H) = V(G)$, then H is a **spanning** $(s; u, v)$**-path-system** (or a **spanning** $(s; u, v)$**-trail-system**, respectively).

Following the Menger Theorems (as seen in Theorems 9.1 and 9.7 of [2]), we define a graph G to be k**-connected** (or k-edge-connected, respectively) if for any pair of distinct vertices u and v, G contains a $(k; u, v)$-path-system (or a $(k; u, v)$-trail-system, respectively). It follows that the connectivity $\kappa(G)$ of a graph G equals the maximum number k such that for every pair of distinct vertices u and v, G has a $(k; u, v)$-path-system. Likewise, the edge-connectivity $\kappa'(G)$ of G equals the maximum number k such that for every pair of distinct vertices u and v, G has a $(k; u, v)$-trail-system. It is natural to consider the research problem when the $(s; u, v)$-path-systems (or the $(s; u, v)$-trail-systems) are required to be spanning subgraphs of G, which leads to the following definition. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for any integer k with $0 \le k \le s$ and for any $u, v \in V(G)$ with $u \ne v$, G has a spanning $(k; u, v)$-path-system. A graph G is s**-spanning connected** if $\kappa^*(G) \ge s$. It follows from definitions that

$$\kappa(G) \ge \kappa^*(G), \text{ for any graph } G. \tag{4}$$

Hsu in [11] initiated the study of spanning connectivity of a network as a way to evaluate communication performance of interconnected networks. Many former studies on spanning connectivity of graphs have been focused on determining the spanning connectivity of special graph families, as summarized in the monograph [12] by Hsu and Lin. The study of spanning connectivity is also motivated by the classical hamiltonian graph problem. By definition, a graph G is hamiltonian (that is, with a spanning cycle) if and only if for any $u, v \in V(G)$, G has a spanning $(2; u, v)$-path-system. As G is Hamilton-connected if and only if $u, v \in V(G)$, G has a spanning $(1; u, v)$-path-system, it follows by definition that G is Hamilton-connected if and only if $\kappa^*(G) \ge 2$. As it is well-known that every Hamilton-connected graph must be 3-connected, we conclude that

$$\text{if a graph } G \text{ satisfies } \kappa^*(G) > 0, \text{ then } \kappa^*(G) \ge 2 \text{ and } \kappa(G) \ge 3. \tag{5}$$

Let $L(G)$ denote the **line graph** of a graph G, which is a simple graph with vertex set $E(G)$, and with edge set $E(L(G)) = \{e'e'' : e', e'' \in E(G)$ and e', e'' are adjacent in $G\}$. A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is a **claw-free** graph. Beineke [1] and Robertson (Page 74 of [10]) showed that line graphs are claw-free graphs. By several ingenious closure concepts developed by Ryjáček [26] and by Ryjáček and Vrána [27], the fascinating conjecture on hamiltonian line graph posed by Thomassen is shown to be equivalent to each of the following.

Conjecture 1.4. *Let G be a graph and let Γ be a claw-free graph.*
(i) (Thomassen [29] and, Kučzel and Xiong [14]) If $\kappa(L(G)) \ge 4$, then $\kappa^*(L(G)) \ge 2$.
(ii) (Matthews and Sumner [24], and Ryjáček and Vrána [27]) If $\kappa(\Gamma) \ge 4$, then $\kappa^*(\Gamma) \ge 2$.

An edge cut X of a graph G is **essential** if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is **essentially** k**-edge-connected** if G is connected and does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G)$ be the largest integer k such that G is essentially k-edge-connected, if at least one such k exists, or $ess'(G) = |E(G)| - 1$ if for any integer k, G does not have an essential edge cut of size k. By the definitions of line graphs and essential edge-connectivity and by (5), for a connected graph G,

$$\kappa(L(G)) = ess'(G), \text{ and if } \kappa^*(L(G)) \ge k > 0, \text{ then } ess'(G) \ge \max\{3, k\}. \tag{6}$$

Another purpose of this research is to investigate spanning connectivity of line graphs. By applying Theorem 1.3, we prove that for any graph G and any positive integer s, if G satisfies a generalized Ore degree condition lower bounded by an arbitrary linear function in the number of vertices, then $\kappa^*(L(G)) \ge s$ with only finitely many contraction obstacles. The main results of this paper are the following.

Theorem 1.5. *Let s be a positive integer and a and b be real numbers with $0 < a < 1$. There exists a family of finitely many graphs $\mathcal{F}_1(a, b, s)$ such that if G is a simple graph on n vertices with $ess'(G) \ge \max\{3, s\}$ and satisfies (3), then one of the following must hold.*
(i) $\kappa^*(L(G)) \ge s$.
(ii) G is contractible to a member in $\mathcal{F}_1(a, b, s)$.

The next example indicates that the graph $K_{3,3}$ plays an important role in the study of 3-spanning-connected line graphs, just like that the Petersen graph is, in some sense, the major obstacle for 2-spanning-connected line graphs.

Example 1.6. Let J be a graph isomorphic to $K_{3,3}$ with vertex set $V(J) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ such that each of $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ is a stable set of J. For any integer $n \ge 16$, let $\mathcal{J}(n)$ denote the family of graphs such that each $J(n) \in \mathcal{J}(n)$ is obtained from J by blowing up each vertex $w \in V(J) - \{v_1\}$ into a 3-edge-connected graph $K(w)$ with order $\lfloor \frac{n-1}{5} \rfloor \le |V(K(w))| \le \lceil \frac{n-1}{5} \rceil$ such that $K(w)$ is either a complete graph or a complete graph minus an edge whose vertices are incident with edges in J and such that $\sum_{w \in V(J)-\{v_1\}} |V(K(w))| = n - 1$. It will be shown in Lemma 5.4 of Section 5 that for any $J(n) \in \mathcal{J}(n)$, $L(J(n))$ is not spanning 3-connected.

Theorem 1.7. *If G is a simple graph on $n \geq 156$ vertices with $ess'(G) \geq 3$ and*

$$m_2(G) \geq \frac{n-6}{5}, \tag{7}$$

then one of the following must hold.
(i) $\kappa^*(L(G)) \geq 3$.
(ii) G is contractible to a member in $\mathcal{J}(n)$.

As commented above, in a graph G, we have $2m_2(G) \geq d_G(u) + d_G(v)$, for any $u, v \in V(G)$ with $uv \notin E(G)$. Theorems 1.5 and 1.7 can be routinely stated in terms of Ore degree conditions, with the corresponding contraction obstacle family being subfamilies $\mathcal{F}_1'$ and $\mathcal{J}'(n)$ of those in Theorems 1.5 and 1.7, respectively.

In the next section, we prove Theorem 1.3, with applications to derive some former results. The rest of the paper will be denoted to the application of Theorem 1.3 to spanning connectivity of line graphs. Some useful tools will be displayed and developed in Section 3, and Theorems 1.5 and 1.7 will be proved in the last few sections.

2. Proof of Theorem 1.3

The purpose of this section is to obtain a reduction of graphs with respect to a given complete family. Some of the properties of graphs, such as graphs with at least s edge-disjoint spanning trees in [3,17,19] can be extended to arbitrary complete families of graphs. For a complete graph family $\mathcal{C}$ and a graph H, if $H \in \mathcal{C}$, then H is also called a $\mathcal{C}$**-graph**. For a graph H and an edge set X with $V(X) \subseteq V(H)$, define $H + X$ to be the graph with vertex set $V(H)$ and edge set $E(H) \cup X$. When $X = \{e\}$, we use $H + e$ for $H + \{e\}$.

Proposition 2.1. *Let $\mathcal{C}$ be a complete family of graphs, and let G be a graph. Each of the following holds.*
(i) If a graph $H \in \mathcal{C}$ and if $u, v \in V(H)$ and $e = uv$, then $H + e \in \mathcal{C}$.
(ii) If H_1, H_2 are $\mathcal{C}$-subgraphs of G with $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2 \in \mathcal{C}$.
(iii) If T is a spanning connected subgraph of G and for any edge $e \in E(T)$, G has a $\mathcal{C}$-subgraph H_e with $e \in E(H_e)$, then $G \in \mathcal{C}$.

Proof. Let $u, v \in V(H)$ and $e = uv$. If $e \in E(H)$, then $H + e = H \in \mathcal{C}$. Assume that $e \notin E(H)$, then as $H \in \mathcal{C}$ and $H + e/H = K_1 \in \mathcal{C}$, it follows by Definition 1.1(C3) that $H + e \in \mathcal{C}$. This justifies (i).

To prove (ii), let $H = H_1 \cup H_2$. By (i), $H[V(H_1)] \in \mathcal{C}$. By (C2), $H/H_2 = H[V(H_1)]/H_2 \in \mathcal{C}$. As $H_2 \in \mathcal{C}$, it follows by Definition 1.1(C3) that $H_1 \cup H_2 \in \mathcal{C}$.

Now let T be a spanning connected subgraph of G satisfying the hypothesis of (iii). We argue by induction on $|V(G)| = |V(T)|$. As $K_1 \in \mathcal{C}$ and by the assumption of T, we conclude that (iii) holds if $|V(G)| \in \{1, 2\}$. Now assume that $|V(G)| \geq 3$ and Proposition 2.1 holds for smaller values of $|V(G)|$. Then $|V(T)| = |V(G)| \geq 3$. As T is connected, there must be an edge $e \in E(T)$, and so G contains a $\mathcal{C}$-subgraph H_e with $e \in E(H_e)$. Let $G' = G/H_e$ and $T' = T/(T \cap H_e)$. Then T' is also a spanning connected subgraph of G'. For any edge $e' \in E(T') \subseteq E(T)$, by assumption, G has a $\mathcal{C}$-subgraph $H_{e'}$. By (C2), $L_{e'} := (H_{e'} \cup H_e)/H_e \in \mathcal{C}$. Hence for any edge $e' \in E(T')$, G' has a $\mathcal{C}$-subgraph $L_{e'}$ with $e' \in E(L_{e'})$. It follows by induction that $G' \in \mathcal{C}$. As $H_e \in \mathcal{C}$ and $G' = G/H_e \in \mathcal{C}$, by Definition 1.1(C3), we conclude that $G \in \mathcal{C}$. □

By Proposition 2.1(ii), every graph G has a unique collection of maximal $\mathcal{C}$-subgraphs $H_1, H_2, ..., H_c$. Define $G/(H_1 \cup H_2 \cup ... \cup H_c)$ to be the $\mathcal{C}$**-reduction** of G. If G equals its own reduction, then G is $\mathcal{C}$**-reduced**. Lemma 2.2 is implied by a classic result of Nash-Williams in [25]. An explicit proof can be found in Theorem 2.4 of [31].

Lemma 2.2. *(Nash-Williams [25]) Let $s > 0$ be an integer. For any graph G with $n = |V(G)|$, if $|E(G)| \geq s(n-1)$, then G contains a nontrivial subgraph H with $H \in \mathcal{T}_s$.*

By definition and by Lemma 2.2, we have the following observations.

Observation 2.3. *Let $\mathcal{C}$ be a complete family.*
(i) A graph G is $\mathcal{C}$-reduced if and only if G does not have a nontrivial $\mathcal{C}$-subgraph.
(ii) If $\mathcal{T}$ is also a complete family satisfying $\mathcal{T} \subseteq \mathcal{C}$, then any $\mathcal{C}$-reduced graph G is also $\mathcal{T}$-reduced.
(iii) Let $s \geq 2$ be an integer. If G is $\mathcal{T}_s$-reduced, then $|E(G)| \leq (s+1)|V(G)| - s - 1$.
(iv) Every nontrivial $\mathcal{C}$-reduced graph is not in $\mathcal{C}$.

