Discrete Mathematics 346 (2023) 113402

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

# On hamiltonian properties of $K_{1,r}$-free split graphs

Xia Liu [a], Sulin Song [b,*], Mingquan Zhan [c], Hong-Jian Lai [d]

[a] Department of Mathematics, Northwest Normal University, Lanzhou, Gansu 730070, PR China
[b] Department of Mathematics, West Texas A&M University, Canyon, TX 79016, USA
[c] Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, USA
[d] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

A R T I C L E I N F O

*Article history:*
Received 20 December 2022
Received in revised form 17 February 2023
Accepted 25 February 2023
Available online xxxx

*Keywords:*
$K_{1,r}$-free graphs
Hamiltonian cycles
Split graphs
Fully cycle extendable

A B S T R A C T

Let $r \geq 3$ be an integer. A graph $G$ is $K_{1,r}$**-free** if $G$ does not have an induced subgraph isomorphic to $K_{1,r}$. A graph $G$ is fully cycle extendable if every vertex in $G$ lies on a cycle of length 3 and every non-hamiltonian cycle in $G$ is extendable. A connected graph $G$ is a split graph if the vertex set of $G$ can be partitioned into a clique and a stable set. Dai et al. (2022) [4] conjectured that every $(r-1)$-connected $K_{1,r}$-free split graph is hamiltonian, and they proved this conjecture when $r = 4$ while Renjith and Sadagopan proved the case when $r = 3$. In this paper, we introduce a special type of alternating paths in the study of hamiltonian properties of split graphs and prove that a split graph $G$ is hamiltonian if and only if $G$ is fully cycle extendable. Consequently, for $r \in \{3, 4\}$, every $r$-connected $K_{1,r}$-free split graph is Hamilton-connected and every $(r-1)$-connected $K_{1,r}$-free split graph is fully cycle extendable.



## 1. The problem

In this paper, we consider only finite simple graphs and refer to [2] for notation and terminologies not locally defined here. We call a graph $G$ **hamiltonian** if it contains a hamiltonian cycle, i.e., a cycle contains all vertices of $G$. Furthermore, a graph $G$ of order $n \geq 3$ is **pancyclic** if $G$ contains a cycle of each possible length from 3 to $n$; $G$ is **vertex pancyclic** if each vertex is contained on a cycle of each possible length from 3 to $n$.

Let $r \geq 3$ be an integer. A graph $G$ is $K_{1,r}$**-free** if $G$ does not have an induced subgraph isomorphic to $K_{1,r}$. A connected graph $G$ is called a **split graph** if its vertex set $V(G)$ can be partitioned as the disjoint union of $S$ and $J$ (either of which may be empty) such that $S$ is a maximum clique of $G$ whereas $J$ is a stable set of $G$. Split graphs were introduced by Foldes and Hammer [6] in 1977, and were studied further in [3], [5], [8], [9], [10], [11].

**Theorem 1.1** *(Renjith and Sadagopan [10]). Let $G$ be a $K_{1,3}$-free split graph. Then $G$ is hamiltonian if and only if $G$ is 2-connected.*

Very recently, Dai et al. [4] proposed conditions for $K_{1,3}$-free split graphs to be pancyclic and $K_{1,4}$-free split graphs to be hamiltonian, respectively.

**Theorem 1.2** *(Dai, Zhang, Broerama and Zhang [4]). Let $G$ be a $K_{1,3}$-free split graph. Then $G$ is pancyclic if and only if $G$ is 2-connected.*

* Corresponding author.
*E-mail addresses:* liuxia_90@163.com (X. Liu), ssong@wtamu.edu (S. Song), Mingquan.Zhan@millersville.edu (M. Zhan), hjlai@math.wvu.edu (H.-J. Lai).

https://doi.org/10.1016/j.disc.2023.113402
0012-365X/

**Theorem 1.3** *(Dai, Zhang, Broerama and Zhang [4]). Let $G$ be a $K_{1,4}$-free split graph. If $G$ is 3-connected, then $G$ is hamiltonian.*

The following conjecture is posed in [4].