Throughout the discussions, we continue using (2) to define $Z_{a,b}(G)$. This if G satisfies (3), then $G[Z_{a,b}]$ must be a complete subgraph of G. The following lemma is also useful.

Lemma 2.4. *(Liu et al. Lemma 3.1 of [22]) Let G be a simple graph and $W \subseteq V(G)$ be a subset such that $d = \min\{d_G(w) : w \in W\}$. If $|\partial_G(W)| < d$, then $|W| \geq d + 1$.*

We will prove a slightly stronger version of Theorem 1.3, stated as Theorem 2.5 below. This indicates that the finite obstacle family consists of only nontrivial $\mathcal{C}$-reduced graphs. As $K_n \in \mathcal{T}_{s+1}$ when $n \geq 2s+2$, we have $M_0 \leq 2s+1$ for the constant M_0 in the next theorem.

Theorem 2.5. *Let $s \geq 1$ be an integer, a and b be real numbers with $0 < a < 1$, $\mathcal{C}$ be a complete family such that $\mathcal{T}_{s+1} \subseteq \mathcal{C}$, and let G be a simple graph of order n satisfying (3). If for some integer M_0, we have $|Z_{a,b}(G)| \leq M_0$, then there exists a finite graph family $\mathcal{F} = \mathcal{F}(a, b, s, \mathcal{C})$ consisting of nontrivial $\mathcal{C}$-reduced graphs such that exactly one of the following holds.*
(i) $G \in \mathcal{C}$.
(ii) G is contractible to a member in $\mathcal{F}$.

Proof. Let c and N be integers satisfying

$$c = \max\{2 - |b|, 2s+2\},$$
$$N > \max\left\{\frac{2|b|}{a}, \frac{c+1-|b|}{a}, \frac{(1+a+aM_0)(c+1)-2as}{a(c-2s-1)}\right\} \tag{8}$$

We shall argue by contradiction to prove the theorem with $\mathcal{F}$ being the family of all $\mathcal{C}$-reduced graphs with order at least 2 and at most N. As $\mathcal{T}_{s+1} \subseteq \mathcal{C}$, by Observation 2.3, graphs in $\mathcal{F}$ are also $\mathcal{T}_{s+1}$-reduced. Hence every graph in $\mathcal{F}$ does not contain an $(s+1)K_2$ as a subgraph, which implies that $\mathcal{F}$ is a graph family of finitely many graphs. Assume that there exists a graph G with n vertices satisfying (3) and $G \notin \mathcal{C}$. Let G' denote the $\mathcal{C}$-reduction of G with $n' = |V(G')|$. We have the following claims.

Claim 1. *Each of the following holds.*
(i) $G' \notin \mathcal{F}$, $n \geq n' > N$ and $an + b > 0$.
(ii) $|E(G')| \leq (s+1)|V(G')| - s$.

Since G' is $\mathcal{C}$-reduced and $\mathcal{T}_{s+1} \subseteq \mathcal{C}$, it follows by Observation 2.3 that Claim 1(ii) must hold. It remains to prove Claim 1(i). Since G is a counterexample, $G' \notin \mathcal{F}$. If $n' = 1$, then as $K_1 \in \mathcal{C}$, it follows by the assumption that $\mathcal{C}$ is a complete family and Definition 1.1(C3) that $G \in \mathcal{C}$, whence G is not a counterexample. Hence $n' > 1$. If $n' \leq N$, then $G' \in \mathcal{F}$. Thus $n \geq n' > N$. By (8), $n > N \geq \frac{|b|}{a}$, and so $an + b > 0$. This completes the proof for Claim 1(i).

For the value of c as defined above, we define

$$X_c = \cup_{i \leq c} D_i(G') \text{ and } X'_c = \{v' \in X_c : PI_G(v') \cap Z_{a,b} \neq \emptyset\}. \tag{9}$$

Claim 2. *Each of the following holds.*
(i) $|X'_c| \leq M_0$.
(ii) $|X_c - X'_c| \leq \frac{1}{a} + 1$.
(iii) $n' - |X_c| \leq \frac{2(s+1)n'-2s}{c+1}$.

As Claim 2(i) follows from the fact that $|X'_c| \leq |Z_{a,b}| \leq M_0$, it remains to prove Claim 2(ii) and (iii). For any vertex $x \in X_c$, let $H_x = PI_G(x)$ denote the preimage of x in G. As $x \notin X'_c$, every vertex $v \in V(H_x)$ is not in $Z_{a,b}$, and so by (3) and by (8), we have $d_G(v) \geq an + b > c$. By Lemma 2.4, $|V(H_x)| \geq an + b + 1$, and so

$$n = |V(G)| \geq \sum_{x \in X_c - X'_c} |V(H_x)| \geq (an + b + 1)|X_c - X'_c|.$$

By (8), we have $n > \frac{2|b|}{a} \geq \frac{|b|-b}{a}$, and so $\frac{|b|+1}{an+b+1} < 1$. It follows by algebraic manipulations that

$$|X_c - X'_c| \leq \frac{n}{an+b+1} = \frac{an+b+1-(b+1)}{a(an+b+1)} \leq \frac{1}{a} + \frac{|b|+1}{an+b+1} \leq \frac{1}{a} + 1.$$

This justifies Claim 2(ii).

By (9), for any $z \in V(G') - X_c$, we have $d_{G'}(z) \geq c+1$. It follows by Claim 1(ii) that

$$(c+1)|V(G') - X_c| \leq \sum_{z \in V(G')} d_{G'}(z) = 2|E(G')| \leq 2(s+1)n' - 2s,$$

which implies that Claim 2(iii).

We are going to find a contradiction. By Claim 2(i) and (ii), we have $|X_c| \leq \frac{1}{a} + 1 + M_0$. This, together with Claim 2(iii), implies that

$$\frac{1+aM_0+a}{a} \geq |X_c| = n' - \frac{2(s+1)n'-2s}{c+1} = n'\left(1-\frac{2(s+1)}{c+1}\right) + \frac{2s}{c+1}.$$

With algebraic manipulations, and by (8), we conclude that

$$n' \leq \left(\frac{1+aM_0+a}{a} - \frac{2s}{c+1}\right)\left(1-\frac{2(s+1)}{c+1}\right)^{-1} = \frac{(1+a+aM_0)(c+1)-2as}{a(c-2s-1)} \leq N,$$

leading to a contradiction that $n > N$. This completes the proof of Theorem 2.5. □

Theorem 2.5 can be applied to obtain a former result on strongly $\mathbb{Z}_{2p+1}$-connected graphs. As this paper is not focused on the study of group connectivity of graphs, we refer to [18] for the related definitions of strongly $\mathbb{Z}_{2p+1}$-connected graphs. Other applications of Theorem 2.5 will be discussed in Subsection 3.4 as well as in the last two sections of this paper.

Theorem 2.6. *(Theorem 1.6 of [18]) Let G be a simple graph on n vertices. For any integer $p > 0$ and for any real numbers a and b with $0 < a < 1$, there exist an integer $N = N(a, p)$ and a finite family $\mathcal{F}_1(a, p)$ of graphs that are not strongly $\mathbb{Z}_{2p+1}$-connected such that if $n \geq N$ and if for every pair of nonadjacent vertices u and v in G, $d_G(u) + d_G(v) \geq an + b$, then either G is strongly $\mathbb{Z}_{2p+1}$-connected or G can be contracted to a member in $\mathcal{F}_1(a, p)$.*

Proof. Let $\mathcal{C}$ denote the family of all graphs that are strongly $\mathbb{Z}_{2p+1}$-connected. By Theorem 1.12 of [23], $\mathcal{T}_{s+1} \subseteq \mathcal{C}$ with $s = 6p - 1$. In Proposition 2.1 and Lemma 2.2 of [18], it is known that $\mathcal{C}$ is a complete family and no $\mathcal{C}$-reduced graph contains a complete graph of order at least $4p + 1$. Therefore, Theorem 2.6 can be obtained by applying Theorem 2.5 with $M_0 = 4p + 1$. □

3. Mechanisms for spanning connectivity of line graphs

3.1. Trail systems in G and spanning connectivity of $L(G)$

Harary and Nash-William [10] characterized graphs whose line graphs are hamiltonian. Chen et al. in [9] extended this characterization by displaying a relationship between spanning connectivity in $L(G)$ and certain type of dominating trail systems in G. For an edge subset $X \subseteq E(G)$ of a graph G, we also use X to denote the subgraph $G[X]$ induced by X. Thus $V(X)$ is a subset of $V(G)$ consisting of all vertices incident with at least one edge in X. In particular, if $e = uv \in E(G)$, we define $V(e) = \{u, v\}$. Let

$$T = v_0, e_1, v_1, e_2, \cdots, e_k, v_k \tag{10}$$

denote a trail such that for each i with $1 \leq i \leq k$, $V(e_i) = \{v_{i-1}, v_i\}$, and such that if $1 \leq i < j \leq k$, then $e_i \neq e_j$. A trial T (with the notation in (10)) is **open** (or **closed**, respectively) if $v_0 \neq v_k$ (or $v_0 = v_k$, respectively). We define the **internal vertices** of the trail in (10) to be the multiset $T^o = \{v_1, v_2, \cdots, v_{k-1}\}$ if T is open, and to be $V(T)$ if T is closed. As in an open trail, vertices may occur more than once, it is also possible for the end vertices v_0 or v_k in (10) to be internal. A trail T of G is **dominating** if T^o is a vertex cover of G. That is, $G - T^o$ is edgeless.

Let $e', e'' \in E(G)$ be two edges of G. A trail T of G is an **(e', e'')-trail** of G if the two end edges of T are e' and e'', respectively. As an example, the trail in (10) is an (e_1, e_k)-trail. Two (e', e'')-trails T_1 and T_2 are **internally edge-disjoint** if $E(T_1) \cap E(T_2) = \{e', e''\}$. For a given integer $s \geq 0$, an $(s; e', e'')$-trail system in G is a subgraph J consisting of s internally edge-disjoint (e', e'')-trails $(T_1, T_2, \cdots, T_s)$. A vertex v is an **internal vertex of** J if for some i with $1 \leq i \leq s$, v is an internal vertex of T_i. For an $(s; e', e'')$-trail system J, define

$$\partial_G(J) = \{e \in E(G) - E(J) : e \text{ is incident with an internal vertex of } J\}.$$

An $(s; e', e'')$-trail system J is **dominating** if $E(G) - E(J) = \partial_G(J)$, and is **spanning** if it is dominating with $V(G) = V(J)$. Let $u', u'' \in V(G)$ and $e', e'' \in E(G)$. By definition, every spanning $(k; u', u'')$-trail system is also a dominating (k, u', u'')-trail system; and every spanning $(k; e', e'')$-trail system is also a dominating $(k; e', e'')$-trail system.

Theorem 3.1. *(Chen et al., Theorem 2.1 of [9]) Let G be a graph with $|E(G)| \geq 3$ and let $s \geq 3$ be an integer. Then $\kappa^*(L(G)) \geq s$ if and only if for any edge $e', e'' \in E(G)$, and for each integer k with $1 \leq k \leq s$, G has a dominating $(k; e', e'')$-trail-system.*

To facilitate the discussions, we often use a vertex sequence $T = v_0v_1...v_i...v_j...v_k$ to denote a trail with the understanding that every edge may occur at most once in the trail. To emphasize the end vertices, we also use $T[v_0, v_k]$ to denote this (v_0, v_k)-trail T together with its orientation, and use $T[v_k, v_0] = v_kv_{k-1}...v_j...v_i...v_2v_1v_0$ to denote the trail with the same edge set but with reversed orientation. For indices $1 \leq i < j \leq k$, we define $T[v_i, v_j] := v_iv_{i+1}...v_j$ to be the subtrail of T, and so $T[v_j, v_i] := v_jv_{j-1}...v_i$ denotes that trail formed by reversing the orientation of $T[v_i, v_j]$. Thus when $v_i = v_j$, as subgraphs, $T[v_i, v_j]$ and $T[v_j, v_i]$ represent the same closed trail but with opposite orientations. If $T_1 = T_1[u, v]$,

$T_2 = T_2[v, w]$ are edge-disjoint trails, we adopt the amalgamation notation $T_1[u, v]T_2[v, w]$ to denote an (u, w)-trail that traverses along T_1 to reach v, then traverses along T_2 to stop at w. If T_1 consists of only one edge uv, then we use $uvT_2[v, w]$ for $T_1[u, v]T_2[v, w]$. The trail $T_1[u, v]vw$ is defined similarly. A trail T that starts from a vertex v with e being its last edge is often denoted by a (v, e)-trail. Likewise, a (e, v)-trail is one whose first edge is e and the last vertex is v.

3.2. Supereulerian width of a graph

A graph G is **spanning trailable** if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail.

Notation 3.2. *Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where v_e, $v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$.*

As defined in [20], a graph G is **strongly spanning trailable (SST in short)** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. Since $e = e'$ is possible, SST graphs are both spanning trailable and supereulerian.

As a generalization of supereulerian graphs, the notion of the supereulerian width of a graph is introduced by Li et al. in [19]. Let $s \geq 0$ be an integer. A graph G is **supereulerian with width** s if for any $u, v \in V(G)$, G contains a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \leq k \leq s$. The following former results are useful.

Lemma 3.3. *(Proposition 2.2 of [16]) For a nontrivial graph G, $\kappa^*(L(G)) \geq 2$ if and only if for any pair of edges $e, e' \in E(G)$, G has a dominating (e, e')-trail. In particular, if G is spanning trailable, then $\kappa^*(L(G)) \geq 2$.*

Lemma 3.4. *Let $e' = u'v'$, $e'' = u''v'' \in E(G)$ be two edges, and let Q_1, Q_2 be two edge-disjoint (u', u'')-trails of G. Each of the following holds.*
(i) If $e'' \in E(Q_1)$, then either Q_1 can be viewed as an (u', e'')-trail, or G has an (u', e'')-trail Q_1' and a nontrivial closed trail $Q_1''[u'', u'']$ such that $E(Q_1) = E(Q_1') \cup E(Q_1'')$ and $E(Q_1') \cap E(Q_1'') = \emptyset$.
(ii) If Q_1 is an (u', e'')-trail and $e' \in E(Q_1)$, then either Q_1 is an (e', e'')-trail, or G has an (e', e'')-trail Q_1' and a nontrivial closed trail $Q_1''[u', u']$ such that $E(Q_1) = E(Q_1') \cup E(Q_1'')$ and $E(Q_1') \cap E(Q_1'') = \emptyset$.
(iii) If $e' \in E(Q_1)$ and $e'' \in E(Q_2)$, then G has internally-edge-disjoint (e', e'')-trails T_1 and T_2 with $E(T_1) \cup E(T_2) = E(Q_1) \cup E(Q_2)$.

Proof. Assume that Q_1 is not an (u', e'')-trail. Denote $Q_1 = v_1v_2...v_iv_{i+1}...v_t$ with $v_1 = u'$, $v_t = u''$ and $e'' = v_iv_{i+1}$ for some $i + 1 < t$. Hence $v_t \in \{u'', v''\} = \{v_i, v_{i+1}\}$. If $v_i = u'' = v_t$, then $Q_1[v_1, v_i]Q_1[v_t, v_{i+1}]v_{i+1}v_i$ is an (u', e'')-trail. Hence we assume that $v_{i+1} = u'' = v_t$. In this case, let $Q_1' = Q_1[v_1, v_{i+1}]$ and $Q_1'' = Q_1[v_{i+1}, v_t]$. Then Q_1' is an (u', e'')-trail and Q_1'' is a nontrivial closed trail satisfying the expected properties stated in (i). This justifies Lemma 3.4(i).

Now let Q_1 be an (u', e'')-trail and $e' \in E(Q_1)$. We may assume that $Q_1 = v_1v_2...v_iv_{i+1}...v_t$ with $v_1 = u'$, $e' = v_iv_{i+1}$ and $e'' = v_{t-1}v_t$. If $i = 1$, then Q_1 is an (e', e'')-trail. Hence we assume that $1 < i$, and so $v_1 \in \{u', v'\} = \{v_i, v_{i+1}\}$. If $v_1 = v_{i+1}$, then $v_{i+1}v_iQ_1[v_i, v_1]Q_1[v_{i+1}, v_t]$ is an (e', e'')-trail. Hence we may assume that $v_1 = v_i$. In this case, we let $Q_1' = Q_1[v_i, v_t]$ and $Q_1'' = Q_1[v_1, v_i]$. Then Q_1' is an (e', e'')-trail and Q_1'' is a nontrivial closed trail satisfying the expected properties stated in (ii). This justifies Lemma 3.4(ii).

Finally, we assume that $e' \in E(Q_1)$ and $e'' \in E(Q_2)$. Applying Lemma 3.4(i) to both Q_1 and Q_2, there are edge-disjoint trails Q_1', Q_1'', Q_2', Q_2'', such that $Q_1' = Q_1'[u', u'']$ is an (e', u'')-trail, $Q_2' = Q_2'[u', u'']$ is an (u', e'')-trail, $Q_1'' = Q_1''[u', u']$ and $Q_2'' = Q_2''[u'', u'']$ are (possibly edgeless) closed trails. Define $T_1 = Q_1'[v_1', u'']Q_2''[u'', u'']u''v''$, and $T_2 = v'u'Q_1''[u', u']Q_2'[u', u'']$. Then T_1 and T_2 are internally edge-disjoint (e', e'')-trails in G with $E(T_1) \cup E(T_2) \subseteq E(Q_1) \cup E(Q_2)$. This validates Lemma 3.4(iii). □

Theorem 3.5. *Let G be a connected graph with $|E(G)| \geq 3$.*
(i) Let k be an integer with $k \geq 3$, and let $e' = u'v'$ and $e'' = u''v''$ be two edges in G. If G has a $(k; u', u'')$-trail-system Q, then G has a $(k; e', e'')$-trail-system Q' with $E(Q) \subseteq E(Q')$ and $\partial_G(Q) \subseteq \partial_G(Q')$.
(ii) If G is spanning trailable, then $\kappa^*(L(G)) \geq \mu'(G)$.

Proof. To prove (i), we let $e' = u'v'$ and $e'' = u''v''$ be two edges in G. As G is loopless, we may assume that $u' \neq u''$. Suppose that G contains a $(k; u', u'')$-trail-system Q consisting of k mutually edge-disjoint (u', u'')-trails $\{Q_1, Q_2, ..., Q_k\}$. If $\{e', e''\} \cap (\cup_{i=1}^k E(Q_i)) = \emptyset$, then for each i with $1 \leq i \leq k$, letting $Q_i^1 = v'u'Q_i[u', u'']u''v''$, we have found a spanning $(k; e', e'')$-trail-system Q^1 consisting of the trials in $\{Q_1^1, Q_2^1, ..., Q_k^1\}$ such that $E(Q) \subseteq E(Q^1)$ and $\partial_G(Q) \subseteq \partial_G(Q^1)$.

Next we assume that $|\{e', e''\} \cap (\cup_{i=1}^k E(Q_i))| = 1$, and by symmetry, that $\{e', e''\} \cap (\cup_{i=1}^k E(Q_i)) = \{e''\}$. As $e'' \in \cup_{i=1}^k E(Q_i))$ and as the Q_i's are mutually edge-disjoint, we may assume that $e'' \in E(Q_1)$, and so $e'' \notin E(Q_i)$ for all $i \geq 2$. By Lemma 3.4(i), G has an (u', e'')-trail Q_1' and a (possibly trivial) closed trail $Q_1''[u'', u'']$ such that $E(Q_1) = E(Q_1') \cup E(Q_1'')$ and $E(Q_1') \cap E(Q_1'') = \emptyset$. Define $Q_1^2 = v'u'Q_1'$, $Q_2^2 = v'u'Q_2[u', u'']Q_1''[u'', u'']u''v''$, and for $3 \leq i \leq k$, $Q_i^2 = v'u'Q_i[u', u'']u''v''$. Thus we have found a spanning $(k; e', e'')$-trail-system Q^2 consisting of the trials in $\{Q_1^2, Q_2^2, ..., Q_k^2\}$ and satisfying $E(Q) \subseteq E(Q^2)$ and $\partial_G(Q) \subseteq \partial_G(Q^2)$.

Finally we assume that $\{e', e''\} \subset \cup_{i=1}^k E(Q_i)$. If e', e'' are both in the same trail, then by symmetry, we may assume that $e', e'' \in E(Q_1)$. By Lemma 3.4(i) and (ii), G has an (e', e'')-trail Q_1', and (possibly trivial) closed trails $Q_1''[u', u']$ $Q_1'''[u'', u'']$ such that $E(Q_1) = E(Q_1') \cup E(Q_1'') \cup E(Q_1''')$, and Q_1', Q_1'', Q_1''' are mutually edge-disjoint. In this case, we define $Q_1^3 = Q_1'$, $Q_2^3 = v'u'Q_1''[u', u'']Q_2[u', u'']Q_1''[u'', u'']u''v''$, and for $3 \le i \le k$, $Q_i^3 = v'u'Q_i[u', u'']u''v''$. Thus we have found a spanning $(k; e', e'')$-trail-system Q^3 consisting of the trials in $\{Q_1^3, Q_2^3, ..., Q_k^3\}$ and satisfying $E(Q) \subseteq E(Q^3)$ and $\partial_G(Q) \subseteq \partial_G(Q^3)$. Hence we may assume that e', e'' are not in the same trail. By symmetry, we may assume that $e' \in E(Q_1)$ and $e'' \in E(Q_2)$. By Lemma 3.4(iii), then G has internally edge-disjoint (e', e'')-trails T_1 and T_2 with $E(T_1) \cup E(T_2) = E(Q_1) \cup E(Q_2)$. In this case, we define $Q_1^4 = T_1$, $Q_2^4 = T_2$, and for $3 \le i \le k$, $Q_i^4 = v'u'Q_i[u', u'']u''v''$. Thus we have found a spanning $(k; e', e'')$-trail-system Q^4 consisting of the trials in $\{Q_1^4, Q_2^4, ..., Q_k^4\}$ and satisfying $E(Q) \subseteq E(Q^4)$ and $\partial_G(Q) \subseteq \partial_G(Q^4)$. Therefore, in any case, a $(k; e', e'')$-trail-system Q' of G satisfying $E(Q) \subseteq E(Q')$ and $\partial_G(Q) \subseteq \partial_G(Q')$ can always be found, and so Theorem 3.5(i) is proved.