**Conjecture 1.4** *(Dai, Zhang, Broerama and Zhang [4]). Let $r \geq 2$ be an integer. Every $r$-connected $K_{1,r+1}$-free split graph is hamiltonian.*

Theorems 1.1 and 1.3 indicate that Conjecture 1.4 is valid for $r \in \{2, 3\}$. This motivates this research. In Section 2, we introduce a certain type of alternating paths in split graphs, which will be utilized to study the hamiltonian properties of split graphs. We investigate the fully cycle extendability and Hamilton-connectedness for $K_{1,3}$-free split graphs in Section 3, and those for $K_{1,4}$-free split graphs in Section 4. Our results extend Theorems 1.1, 1.2 and 1.3 to vertex pancyclicity and Hamilton-connectedness.

## 2. Alternating paths

For two graphs $G_1$ and $G_2$, let $G_1 \cup G_2$ be a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. We can consider a collection of subgraphs $G_1, G_2, \ldots, G_h$ in $G$ as a subgraph $\bigcup_{i=1}^{h} G_i$ of $G$. If $S$ is an edge (or a vertex) subset of $G$, $G[S]$ is the subgraph induced in $G$ by $S$. For a subset $A \subseteq V(G)$, we denote $G - A = G[V(G) - A]$; for a subset $X \subseteq E(G)$ and a subgraph $H \subseteq G$ with $X \cap E(H) = \emptyset$, we denote $G - X = G[E(G) - X]$ and write $H + X$ for $H \cup G[X]$.

Throughout this section, let $r \geq 2$ denote an integer. A path with endpoints $u$ and $v$ is often referred as to a $(u, v)$-path. Let $G$ be a split graph with $V(G) = S \cup J$, where $S$ is a maximum clique of $G$, and $J$ is a stable set of $G$. We shall call such an ordered pair $(S, J)$ a **split partition** of $G$. Denote $s = |S|$ and $j = |J|$. Then $G[S] \cong K_s$. Since complete graphs are hamiltonian, we may assume that $j > 0$. Let $J = \{u_1, u_2, ..., u_j\}$. Then $N_G(u) \subseteq S$ for any $u \in J$. Suppose that $\kappa(G) \geq r$. As $j > 0$, every vertex $u \in J$ must be adjacent to at least $r$ distinct vertices in $S$, and so $s \geq r$. By the maximality of $|S|$, $s \geq r + 1$. We define the **interior** of a path $P = v_1 v_2 ... v_{2t+1}$ to be $P^o = V(P) - \{v_1, v_{2t+1}\}$, and $P$ to be an **$(S, J)$-alternating path** in $G$ if $v_1 \neq v_{2t+1}$, $\{v_1, v_3, \ldots, v_{2t+1}\} \subseteq S$ and $\{v_2, v_4, \ldots, v_{2t}\} \subseteq J$. If we allow $v_1 = v_{2t+1}$ while keeping both $\{v_1, v_3, \ldots, v_{2t+1}\} \subseteq S$ and $\{v_2, v_4, \ldots, v_{2t}\} \subseteq J$, then $P$ is an **$(S, J)$-alternating cycle**.

A collection $\mathcal{P} = \{P_1, P_2, \ldots, P_h\}$ of $(S, J)$-alternating paths in $G$ is a **$J$-cover** if both of the following hold:

(A1) for any $\{i, j\} \subseteq \{1, 2, \ldots, h\}$, $V(P_i) \cap V(P_j) = \emptyset$, and
(A2) $J \subseteq \bigcup_{i=1}^{h} P_i^o$.

For a collection $\mathcal{P} = \{P_1, P_2, \ldots, P_h\}$ of $(S, J)$-alternating paths in $G$, let $End(\mathcal{P})$ denote the collection of the endpoints of all these alternating paths in $\mathcal{P}$, and $Inn(\mathcal{P}) = (V(\mathcal{P}) \cap S) - End(\mathcal{P}) = S \cap (\bigcup_{i=1}^{h} P_i^o)$. We also use $\mathcal{P}$ to denote the subgraph $\bigcup_{i=1}^{h} P_i$.

**Lemma 2.1.** *Let $G$ be a split graph with a split partition $(S, J)$. Then, $G$ is hamiltonian if and only if $G$ has either a $J$-cover or a spanning $(S, J)$-alternating cycle.*

**Proof.** Assume that $C$ is a hamiltonian cycle of $G$. Then the subgraph $C - J$ consists of nontrivial paths and trivial paths. Let $A$ be the set of all degree 2 vertices in $C - J$, and $E'$ be the collection of edges of all nontrivial paths in $C - J$. Then $(C - A) - E'$ is either a $J$-cover or a spanning $(S, J)$-alternating cycle.