To prove (ii), we observe that since G is spanning trailable, it follows from Lemma 3.3 that $\kappa^*(L(G)) \ge 2$. Thus Theorem 3.5(ii) holds if $\mu'(G) \le 2$. Hence we may assume that $s := \mu'(G) \ge 3$. By Theorem 3.1, for each integer k with $3 \le k \le s$, and for any edge $e', e'' \in E(G)$, we are to find a dominating $(k; e', e'')$-trail-system of G. Again denote $e' = u'v'$ and $e'' = u''v''$. Since $s \ge k$, G contains a spanning $(k; u', u'')$-trail-system Q consisting of k mutually edge-disjoint (u', u'')-trails. By Theorem 3.5(i), G contains a $(k; e', e'')$-trail-system Q' with $E(Q) \subseteq E(Q')$ and $\partial_G(Q) \subseteq \partial_G(Q')$. As Q is spanning, Q' is also spanning, and so dominating. Hence by Theorem 3.1 and by definition, $\kappa^*(L(G)) \ge \mu'(G)$. □

It is a natural question whether there exist graphs G with $\mu'(G) < \kappa^*(L(G))$. Further preparations would be needed to address this question. In Example 5.9, we shall present 3-edge-connected graphs G with $\mu'(G) < 3$ but $\kappa^*(L(G)) \ge 3$.

3.3. Basics in s-collapsible reductions

In this subsection, we will introduce a reduction method that associates with the study in spanning connectivity of line graphs. For a graph G and an integer i, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}, \text{ and } O(G) = \cup_{j \ge 0} D_{2j+1}(G).$$

Let $s \ge 1$ be an integer. Following Proposition 2.2 of [19], we define a nonempty graph G to be ***s*-collapsible** if for any vertex subset $X \subseteq V(G)$ with $|X| \equiv 0 \pmod 2$, G contains a spanning connected subgraph L_X satisfying $O(L_X) = X$ and $\kappa'(G - E(L_X)) \ge s - 1$. Let $\mathcal{C}_s$ denote the family of all s-collapsible graphs. An s-collapsible graph is often referred as to a $\mathcal{C}_s$**-graph**. As indicated in Section 2 of [19], $\mathcal{C}_1$ consists of precisely the collapsible graphs defined by Caltin in [3].

Let $H_1, H_2, ..., H_c$ denote the maximal s-collapsible subgraph of G. Then $G/(\cup_{i=1}^c H_i)$ is the $\mathcal{C}_s$**-reduction** of G. A graph G is $\mathcal{C}_s$**-reduced** if G equals to its own $\mathcal{C}_s$-reduction. The follow result collects some useful facts on $\mathcal{C}_s$-reductions.

Theorem 3.6. *Let $s \ge 1$ be an integer, G be a nontrivial connected graph and let H be a $\mathcal{C}_s$-subgraph of G. Each of the following holds.*
(i) (Corollary 2.9 of [19]) G is s-collapsible if and only if G/H is s-collapsible. Moreover, $\mu'(G) \ge s+1$ if and only if $\mu'(G/H) \ge s+1$. In particular, every $\mathcal{C}_s$ graph G satisfies $\mu'(G) \ge s+1$.
(ii) (Theorem 2.11 of [19]) Let $F(G, s)$ be the minimum number of new edges that must be added to G to result in a graph with s edge-disjoint spanning trees. If $F(G, s+1) \le 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \ge s+1$.
(iii) (Corollary 2.13 of [19]) If $\kappa'(G) \ge s+1$ and G is $\mathcal{C}_s$-reduced, then

$$F(G, s+1) = (s+1)(|V(G)| - 1) - |E(G)| \ge 2.$$

(iv) (Theorem 1.3 of [6]) If $F(G, 2) \le 2$ and the $\mathcal{C}_1$-reduction of G is not a K_2 not a member in $\{K_{2,t} : t \ge 1\}$, then $G \in \mathcal{C}_1$.
(v) (Theorem 4 of [5]) Suppose that $F(G, 2) = 0$. Then G is spanning trailable if and only if $ess'(G) \ge 3$. Moreover, if in addition, $\kappa'(G) \ge 3$, then G is strongly spanning trailable.

Proof. We only need to justify the second half of (v). To do that, we follow Notation 3.2. Assume that $F(G, 2) = 0$ and $\kappa'(G) \ge 3$. Then for any $e, e' \in E(G)$, we observe that $F(G(e', e''), 2) \le 2$. As $\kappa'(G) \ge 3$, $G(e', e'')$ cannot be contracted to K_2 nor to a member in $\{K_{2,t} : t \ge 1\}$. Hence $G(e, e') \in \mathcal{C}_1$, and so by (i) with $s = 1$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail, and so G is strongly spanning trailable. □

Corollary 3.7. *Let $k \ge 2$ be an integer and G be a graph.*
(i) (P. Huang and L. Hsu [13], Chen et al., Theorem 1.4 of [9]) Suppose that $F(G, k) = 0$. Then $\kappa^*(L(G)) \ge k$ if and only if $\kappa(L(G)) \ge \max\{k, 3\}$.
(ii) Suppose that $F(G, 2) = 0$ and $ess'(G) \ge 3$. Let G' denote the $\mathcal{C}_{k-1}$-reduction of G. Then $\kappa^*(L(G)) \ge 2$. Moreover, if $k \ge 3$, then $\kappa^*(L(G)) \ge \min\{k, \mu'(G')\}$.

Proof. By (4), it suffices to assume $\kappa(L(G)) \geq k$ to prove that $\kappa^*(L(G)) \geq k$. By (6), G is essentially k-edge-connected. If $k = 2$, then by Theorem 3.6(v) and by Lemma 3.3, $L(G)$ is Hamilton-connected and so $\kappa^*(L(G)) \geq 2$. Thus we assume that $k \geq 3$. By Theorem 3.6(ii) with $s = k - 1$, we have $\mu'(G) \geq k \geq 3$. It follows by Theorem 3.5 that $\kappa^*(L(G)) \geq k$. This proves (i).

Assume now that $F(G, 2) = 0$ and $ess'(G) \geq 3$. By Theorem 3.6(v) and by Lemma 3.3, we have $\kappa^*(L(G)) \geq 2$. Let $k' = \min\{k, \mu'(G')\}$ and assume that $k' \geq 3$. By Theorem 3.6(i), $\mu'(G) \geq \mu'(G') \geq k'$. It follows from Theorem 3.5 that $\kappa^*(L(G)) \geq \mu'(G) \geq \min\{k, \mu'(G')\}$. □

Lemma 3.8. *Let G be a graph and $m \geq 1$ be an integer. Define mG to be the graph obtained from G be replacing each edge e of G by a set of m parallel edges joining the two vertices in $V(e)$. Each of the following holds.*
(i) (Corollary 3.1 of [19]) Let $\ell \geq 2$, $s \geq 1$ be integers. Then $\ell K_2 \in \mathcal{C}_s$ if and only if $\ell \geq s + 1$.
(ii) (Theorem 3.3 of [19]) Let $n \geq 2$, $s \geq 2$ be integers. Then $K_n \in \mathcal{C}_s$ if and only if $\ell \geq s + 3$.

We now can comment a bit more on the complete families that will be used in our arguments.

Lemma 3.9. *Let $s \geq 0$ be an integer.*
(i) (Corollary 2.4 of [19]) Let $\mathcal{C}_s$ denote the family of all s-collapsible graphs. Then $\mathcal{C}_s$ is a complete family.
(ii) (Theorem 2.10 of [19]) $\mathcal{T}_{s+1} \subseteq \mathcal{C}_s$.

3.4. An application of Theorem 2.5

Theorem 2.5 can be applied to prove a former result on supereulerian width, using this strengthened Ore type condition m_2 to study the supereulerian width of graphs.

Theorem 3.10. *(Xiong et al., Theorem 1.2 of [30]) For any real numbers a, b with $0 < a < 1$ and any integer $s > 1$, there exists a finite family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for any simple graph G with $n = |V(G)|$, if $m_2(G) \geq an + b$, then either $\mu'(G) \geq s$, or G is not contractible to a member in $\mathcal{F}$.*

Proof. Let $\mathcal{C} = \mathcal{C}_{s-1}$. By Lemma 3.9, $\mathcal{C}$ is a complete family. By Theorem 3.6(iii), $\mathcal{T}_{s+1} \subseteq \mathcal{C}$. By Lemma 3.8(ii), and as $Z_{a,b}$ induces a complete subgraph in any graph satisfying (3), we conclude that the $\mathcal{C}$-reduction of any graph satisfying (3) does not contain a complete subgraph of order $s+2$, and so we set $M_0 = s+2$. It follows by Theorem 2.5 that for any real values a, b, and for this complete family $\mathcal{C} = \mathcal{C}_{s-1}$, there exists a finite graph family $\mathcal{F}$ such that either G is in $\mathcal{C}_{s-1}$, whence by Theorem 3.6(i) that $\mu'(G) \geq s$, or G is contracted to a member in $\mathcal{F}$. □

4. Proof of Theorem 1.5

By Theorem 3.5, it is natural to consider applying Theorem 3.10 to prove Theorem 1.5. To do that, we need to demonstrate that graphs satisfying the conditions of Theorem 1.5 must be strongly spanning trailable. Theorem 2.5 will be applied for this purpose.

Proof of Theorem 1.5. By Example 1.2, $\mathcal{T}_2$ is a complete family. As every complete graph of order at least 4 is in $\mathcal{T}_2$, a $\mathcal{T}_2$-reduced graph does not have any complete subgraph of order at least 4. It follows that if a graph G satisfies (3), then $|Z_{a,b}(G)| \leq 3$. By Theorem 2.5 with $\mathcal{C} = \mathcal{T}_2$ and $M_0 = 3$, we have the following conclusion stated as the claim below.

Claim 3. *There exists a finite family $\mathcal{F}_{11}$ such that any graph G satisfying (3) is either in $\mathcal{T}_2$ or is contractible to a member in $\mathcal{F}_{11}$.*

Let $\mathcal{F}_{12}$ be the finite obstacle family whose existence is warranted by Theorem 3.10, and let $\mathcal{F}_1 = \mathcal{F}_{11} \cup \mathcal{F}_{12}$. Then $\mathcal{F}_1$ is also a finite family of graphs. Let G be a graph satisfying (3) with $ess'(G) \geq \max\{3, s\}$. We are to show that either $\kappa^*(L(G)) \geq s$ or G is contractible to a member in $\mathcal{F}_1$. To do that, we assume that G is not contractible to a member in $\mathcal{F}_1$ to prove that we must have $\kappa^*(L(G)) \geq s$. Since G is not contractible to a member in $\mathcal{F}_{11} \subseteq \mathcal{F}_1$, we conclude by Claim 3 that $G \in \mathcal{T}_2$, and so as $ess'(G) \geq \max\{3, s\}$ and by Theorem 3.6(v) and Corollary 3.7,

$$G \text{ is spanning trailable and } \kappa^*(L(G)) \geq 2. \tag{11}$$

Let $s \geq 3$. Since G is not contractible to a member in $\mathcal{F}_{12} \subseteq \mathcal{F}_1$, it follows by Theorem 3.10 that $\mu'(G) \geq s$. This, together with (11), implies that G is a spanning trailable graph with $\mu'(G) \geq s$. By Theorem 3.5, we conclude that $\kappa^*(L(G)) \geq s$, and complete the proof of the theorem. □

5. The core of a graph and a $K_{3,3}$-family

Symmetric difference of sets will be used in this section. For two sets X and Y, define

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

A related degree condition for a 3-edge-connected graph to be of supereulerian width at least 3 is obtained in [30].

Theorem 5.1. *(Xiong et al., Theorem 1.3 of [30]) For a simple graph G with $|V(G)| = n \geq 141$ and $\kappa'(G) \geq 3$, if for any pair of nonadjacent vertices u and v, $\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$, then $\mu'(G) \geq 3$ if and only if G is not contractible to $K_{3,3}$.*

It is natural to consider applying Theorem 5.1 in conjunction of Theorem 2.5 to prove Theorem 1.7. However, Theorem 1.7 does not assume that graphs under considerations are 3-edge-connected. Therefore, we turn to the assistance of some former results on the core of an essentially 3-edge-connected graph to prove Theorem 1.7, instead of taking the approach of applying Theorem 5.1.

Given a graph G, and a vertex $v \in V(G)$, define $E_G(v)$ to be the set of edges incident with v in G. Suppose that G is an essentially 3-edge-connected graph. For every vertex $v \in D_2(G)$, denote $E_G(v) = \{e'_v, e''_v\}$, and

$$X_2(G) = \{e''_v : v \in D_2(G)\}.$$

Following [28], we define the **core** of G as follows:

$$G_0 = (G - D_1(G))/X_2(G).$$

As $G - D_1(G)$ can also be viewed as contracting the edges incident with vertices in $D_1(G)$, we use ϕ_1 to denote the mapping arisen in the contraction process from G to G_0, which maps a subgraph of G to a subgraph of G_0. In the rest of this paper, we often let G'_0 denote the $\mathcal{T}_3$-reduction of G_0, let $H'_z = PI_{G_0}(z)$, and let $H_z = \phi_1^{-1}(H'_z)$ denote the pull-back image of H'_z in G, which will be called **the restoration** of z in G. The **restoration of subgraphs** of G'_0 in G can be similarly defined. The following lemmas are useful.

Lemma 5.2. *(Shao, [28]) Let G be a connected nontrivial graph such that $ess'(G) \geq 3$, and let G_0 denote the core of G.*
(i) G_0 is uniquely determined by G with $\kappa'(G_0) \geq 3$.
(ii) (see also Lemma 2.9 of [16]) If for any $e, e' \in E(G_0)$, $G_0(e, e')$ has a spanning $(v_e, v_{e'})$-trail, then $\kappa^*(L(G)) \geq 2$.

Lemma 5.3. *Let G be a simple graph on n vertices with $\kappa'(G) \geq 2$.*
(i) (Li et al., Lemma 2.1 of [21]) If $n \leq 8$, $D_1(G) = \emptyset$ and $|D_2(G)| \leq 2$, then either $G \in \mathcal{C}_1$, or G is contractible to a $K_{2,3}$.
(ii) (Chen and Chen [8]) If $n \leq 9$ and $|D_2(G)| \leq 2$, then either $G \in \mathcal{C}_1$, or G is contractible to a $K_{2,3}$.

Lemma 5.4. *Let G be a graph on n vertices.*
(i) (Chen et al., Theorem 1.4 of [9]) Suppose that $k \geq 2$ is an integer, $ess'(G) \geq 3$ and G_0 is the core of G. If G_0 has k-edge-disjoint spanning trees, then $\kappa^*(L(G)) \geq k$ if and only if $\kappa(L(G)) \geq k$.
(ii) (Li et al., Lemma 4.2 of [19]) Denote $V(K_{3,3}) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ such that each of $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ is a stable set of $K_{3,3}$. Then $K_{3,3}$ does not have a $(3; u_1, u_2)$-trail system that contains u_3.
(iii) Let $\mathcal{J}(n)$ be the family of graphs defined in Example 1.6. For any $J(n) \in \mathcal{J}(n)$, both $\kappa(L(J(n))) \geq 3$ and $\kappa^*(L(J(n))) < 3$.
(iv) Define $\mathcal{P}_2$ to be the family of graphs such that a graph G is in $\mathcal{P}_2$ if and only if $G \in \mathcal{T}_2$, and for any $u, v \in V(G)$), G contains a (u, v)-path P such that $(G - E(P)) - D_1(G - E(P)) \in \mathcal{C}_1$. Then every graph $G \in \mathcal{P}_2$ satisfies $\mu'(G) \geq 3$.
(v) Suppose that $\kappa'(G) \geq 3$. If $n \leq 6$, then G is strongly spanning trailable.
(vi) (Li et al., Theorem 4.4 of [19]) If $|V(G)| \leq 6$ and $\kappa'(G) \geq 3$, then $\mu'(G) < 3$ if and only if $G \cong K_{3,3}$.

Proof. It suffices to prove (iii), (iv) and (v). By Example 1.6, every $J(n) \in \mathcal{J}(n)$ is 3-edge-connected, and so $\kappa(L(J(n))) \geq 3$. We shall use the notation in Example 1.6 in the arguments and let $J \cong K_{3,3}$. For each $w \in V(J) - \{v_1\}$, $H(w)$ is a complete graph $|E(H(w))| > 0$. Choose edges $e' \in E(H(u_1))$ and $e'' \in E(H(u_2))$. Then $J(n)$ does not have a dominating $(3; e', e'')$-trail system Q. If it does, then since $H(u_3)$ is nontrivial, Q must use at least one vertex in $V(H(u_3))$. It follows that by contracting edges in $E(Q) \cap E(H(w))$ for each vertex $w \in V(J)$, we would obtain a $(3; u_1, u_2)$-trail system in $K_{3,3}$ that contains u_3, which is a violation to Lemma 5.4(ii). Thus by Theorem 3.1, $\kappa^*(L(J(n))) < 3$. This validates (iii).

To prove (iv), we assume that $G \in \mathcal{P}_2$. To avoid triviality, we assume that $|V(G)| \geq 2$. As $G \in \mathcal{T}_2 \subseteq \mathcal{C}_1$, by Theorem 3.6(i), $\mu'(G) \geq 2$. Randomly pick two vertices $u, v \in V(G)$. As $G \in \mathcal{P}_2$, G contains a (u, v)-path P such that $(G - E(P)) - D_1(G - E(P)) \in \mathcal{C}_1$. Thus by Theorem 3.6(i) again, $(G - E(P)) - D_1(G - E(P))$ has a spanning $(2; u, v)$-trail system consisting of two edge-disjoint (u, v)-trails T_1 and T_2. It follows that the collection $\{P, T_1, T_2\}$ forms a spanning $(3; u, v)$-trail system of G. As u and v are arbitrary, we conclude that $\mu'(G) \geq 3$, and so (iv) holds.

Assume that $|V(G)| \leq 6$ and $\kappa'(G) \geq 3$. Let $e', e'' \in E(G)$. Then $|V(G(e', e''))| \leq 8$ without vertices of degree one and with at most two vertices of degree 2. As $\kappa'(G) \geq 3$, $G(e', e'')$ cannot be contracted to a $K_{2,3}$, and so by Lemma 5.3(i), $G(e', e'')$ is a $\mathcal{C}_1$-graph, and so it has a spanning connected subgraph H with $O(H) = \{v_{e'}, v_{e''}\}$. Hence G is strongly spanning trailable. □

Lemma 5.5. *(Li et al., Lemma 2.8 of [19]) Let $s \geq 1$ be an integer. Suppose that H is a connected subgraph of a given graph G, and let v_H denote the vertex in G/H onto which H is contracted. For any $x \in V(G)$, define $x' = x$ if $x \in V(G) - V(H)$ and $x' = v_H$ if $x \in V(H)$. If $H \in \mathcal{C}_s$, then for any $u, v \in V(G)$ with $u \neq v$, the following are equivalent.*
(i) G has a spanning $(s+1; u, v)$-trail-system.
(ii) If $u' \neq v'$, then G/H has a spanning $(s+1; u', v')$-trail-system; and if $u' = v' = v_H$, then G/H is supereulerian.

Lemma 5.6. *Let G be an essentially 3-edge-connected graph, G_0 the core of G and let G_0' be the $\mathcal{T}_3$-reduction of G_0.*
(i) If $G_0' \in \mathcal{P}_2$, then $\kappa^*(L(G)) \geq 3$.
(ii) If G_0' is strongly spanning trailable, then G_0 is also strongly spanning trailable, and $\kappa^*(L(G)) \geq \mu'(G_0)$.

Proof. Since G_0' is the $\mathcal{T}_3$-reduction of G_0 and since G_0' is a $\mathcal{T}_2$-graph, it follows by $\mathcal{T}_3 \subset \mathcal{T}_2$ and by Definition 1.1(C3) that G_0 is also a $\mathcal{T}_2$-graph. Applying Lemma 5.4(i) with $k = 2$, we conclude that $\kappa^*(L(G)) \geq 2$. To show that $\kappa^*(L(G)) \geq 3$, by Theorem 3.1, it suffices to show that for any $e', e'' \in E(G)$, G has a dominating $(3; e', e'')$-trail system.

Let $e' = u'v'$ and $e'' = u''v''$ be two edges in G. Without lost of generality, we assume that $d_G(u') \geq d_G(v')$ and $d_G(u'') \geq d_G(v'')$. As $ess'(G) \geq 3$, we conclude that $d_G(u') \geq 3$ and $d_G(u'') \geq 3$, and so we can view u', u'' as vertices in G_0. Further, if e' (or e'', respectively) is incident with a vertex of degree 2 in G, then we take the convention that e' (or e'', respectively) is an edge in G_0, as a spanning trail in G_0 starting with an edge incident with a vertex v of degree 2 in G can always be restored into a dominating trail in G that starts with either edge incident with v.

Since $G_0' \in \mathcal{P}_2$, by Lemma 5.4(iv), $\mu'(G_0') \geq 3$. As G_0' is a $\mathcal{T}_3$-reduction of G_0 and so every subgraph of G_0 being contracted to get G_0' is a $\mathcal{T}_3$-graph. By Example 1.2, every $\mathcal{T}_3$ graph is a $\mathcal{C}_2$-graph. It follows by Lemma 5.5 with $s = 2$ that $\mu'(G_0) \geq 3$ as well. Hence G_0 has a spanning $(3; u', u'')$-trail system Q''. Let Q denote the restoration of Q''. Then, as Q'' is spanning in G_0, Q is a $(3; u', u'')$-trail system of G containing all vertices of degree 3 in G. By Theorem 3.5(i), G has a $(3; e', e'')$-trail-system Q' with $E(Q) \subseteq E(Q')$ and $\partial_G(Q) \subseteq \partial_G(Q')$. It follows that Q' is a dominating $(3; e', e'')$-trail-system of G. This completes the proof of Lemma 5.6 (i).

To prove (ii), we shall apply Lemma 5.5 with $s = 1$. We first justify the following claim.