Conversely, it suffices to show that $G$ is hamiltonian if $G$ has a $J$-cover $\{P_1, P_2, \ldots, P_h\}$. Let $z_i', z_i''$ be the endpoints of each $P_i$. Since $\bigcup_{i=1}^{h}\{z_i', z_i''\} \subseteq S$ is a clique of $G$, $\bigcup_{i=1}^{h-1}\{z_i'' z_{i+1}'\} \subseteq E(G)$. Then $P = (\bigcup_{i=1}^{h} P_i) \cup (\bigcup_{i=1}^{h-1}\{z_i'' z_{i+1}'\})$ is a $(z_1', z_h'')$-path with $J \subseteq V(P)$ by (A1) and (A2). As $V(G) - P^o$ is a clique of $G$, it follows that $G - P^o$ has a spanning $(z_h'', z_1')$-path $P'$. Thus, $P \cup P'$ is a hamiltonian cycle of $G$. $\square$

A graph $G$ is **Hamilton-connected**, if for every pair of distinct vertices $u, v$ in $G$, there exists a hamiltonian path from $u$ to $v$.

**Lemma 2.2.** *Let $G$ be a split graph with a split partition $(S, J)$ and a $J$-cover $\mathcal{P}$, and let $u, v$ be any two distinct vertices in $V(G) - Inn(\mathcal{P})$. If $u, v$ are not the endpoints of an $(S, J)$-alternating path in $\mathcal{P}$, then $G$ has a hamiltonian $(u, v)$-path.*

**Proof.** Let $S_1 = \bigcup_{P \in \mathcal{P}} V(P) \cap S$, $S_2 = S - S_1$, and let $w$ be a new vertex distinct from $V(G)$. Without loss of generality, we assume first that $u \in J$. If $v \in J$, we assume that $a \in N_G(u) \cap S_1$ and $b \in N_G(v) \cap S_1$ and $a \neq b$. Let $G_1 = G - \{u, v\} + \{aw, bw\}$. Then $G_1$ is a split graph with a split partition $(S, J')$, where $J' = J \cup \{w\} - \{u, v\}$, and $\mathcal{P} - \{u, v\} + \{aw, bw\}$ is a $J'$-cover of $G_1$. Thus $G_1$ is hamiltonian. Let $C$ be a hamiltonian cycle of $G_1$. Then $C - \{w\} + \{au, bv\}$ is a hamiltonian $(u, v)$-path in $G$. If $v \in S_2 \cup End(\mathcal{P})$, we assume that $a \in N_G(u) \cap S_1$ so that $a$ does not belong to the alternating path in $\mathcal{P} - \{u\}$ that contains $v$ if $v \in End(\mathcal{P})$. Let $G_2 = G - \{u\} + \{aw, vw\}$. Then $\mathcal{P} - \{u\} + \{aw, vw\}$ is a $(J \cup \{w\} - \{u\})$-cover of the split graph $G_2$. Thus $G_2$ is hamiltonian. Let $C$ be a hamiltonian cycle of $G_2$. Then $C - \{w\} + \{au\}$ is a hamiltonian $(u, v)$-path in $G$.

We then assume that $u \in S_2$ and $v \in S_2 \cup End(\mathcal{P})$, or $u, v$ are the endpoints of different alternating paths in $\mathcal{P}$, then we set $G_3 = G + \{uw, vw\}$. Thus $\mathcal{P} + \{uw, vw\}$ is a $(J \cup \{w\})$-cover of the split graph $G_3$. So $G_3$ is hamiltonian. Let $C$ be a hamiltonian cycle of $G_3$. Then $C - \{w\}$ is a hamiltonian $(u, v)$-path in $G$. $\square$

**Lemma 2.3.** *Let $G$ be a split graph with a split partition $(S, J)$, a $J$-cover $\mathcal{P}$ and $\delta(G) \geq 3$. If the length of each alternating $(S, J)$-path in $\mathcal{P}$ is 2, then $G$ is Hamilton-connected.*