Claim 4. *If a connected graph Γ is strongly spanning trailable, then $\mu'(\Gamma) \geq 2$.*

Proof. For any $u, v \in V(\Gamma)$, let $e_u, e_v \in E(\Gamma)$ such that $u \in V(e_u)$ and $v \in V(e_v)$. We allow the possibility of $e_u = e_v$, and when $e_u = e_v$, we set $\Gamma(e_u, e_v) = \Gamma(e_u)$. Since Γ is strongly spanning trailable, $\Gamma(e_u, e_v)$ has a spanning (v_{e_u}, v_{e_v})-trail J'. Replacing the end edges of J' containing v_{e_u} (v_{e_v}, respectively) by e_u (by e_v, respectively), we obtain a spanning (u, v)-trail of Γ. If we choose $e_u = e_v = uv$, then we obtain a spanning closed trail, which is a spanning $(2; u, v)$-trail system. Hence $\mu'(\Gamma) \geq 2$, and the claim is verified. □

As G_0' is strongly spanning trailable, by Claim 4, $\mu'(G_0') \geq 2$ and for any $e', e'' \in E(G_0')$, we also have $\mu'(G_0'(e', e'')) \geq 2$. Applying Lemma 5.5, we conclude that $\mu'(G_0) \geq 2$ and for any $e', e'' \in E(G_0)$, $\mu'(G_0(e', e'')) \geq 2$. Thus G_0 is also strongly spanning trailable. By Lemma 3.3, $\kappa^*(L(G)) \geq 2$. Let k be an integer with $3 \leq k \leq \mu'(G_0)$. As every spanning $(k; u, v)$-trail system of G_0 can be restored as a dominating $(k; u, v)$-trail system in G, it follows that for any $e', e'' \in E(G)$, G has a dominating $(k; e', e'')$-trail system. By Theorem 3.1, $\kappa^*(L(G)) \geq \mu'(G_0)$. □

Lemma 5.7. *Let G be a graph and let H be a $\mathcal{C}_1$-subgraph of G. If there exist vertices $v_1, v_2, \ldots, v_\ell \in V(G) - V(H)$ such that $|N_G(v_1) \cap V(H)| \geq 2$ and for any i with $2 \leq i \leq \ell$, $|N_G(v_i) \cap (V(H) \cup \{v_1, \ldots, v_{i-1}\})| \geq 2$, then $G[V(H) \cup \{v_1, v_2, \ldots, v_\ell\}]$ is a $\mathcal{C}_1$-subgraph of G.*

Proof. Let $H_0 = H$, and for each $i \geq 1$, define $H_i = G[V(H_{i-1}) \cup \{v_1, \ldots, v_{i-1}\}]$. We argue by induction on i to show that $H_i \in \mathcal{C}_1$. As $H_0 = H$ is assumed to be a $\mathcal{C}_1$-graph, as assume that $i \geq 1$ and that $H_{i-1} \in \mathcal{C}_1$. Since $|N_G(v_i) \cap (V(H) \cup \{v_1, \ldots, v_{i-1}\})| \geq 2$, it follows that H_i/H_{i-1} has two vertices and there are at least two (parallel) edges joining these two vertices. Hence $H_i/H_{i-1} \in \mathcal{T}_2 \subseteq \mathcal{C}_1$. By Example 1.2, $\mathcal{C}_1$ is a complete family and so as $H_{i-1} \in \mathcal{C}_1$ and by Definition 1.1 (C3), we conclude that $H_i \in \mathcal{C}_1$, and the lemma is justified by induction. □

As we remarked earlier, there exist 3-edge-connected graphs G with $\mu'(G) < \kappa^*(L(G))$. We close this subsection with such examples.

Lemma 5.8. *Let H be a maximal $\mathcal{C}_2$-subgraph of a graph G, $G' = G/H$, v_H be the vertex in G' whose preimage is H, and let $e', e'' \in E(G)$ be two edges.*
(i) If $e', e'' \in E(G')$ and G' has a $(3; e', e'')$-trail system Q' with v_H being an internal vertex of Q', then G has a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(Q') \subseteq E(Q)$.
(ii) If $e', e'' \in E(H)$ and G' has a closed trail J' containing v_H, then G has a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(J') \subseteq E(Q)$.

(iii) If $e' \in E(G')$ and $e'' \in E(H)$, and G' has a $(3; e', v_H)$-trail system Q', then G has a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(Q') \subseteq E(Q)$.

Proof. Throughout the proof of this lemma, we assume that $e' = u'v'$ and $e'' = u''v''$ are edges in G. For notational convenience in the proof of this lemma, we also view any $z \in \{u', v', u'', v''\}$ as the vertex in G' such that $PI_G(z)$ contains the vertex z in G.

To prove (i), choose a $(3; e', e'')$-trail system containing v_H and consisting of internally edge-disjoint (e', e'')-trails T_1', T_2', T_3' such that $d_{T_1'}(v_H) \geq d_{T_2'}(v_H) \geq d_{T_3'}(v_H)$ with $d_{T_1'}(v_H)$ maximized. Thus $d_{T_1'}(v_H) > 0$. If $d_{T_2'}(v_H) > 2$, then T_2' contains a closed subtrail T_2'' with $v_H \in V(T_2'')$. Hence moving T_2'' from T_2' to T_1' will increase the degree of v_H in the resulting trail, contrary to the maximality of $d_{T_1'}(v_H)$. Therefore the maximality of $d_{T_1'}(v_H)$ implies that $d_{T_3'}(v_H) \leq d_{T_2'}(v_H) \leq 2$. As each T_i' is a (e', e'')-trail, there exist some vertices $x_i \in \{u', v'\}$ and $y_i \in \{u'', v''\}$ such that each T_i' is an (x_i, y_i)-trail in G'. Since T_1' and T_2' are edge-disjoint (e', e'')-trails, depending on whether each of $x_1 = x_2$ and $y_1 = y_2$ holds or not, $G'[E(T_1') \cup E(T_2')]$ consists of a closed trail J' which possibly does not contain e' or e'', or both e' and e'' in J'. As v_H is an internal vertex of Q', we conclude that $v_H \in V(J')$. As J' is a closed trail in G', $G[E(J')]$ is a subgraph of G with $O(G[E(J')]) \subseteq V(H)$. By the assumption that $H \in \mathcal{T}_3 \subset \mathcal{C}_2$, H has a spanning connected subgraph J'' such that $O(J'') = O(G[E(J')])$ such that $G - E(J'')$ is connected. It follows that $G[E(J') \cup E(J'')]$ is a closed trail of G containing all vertices of H and at least a vertex in $V(e')$ and a vertex in $V(e'')$. Hence $G[E(J') \cup E(J'') \cup \{e', e''\}]$ is an internally edge-disjoint union of two (e', e'')-trails T_1 and T_2. If $v_H \notin V(T_3')$, then set $T_3 = T_3'$. Suppose that $v_H \in V(T_3')$. Then there exist vertices $u_3, v_3 \in V(H)$ (possibly $x_3 = u_3$ or $y_3 = v_3$) such that $G[E(T_3')]$ contains a (x_3, u_3)-trail J_3 and a (v_3, y_3)-trail J_3' with $E(J_3) \cap E(J_3') = \emptyset$. As $G - E(J'')$ is connected, there exists a (u_3, v_3)-path T_3'' in $G - E(J'')$. It follows that $T_3 = G[E(J_3) \cup E(T_3'') \cup E(J_3') \cup \{e', e''\}]$ is an (e', e'')-trail internally edge-disjoint from each of T_1 and T_2. Thus the collection $\{T_1, T_2, T_3\}$ forms a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(Q') \subseteq E(Q)$. This proves (i).

The proof for Lemma 5.8(ii) is similar to that for (i) with minor modifications. Form a graph $H^3(e', e'')$ from $H(e', e'')$ by adding a new edge parallel to the edge $v_{e'}v'$, and by adding another new edge parallel to the edge $v_{e''}v''$. Thus the newly inserted vertices $v_{e'}, v_{e''} \in V(H(e', e'')) - V(H)$ are of degree 3 in $H^3(e', e'')$. Since $H \in \mathcal{T}_2$, it is routing to verify that $H^3(e', e'') \in \mathcal{T}_3$. By replacing H by $H^3(e', e'')$ in G, we form $G^3(e', e'')$. Thus $H^3(e', e'')$ is a subgraph of $G^3(e', e'')$ and $G^3(e', e'')/H^3(e', e'') = G/H = G'$. As J' is a closed trail in G', $G^3(e', e'')[E(J')]$ is a subgraph of $G^3(e', e'')$ with $O(G^3(e', e'')[E(J')]) \subseteq V(H^3(e', e''))$. Since $H^3(e', e'') \in \mathcal{T}_3 \subset \mathcal{C}_2$, $H^3(e', e'')$ has a spanning connected subgraph J'' such that $O(J'') = O(G^3(e', e'')[E(J')])$ and $G^3(e', e'') - E(J'')$ is connected. It follows that $G^3(e', e'')[E(J') \cup E(J'')]$ is a closed trail of $G^3(e', e'')$ containing all vertices of $H^3(e', e'')$, which consists of two edge-disjoint $(v_{e'}, v_{e''})$-trials F_1', F_2'. Since $G^3(e', e'') - E(J'')$ is connected, it has a $(v_{e'}, v_{e''})$-trial F_3'. Replacing the terminal edge containing $v_{e'}$ (or $v_{e''}$, respectively) of the trails F_1', F_2', F_3' by e' (or by e'', respectively), we form a collection of internally edge-disjoint trials $\{T_1, T_2, T_3\}$, which is a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(J') \subseteq E(Q)$. This justifies (ii).

It remains to prove (iii). Suppose that G' has a $(3; e', v_H)$-trail system Q', consisting of internally edge-disjoint T_1', T_2', T_3'. Hence for each $i \in \{1, 2, 3\}$, there exists a vertex $x_i \in V(e')$ such that T_i' is an (x_i, v_H)-trail. As in the proof for (i), we choose these trails so that $d_{T_1'}(v_H) \geq d_{T_2'}(v_H) \geq d_{T_3'}(v_H)$ with $d_{T_1'}(v_H)$ maximized, which implies that $d_{T_3'}(v_H) = 1$. As in the proof for (i), $G'[E(T_1') \cup E(T_2')]$ consists of a closed trail J' with possibly e' or e'' or both not included in J'. Form a graph $G^3(e'')$ ($H^3(e'')$, respectively) from $G(e'')$ ($H(e'')$, respectively) by adding a new edge parallel to the edge $v_{e''}v''$. Thus $H^3(e'')$ is a subgraph of $G^3(e'')$ and $G^3(e'')/H^3(e'') = G/H = G'$. Moreover, as $H \in \mathcal{T}_3$, we also have $H^3(e'') \in \mathcal{T}_3$. As J' is a closed trail in G', $G^3(e'')[E(J')]$ is a subgraph of $G^3(e'')$ with $O(G^3(e'')[E(J')]) \subseteq V(H^3(e''))$. By the fact that $H^3(e'') \in \mathcal{T}_3 \subset \mathcal{C}_2$, $H^3(e'')$ has a spanning connected subgraph J'' such that $O(J'') = O(G^3(e'')[E(J')])$ such that $H^3(e'') - E(J'')$ is connected. It follows that $G^3(e'')[E(J') \cup E(J'')]$ is a closed trail of G containing all vertices of $H^3(e'')$ and at least a vertex in $V(e')$. Hence $G[E(J') \cup E(J'') \cup \{e'\}]$ is an internally edge-disjoint union of two $(e', v_{e''})$-trails F_1' and F_2'. As T_3' is a (e', v_H)-trail with $d_{T_3'}(v_H) = 1$, there exists a vertex $u_3 \in V(H)$ (possibly $x_3 = u_3$) such that $G[E(T_3')]$ contains a (x_3, u_3)-trail J_3. Since $H^3(e'') - E(J'')$ is connected, $H^3(e'') - E(J'')$ has a $(u_3, v_{e''})$-path T_3'', and so $F_3' = G^3(e'')[E(J_3') \cup E(T_3'')]$ is an $(e', v_{e''})$-trail internally edge-disjoint from F_1' and F_2'. Replacing the terminal edge containing $v_{e''}$ of the trails F_1', F_2', F_3' by e'', we form a collection of internally edge-disjoint trials $\{T_1, T_2, T_3\}$, which is a $(3; e', e'')$-trail system Q with $V(H) \subseteq V(Q)$ and $E(J') \subseteq E(Q)$. This completes the proof of (iii), as well as the lemma. □

Example 5.9. Let J be a graph isomorphic to $K_{3,3}$ with vertex set $V(J) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ such that each of $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ is a stable set of J. For any integer $n \geq 16$, let $\mathcal{J}'(n)$ denote the family of graphs such that every graph G in $\mathcal{J}'(n)$ is obtained from J by replacing each vertex $w \in V(J) - \{u_1, v_1\}$ by a nontrivial trivial $\mathcal{T}_3$-graph $H(w)$ in such a way that $\kappa'(G) \geq 3$ and $\sum_{w \in V(J)} |V(H(w))| = n$. We shall show, using Lemma 5.10, that for any $G \in \mathcal{J}'(n)$, both $\mu'(G) \leq 2$ and $\kappa^*(L(G)) \geq 3$.

Lemma 5.10. *let $\mathcal{J}'(n)$ be the graph family defined in Example 5.9 and let $G \in \mathcal{J}'(n)$. Each of the following holds.*
(i) $\mu'(G) \leq 2$.
(ii) G is $\mathcal{C}_1$, and for any $e', e'' \in E(G)$, $G(e', e'')$ is a $\mathcal{C}_1$-graph.
(iii) For any $e', e'' \in E(G)$, G has a dominating $(3; e', e'')$-trail system.
(iv) $\kappa^*(L(G)) \geq 3$.

Proof. As G is contractible to $K_{3,3}$, we have $\mu'(G) \leq \mu'(K_{3,3})$ (see for example, Corollary 2.9 of [19]). By Lemma 5.4(ii), $\mu'(K_{3,3}) \leq 2$, and so $\mu'(G) \leq 2$, justifying (i).

To prove (ii), we first observe that by Lemma 5.3, for any edge $e_1, e_2 \in E(K_{3,3})$, both $K_{3,3}(e_1)$ and $K_{3,3}(e_1, e_2)$ are $\mathcal{C}_1$-graphs as they cannot be contracted to $K_{2,3}$. Now we fix two randomly chosen edges $e', e'' \in E(G)$, and observe that G is a contraction image of $G(e', e'')$. As $\mathcal{C}_1$ is a complete family, which is closed under contraction, it suffices to show that $G(e', e'')$ is a $\mathcal{C}_1$-graph.

By the definition of $\mathcal{J}'(n)$, the $\mathcal{T}_3$-reduction of G is $K_{3,3}$, and by Theorem 3.6(iv), $K_{3,3} \in \mathcal{C}_1$. If $e', e'' \in E(K_{3,3})$, then by contracting the $\mathcal{T}_3$-subgraphs $H(w)$'s, we obtained $K_{3,3}(e', e'')$ as the $\mathcal{T}_3$-reduction of $G(e', e'')$. As $K_{3,3}(e', e'') \in \mathcal{C}_1$, we conclude that $G(e', e'') \in \mathcal{C}_1$, by the facts that $\mathcal{C}_1$ is a complete family and $H(w) \in \mathcal{T}_3 \subseteq \mathcal{C}_1$. Thus we assume that at least one of e', e'' is in one of the $\mathcal{T}_3$-subgraphs $H(w)$'s. For any $w \in V(K_{3,3})$, if e' or e'' is in $E(H(w))$, then since $H(w) \in \mathcal{T}_3$, $H(w)(e')$ or $H(w)(e', e'')$ (if both $e', e'' \in E(H(w))$) cannot be contracted to a $K_{2,t}$, and so by Theorem 3.6(iv), $H(w)(e', e'')$ is a $\mathcal{C}_1$-graph. As $\mathcal{C}_1$ is a complete family, and as all $\mathcal{T}_3$-graphs are $\mathcal{C}_1$-graphs, we conclude that we always have $G(e', e'') \in \mathcal{C}_1$. This proves (ii).

By Theorem 3.1, Lemma 5.10(iv) follows from (ii) and (iii), and so it suffices to prove (iii). We shall adopt the vertex notation of $K_{3,3}$ in Example 5.9 in the arguments to prove (iii). Let e', e'' denote two randomly chosen edges in G. As indicated in Example 5.9, $K_{3,3} = G/(H(u_2) \cup H(u_3) \cup H(v_2) \cup H(v_3))$. We take the convention that an edge $e \in E(K_{3,3})$ if and only if $e \notin E(H(u_2) \cup H(u_3) \cup H(v_2) \cup H(v_3))$.

Case 1. $|\{e', e''\} \cap E(K_{3,3})| \geq 1$.

If both $e', e'' \in E(K_{3,3})$, then depending on whether $\{e', e''\}$ is a matching of $K_{3,3}$ or not, we observe, by inspection, that $K_{3,3}$ always have a spanning $(3; e', e'')$-trail system Q'_1. By Lemma 5.8(i), G has a spanning $(3; e', e'')$-trail system Q_1. Hence we may assume that $\{e', e''\} \cap E(K_{3,3}) = \{e'\}$, and so for some vertex $w \in V(K_{3,3})$, $e'' \in E(H(w))$. In this case, it is also routine by inspection to verify that $K_{3,3}$ has a spanning $(3; e', w)$-trail system Q'_2. By Lemma 5.8(iii), G has a spanning $(3; e', e'')$-trail system Q_2. Hence Lemma 5.10(iii) holds when Case 2 occurs.

Case 2. $\{e', e''\} \cap E(K_{3,3})| = 0$.

If there exists a vertex $w \in V(K_{3,3})$ such that $e', e'' \in E(H(w))$, then as $K_{3,3}$ has a spanning closed trail, it follows by Lemma 5.10(iii) holds. Hence we assume that there exist distinct vertices $w', w'' \in V(K_{3,3} - \{u_1, v_1\}$ such that $e' \in E(H(w')))$ and $e'' \in E(H(w'')))$. If $w'w'' \notin E(K_{3,3})$, then we may assume by symmetry that $w' = u_2$ and $w'' = u_3$. It follows that $K_{3,3}$ has a $(3; u_2, u_3)$-trail system Q'_3 with $V(K_{3,3}) - \{u_1\} \subseteq V(Q'_3)$. If $w'w'' \in E(K_{3,3})$, then we may assume by symmetry that $w' = u_2$ and $w'' = v_2$. It follows that $K_{3,3}$ has a spanning $(3; u_2, u_3)$-trail system Q'_4.

Form a graph $H^3(w')(e')$ from $H(w')(e')$ by adding a new edge parallel to the edge $v_{e'}v'$, and a graph $H^3(w'')(e'')$ from $H(w'')(e'')$ by adding a new edge parallel to the edge $v_{e''}v''$. Let $G^3(e', e'')$ be the graph obtained from $G(e', e'')$ by replacing $H(w')(e')$ and $H(w'')(e'')$ by $H^3(w')(e')$ and $H^3(w'')(e'')$, respectively. Then $H^3(w')(e')$ and $H^3(w'')(e'')$ are subgraphs of $G^3(e', e'')$ and $G^3(e', e'')/(H^3(w')(e') \cup H^3(w'')(e'') \cup H(v_2) \cup H(v_3)) = G/H \cong K_{3,3}$. Moreover, as $H(u_2), H(u_3) \in \mathcal{T}_3$, we also have $H^3(w')(e'), H^3(w'')(e'') \in \mathcal{T}_3$. Viewing $G^3(e', e'')/(H^3(w')(e') = K_{3,3}$, it is routine to lift the $(3; u_2, u_3)$-trail system Q'_3 with $V(K_{3,3}) - \{u_1\} \subseteq V(Q'_3)$ to a $(3; v_{e'}, v_{e''})$-trail system Q''_3 with $V(G) - \{u_1\} \subseteq V(Q''_3)$; and to lift the spanning $(3; u_2, u_3)$-trail system Q'_4 to a spanning $(3; v_{e'}, v_{e''})$-trail system Q''_4. For $i \in \{3, 4\}$, replacing the terminal edges containing $v_{e'}$ by e' and replace the terminal edges containing $v_{e''}$ by e'', we have formed a dominating $(3; e', e'')$-trail system Q_i of G. This proves this case, as well as the lemma. □

6. Proof of Theorem 1.7

In Example 1.6 and Lemma 5.4, the graph families $\mathcal{J}(n)$ and $\mathcal{P}_2$ are defined respectively. The validity of the next lemma will imply Theorem 1.7.

Lemma 6.1. *Let G be a connected simple graph on n vertices with $ess'(G) \geq 3$, G_0 the core of G and G'_0 the $\mathcal{T}_3$-reduction of G_0. If $n \geq 156$ and G satisfies (7), then one of the following holds.*
(i) $G_0 \in \mathcal{T}_3$.
(ii) $G'_0 \notin \mathcal{T}_3$ and $\kappa^*(L(G)) \geq 3$.
(iii) $G_0 \in \mathcal{J}(n)$.

For a graph G satisfying the hypothesis of Theorem 1.7, by Lemma 6.1, one of the conclusions of Lemma 6.1 must hold. As Lemma 6.1 (i) and (ii) implies Theorem 1.7(i) and Lemma 6.1 (iii) implies Theorem 1.7(ii), we conclude that Theorem 1.7 follows from Lemma 6.1. Thus it remains to justify Lemma 6.1.

Proof of Lemma 6.1. It is known that G_0 is 3-edge-connected. We assume that $G_0 \notin \mathcal{T}_3$ to prove either (ii) or (iii) must hold. As in (2), we define $Z = Z_{\frac{1}{5}, -\frac{6}{5}}(G) = \{v \in V(G) : d_G(v) < \frac{n-6}{5}\}$. Since G satisfies (7), $G[Z]$ induces a complete graph.

Since $G_0 \notin \mathcal{T}_3$, we observe that $G_0' \notin \mathcal{T}_3$ and in particular, $G_0' \neq K_1$. Thus G_0' is a 3-edge-connected nontrivial graph. Throughout the rest of the arguments, for each vertex $z \in V(G_0')$, we always use H_z to denote the restoration of z in G. Let $Z' = \{z \in V(G_0') : V(H_z) \cap Z \neq \emptyset\}$. As $K_6 \in \mathcal{T}_3$, we conclude that $|Z'| \leq 5$.

By Observation 2.3(iii) with $s = 3$, we have $2|E(G_0')| \leq 6|V(G_0')| - 4$. Let $n' = |V(G_0')|$ and for $i \geq 0$, $d_i = |D_i(G_0')|$. Since $\kappa'(G_0') \geq 3$, we have

$$\sum_{i\geq 3} id_i = 2|E(G_0')| \leq 6d_i - 4 \text{ and so } 3d_3 + 2d_4 + d_5 \geq 4 + \sum_{i\geq 7}(i-6)d_i. \tag{12}$$

Claim 5. $\Delta(G_0') \leq 29$.