**Proof.** As the length of each path in $\mathcal{P}$ is 2, $Inn(\mathcal{P}) = \emptyset$. By Lemma 2.2, it suffices to find a hamiltonian $(u, v)$-path if $u, v$ are the endpoints of an alternating path $P_i$ in $\mathcal{P}$. Let $P_i = uzv$. As $\delta(G) \geq 3$, let $b \in N_G(z) - \{u, v\}$. Then $\mathcal{P}' = \mathcal{P} - \{zv\} + \{zb\}$ is a $J$-cover of $G$ with $u \in End(\mathcal{P}')$ and $v \in S - V(\mathcal{P}')$. By Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path. $\square$

For a split graph $G$ with a split partition $(S, J)$, we define a collection $\mathcal{Q} = \{P_1, P_2, \ldots, P_{h_1}, C_1, C_2, \ldots, C_{h_2}\}$, with $h_1 \geq 0$ and $h_2 \geq 0$, of vertex-disjoint subgraphs of $G$ to be a **pseudo $J$-cover** if $\mathcal{Q}$ satisfies each of the following.

(Q1) Each $P_i$ is an $(S, J)$-alternating path and each $C_j$ is an $(S, J)$-alternating cycle.
(Q2) $J \subseteq \left(\bigcup_{i=1}^{h_1} P_i^o\right) \cup \left(\bigcup_{j=1}^{h_2} V(C_j)\right)$.

For a pseudo $J$-cover $\mathcal{Q} = \{P_1, P_2, \ldots, P_{h_1}, C_1, C_2, \ldots, C_{h_2}\}$ of $G$, let $End(\mathcal{Q})$ denote the collection of the endpoints of the alternating paths $\bigcup_{i=1}^{h_1} P_i$, and $Inn(\mathcal{Q}) = S \cap (\bigcup_{i=1}^{h_1} P_i^o)$. If we need to emphasize the values of $h_1$ and $h_2$, we write $\mathcal{Q}(h_1, h_2)$ for $\mathcal{Q}$. By definitions, every $J$-cover is also a pseudo $J$-cover $\mathcal{Q}(h_1, h_2)$ with $h_2 = 0$. We also use $\mathcal{Q}$ to denote the subgraph $(\bigcup_{i=1}^{h_1} P_i) \cup (\bigcup_{i=1}^{h_2} C_i)$. For a set $\mathcal{C}$ of some vertex-disjoint cycles in a graph $G$, we say that $\mathcal{C}$ is a **2-factor** of $G$ if $V(G) = \bigcup_{C \in \mathcal{C}} V(C)$. We shall apply the following theorem to show that a pseudo $J$-cover will exist for an $r$-connected $K_{1,r+1}$-free split graph.

**Theorem 2.4** *(Aldred et al. [1]). If $G$ is an $r$-connected $K_{1,r+1}$-free graph, then $G$ has a 2-factor.*

**Lemma 2.5.** *Let $G$ be an $r$-connected $K_{1,r+1}$-free split graph with a split partition $(S, J)$. Then $G$ contains a pseudo $J$-cover.*

**Proof.** By Theorem 2.4, $G$ has a 2-factor $F$. Then the subgraph $F - J$ consists of some cycles, nontrivial paths, and trivial paths. Let $A$ be the set of all degree 2 vertices in $F - J$, and $E'$ be the collection of edges of all nontrivial paths in $F - J$. Then $(F - A) - E'$ is a pseudo $J$-cover of $G$. $\square$

Following Hendry [7], we call a cycle $C$ in $G$ **extendable** if there is a cycle $C'$ in $G$ such that $|V(C')| = |V(C)| + 1$ and $V(C) \subset V(C')$. If such a cycle $C'$ exists, we say that $C$ can be extended to $C'$ or that $C'$ is an extension of $C$. A graph $G$ is **cycle extendable** if it has at least one cycle and every non-hamiltonian cycle in $G$ is extendable. A graph $G$ is **fully cycle extendable** if $G$ is cycle extendable and every vertex in $G$ lies on a cycle of length 3. By definitions, every fully cycle extendable graph is vertex pancyclic.

**Theorem 2.6.** *Let $G$ be a split graph. Then, $G$ is hamiltonian if and only if $G$ is fully cycle extendable.*

**Proof.** Suppose $G$ is a hamiltonian split graph with a split partition $(S, J)$. If $|S| = 2$, then $G \cong K_3$. It follows that $G$ is fully cycle extendable. Now we assume that $|S| \geq 3$. Thus every vertex in $G$ lies on a cycle of length 3 by the definition of split graphs.