Proof. Suppose that $\Delta(G_0') \geq 30$. Then by (12), we have

$$3(d_3 + d_4 + d_5) \geq 3d_3 + 2d_4 + d_5 \geq 4 + (30-6) = 28, \text{ and so } d_3 + d_4 + d_5 \geq 10. \tag{13}$$

Since $|Z'| \leq 5$, there exist at least $10 - |Z'| \geq 5$ vertices $z_1, z_2, ..., z_5$ in $G_0' - Z'$ of degree at most 5 in G_0'. For each $i \in \{1, 2, ..., 5\}$, pick a vertex $v_i \in V(H_{z_i})$. By symmetry, we assume that

$$|V(H_{z_1})| \leq |V(H_{z_2})| \leq ... \leq |V(H_{z_5})|.$$

Since $z_5 \in G_0' - Z'$, every vertex $v \in V(H_{z_5})$ is not in Z, and so as $n \geq 36$, we have $d_G(v) \geq \frac{n-6}{5} \geq 6 > 5 \geq d_{G_0'}(z_5) = |\partial_G(V(H_{z_5}))|$. Thus by Lemma 2.4, for any i with $1 \leq i \leq 5$, we have $|V(H_{z_i})| \geq |V(H_{z_5})| \geq \frac{n-1}{5}$. It follows that

$$n = |V(G)| \geq \sum_{i=1}^{5} |V(H_{z_i})| + (|V(G_0')| - 5) \geq (n-1) + 5 > n,$$

a contradiction. This proves Claim 5. □

Claim 6. *Each of the following holds.*
(i) $|V(G_0') - Z'| \leq 5$ *and* $n' \leq 9$.
(ii) $|V(G_0') - Z'| = 5$ *if and only if* $|Z'| \leq 1$.

Proof. By Claim 5, $\Delta(G_0') \leq 29$. As $n \geq 156$, for each $z \in V(G_0') - Z'$, $|\partial_G(V(H_z))| \leq \Delta(G_0') \leq 29 < \frac{n-6}{5}$. Thus by Lemma 2.4, we have $|V(H_z)| \geq \frac{n-1}{5}$. If $|V(G_0') - Z'| \geq 6$, then by $n \geq 156$, $n = |V(G)| \geq \sum_{z\in V(G_0')-Z'} |V(H_z)| \geq \frac{6(n-1)}{5} \geq (n-1) + \frac{n-1}{5} > n$, a contradiction. Hence we must have $|V(G_0') - Z'| \leq 5$. Suppose further that $|V(G_0') - Z'| = 5$. Then the same argument leads to $n = |V(G)| \geq \sum_{z\in V(G_0')-Z'} |V(H_z)| \geq n - 1 + |Z'|$, implying $|Z'| \leq 1$. It follows that either $|V(G_0') - Z'| = 5$ and $|Z'| \leq 1$, whence $n' \leq 6$; or $|V(G_0') - Z'| \leq 4$ and $|Z'| \leq 5$, whence $n' \leq 9$. This proves Claim 6. □

Claim 7. *Each of the following holds.*
(i) If $|V(G_0') - Z'| = 5$*, then either Lemma 6.1(ii) or Lemma 6.1(iii) holds.*
(ii) If $|V(G_0')| \leq 6$*, then either Lemma 6.1(ii) or Lemma 6.1(iii) holds.*

Proof. By Claim 6(ii), we have $|Z'| \leq 1$, and so G_0' is a 3-edge-connected graph on at most 6 vertices. By Lemma 5.4(vi), either $\mu'(G_0') \geq 3$, whence by Lemma 5.6(ii), we have $\kappa^*(L(G)) \geq \mu'(G_0) \geq 3$; or $|Z'| = 1$ and $G_0' \cong K_{3,3}$. It follows by (7) that $G \in \mathcal{J}(n)$, and so Lemma 6.1(iii) holds.

Assume that $|V(G_0')| \leq 6$. Since $\kappa'(G_0') \geq \kappa'(G_0) \geq 3$, it follows by Lemma 5.4(vi) that $\mu'(G_0') \geq 3$ unless $G_0' \cong K_{3,3}$. If $\mu'(G_0') \geq 3$, then as $|V(G_0')| \leq 6$ and by Lemma 5.4(v), G_0' is strongly spanning trailable. Thus if $G_0' \not\cong K_{3,3}$, then by Lemma 5.6(ii), we have $\kappa^*(L(G)) \geq \mu'(G_0) \geq 3$. Assume that $G_0' \cong K_{3,3}$. If $|Z'| \geq 2$, then by Lemma 5.10, we also have $\kappa^*(L(G)) \geq \mu'(G_0) \geq 3$. If $|Z'| \leq 1$, then by Claim 6, we must have $|V(G_0') - Z'| = 5$ and so Lemma 6.1(iii) holds. □

By the fact that G_0' is $\mathcal{T}_3$-reduced, by Claims 6 and 7, we may assume that $|V(G_0')| \geq 7$, $|V(G_0') - Z'| \leq 4$, $|Z'| \leq 5$ and so G_0' has a subgraph $G_0'[Z']$ spanned by a complete subgraph of order in $\{3, 4, 5\}$. Let K denote a maximum clique of G_0'. Then $|V(K)| \geq |Z'|$. For any vertex $z \in V(G_0') - V(K)$, let $\ell(z)$ denote the shortest length from z to $V(K)$ in G_0', and let $Z_i = \{z \in V(G_0') - V(K) : \ell(z) = i\}$, for each $i \geq 0$.

Claim 8. *Each of the following holds. Let* $\ell_0 = \max\{\ell(w) : w \in V(G_0') - V(K)\}$.
(i) $\ell_0 \leq 3$.
(ii) If $w_0 \in V(G_0') - V(K)$ *satisfies* $\ell(w_0) = 3$*, then* $Z_3 = \{w_0\}$, $|V(G_0') - V(K)| = 4$, $|N_{G_0'}(w_0)| = 2$ *and* $N_{G_0'}(N_{G_0'}(w_0))$ *consists of only one vertex, which is a cut vertex of* G_0'*; and* $G_0' - V(K)$ *is spanned by a 4-cycle.*

(iii) If $\ell_0 = 2$, then either $|Z_2| = 1$, or in $G_0'[Z_2 \cup N_{G_0'}(Z_2)]$, every vertex in Z_2 has degree at least 3 and lies in a cycle of length at most 3.

Proof. Assume by contradiction that there exists a vertex $w_1 \in V(G_0') - V(K)$ with $\ell(w_1) \geq 4$. Then let $w_1 w_2 ... w_h z$ be a shortest (w_1, z)-path for some vertex $z \in Z'$ with $w_1, ..., w_h \in V(G_0') - Z'$, with $h = |V(G_0') - Z'| \leq 4$. Hence $h = \ell(w_1) \geq 4$. As $\kappa'(G_0') \geq 3$, there must be at least three parallel edges joining w_1 and w_2, contrary to the fact that G_0' is $\mathcal{T}_3$-reduced. Hence we must have $\ell_0 \leq 3$.

Suppose that there exists a vertex $w_0 \in V(G_0') - V(K)$ satisfies $\ell(w_0) = 3$. As G_0' does not have three parallel edges, we conclude that $|N_{G_0'}(w_0)| \geq 2$. If $N_{G_0'}(w_0)$ has three distinct vertices, then by $|V(G_0') - V(K)| \leq |V(G_0') - Z'| \leq 4$ and by the fact that G_0' is connected, one of the vertices in $N_{G_0'}(w_0)$ must be adjacent to a vertex in K, resulting in $\ell(w_0) \leq 2$. Thus $|N_{G_0'}(w_0)| = 2$ and $|V(G_0') - V(K)| = 4$ with the only vertex in $V(G_0') - V(K)$ having distance one to $V(K)$ serving as a cut vertex of G_0'. If $|Z_3| \geq 2$, then by $|V(G_0') - V(K)| = 4$, we must have $|Z_2| = 1$ and there are three parallel edges joining the vertex in Z_2 and the vertex in Z_1, contrary to the fact that G_0' is $\mathcal{T}_3$-reduced. As $|Z_3| = |Z_1| = 1$ and $|Z_2| = 2$ and as G_0' is $\mathcal{T}_3$-reduced, $G_0' - V(K)$ is spanned by a 4-cycle. Hence we must have Claim 8(ii).

To show (iii), we assume that $\ell_0 = 2$ and $|Z_2| > 1$. Then $|Z_2| + |N_{G_0'}(Z_2)| = |V(G_0') - V(K)| \leq 4$. As G_0' is connected, $2 \geq |N_{G_0'}(Z_2)| > 0$. If $|N_{G_0'}(Z_2)| = 1$, then $N_{G_0'}(Z_2)$ consists of a single cut vertex of G_0'. As $\kappa'(G_0') \geq 3$, it follows that every vertex in $G_0'[Z_2 \cup N_{G_0'}(Z_2)]$ lies in a cycle of length at most 3, and so the conclusions of Claim 8(iii) must hold. Now assume that $|N_{G_0'}(Z_2)| = 2$. Then $|Z_2| = 2$. By $\kappa'(G_0') \geq 3$ and the fact that G_0' does not have an edge parallel class of size at least 3, we conclude that every vertex in Z_2 has degree at least 3 and is adjacent to both vertices in Z_2. Hence the conclusions of Claim 8(iii) must hold also. □

By Claims 6 and 7, we have $9 \geq |V(G_0')| \geq 7$ and $|V(G_0') - Z'| \leq 4$. By Lemma 5.6, if $G_0' \in \mathcal{P}_2$, then $\kappa^*(L(G)) \geq \mu'(G_0') \geq 3$, and so Lemma 6.1(ii) holds. To complete the proof of the lemma, we shall show that $G_0' \in \mathcal{P}_2$. Recall that K is a maximum clique of G_0'. By Claim 8, every vertex in $V(G_0') - Z'$ has distance at most 3 to a vertex in K. For any vertices $u, v \in V(G_0')$, if $uv \in E(G_0')$, then as $\kappa'(G_0') \geq 3$, $G_0' - uv$ cannot be contracted to $K_{2,3}$, it follows by Lemma 5.3 that $G_0' - uv$ is a $\mathcal{C}_1$ graph. Therefore, we assume that $uv \notin E(G_0')$, and so we may assume that $v \in V(G_0') - V(K)$. Let P be a shortest (u, v)-path in G_0' and let $J = (G_0' - E(P)) - D_1(G_0' - E(P))$. It suffices to show that J is a $\mathcal{C}_1$-graph to complete the proof.

If $\ell_0 = 1$, then as $v \notin V(K)$, either $u \notin V(K)$ and $P = uz_1z_2v$ with $z_1, z_2 \in V(K)$, or $u \in V(K)$ and $P = uz_1v$ with $z_1, u \in V(K)$. In any case, as $\kappa'(G_0') \geq 3$, J cannot be contracted to a $K_{2,3}$ and so by Lemma 5.3, J is a $\mathcal{C}_1$-graph.

Next, we assume that $\ell_0 = 2$. If $u \notin V(K)$, then $|E(P)| = 2$; and if $u \in V(K)$, then $2 \leq |E(P)| \leq 3$. By Claim 8(iii), in any case, $\kappa'(J) \geq 2$, using either Lemma 5.3 or Lemma 5.7, we conclude that J is a $\mathcal{C}_1$-graph.

Assume now $\ell_0 = 3$. Thus there exists a vertex $w_0 \in V(G_0') - V(K)$ satisfies $\ell(w_0) = 3$. By Claim 8(ii), either $u \in V(G_0') - V(K)$ and so $|E(P)| = 2$; or $u \in V(K)$ with either $E(P) \cap E(K) = \emptyset$ or $w_0 \neq v$, and so $|E(P)| \leq 3$; or $u \in V(K)$ with $|V(K)| \geq 4$, $E(P) \cap E(K) \neq \emptyset$ and $w_0 = v$, and so $|E(P)| = 4$. In any case, by the structure of G_0' as described in Claim 8(ii), $\kappa'(J) \geq 2$, and J contains a cycle of length at most 3. It follows by Lemma 5.7 that J is a $\mathcal{C}_1$-graph. Thus by definition, $G_0' \in \mathcal{P}_2$. As remarked above, the proof of the lemma is complete. □

Declaration of competing interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data availability

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