To show that $G$ is fully cycle extendable, it is enough to prove that $G$ is cycle extendable. Suppose by contrary that $G$ has a non-hamiltonian cycle $C$ such that there is no cycle $C'$ with $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. As $G$ is hamiltonian, $|V(G)| - |V(C)| \geq 2$.

**Claim 1.** $V(G) - V(C) \subseteq J$.

Assume that there is a vertex $w \in S - V(C)$. If there exists $uv \in E(C)$ such that $u, v \in S$, then $C' = C - \{uv\} + \{uw, vw\}$ is a cycle with $V(C') = V(C) \cup \{w\}$, a contradiction. We then may assume that $C = x_1 y_1 x_2 y_2 \ldots x_k y_k x_1$, where $x_1, \ldots, x_k \in S$ and $y_1, \ldots, y_k \in J$. For $y_i$, if there exists $z \in S - \{x_1, \ldots, x_k\}$ such that $zy_i \in E(G)$, then $C' = C - \{x_i y_i\} + \{x_i z, z y_i\}$ is a cycle with $C' = C \cup \{z\}$, a contradiction. So, for each $y_i$, we have $N_G(y_i) \subseteq \{x_1, \ldots, y_k\}$. Therefore, the number of components of $G - \{x_1, \ldots, x_k\}$ is at least $k + 1$, which is contrary to the hamiltonicity of $G$. Claim 1 holds.

Pick a vertex $w \in V(G) - V(C) \subseteq J$. Let $H$ be the subgraph induced by $V(C) \cup \{w\}$. Then $H$ is a split graph with the split partition $(S, J')$, where $J' = V(C) \cap J \cup \{w\}$. Let $J^* = J - J'$. Since $G$ is hamiltonian, by Lemma 2.1, $G$ has either a $J$-cover $\mathcal{P}$ or a spanning $(S, J)$-alternating cycle $C^*$. Thus $\mathcal{P} - J^*$ or $C^* - J^*$ consists of some nontrivial paths and trivial paths. Let $\mathcal{P}'$ be the collection of all those nontrivial paths. Then $\mathcal{P}'$ is a $J'$-cover of $H$. By Lemma 2.1, $H$ has a hamiltonian cycle $C'$ with $V(C') = V(C) \cup \{w\}$, contrary to the choice of $C$. $\square$

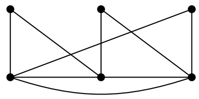

**Fig. 1.** The graph $G_6$.

## 3. $K_{1,3}$-free split graphs

**Theorem 3.1** *(Wu [12]). Let $G$ be a 3-connected $K_{1,3}$-free graph. If $\sum_{v\in X} d_G(v) \geq |V(G)|+1$ for any stable set $X$ with $|X|=3$, then $G$ is Hamilton-connected.*

By the structure property of $K_{1,3}$-free graphs, we have the following lemma.

**Lemma 3.2.** *Let $G$ be a $K_{1,3}$-free split graph with a split partition $(S, J)$. Let $a, b \in J$ such that $N_G(a) \cap N_G(b) \neq \emptyset$. Then for any two vertices $x, y \in J - \{a, b\}$, $N_G(x) \cap N_G(y) = \emptyset$.*

**Lemma 3.3.** *Let $G$ be a 2-connected $K_{1,3}$-free split graph with a split partition $(S, J)$. Then $G$ has a $J$-cover if and only if $G \neq G_6$, depicted in Fig. 1.*

**Proof.** By Lemma 2.5, $G$ has a pseudo $J$-cover $\mathcal{Q} = \{P_1, P_2, \ldots, P_{h_1}, C_1, C_2, \ldots, C_{h_2}\}$. Let $S_2 = S - V(\mathcal{Q})$. By Lemma 3.2, $h_2 \leq 1$. If $h_2 = 0$, then $\mathcal{Q}$ is a $J$-cover of $G$. Next we assume that $h_2 = 1$.

Let $C_1 = a_1b_1a_2b_2\cdots a_kb_ka_1$, where $a_1, \cdots, a_k \in J$ and $b_1, \cdots, b_k \in S$. By Lemma 3.2, $2 \leq k \leq 3$. If either $S_2 \neq \emptyset$ or $h_1 \geq 1$, we choose $x \in S_2 \cup End(\mathcal{Q})$. Since $G[\{b_1, a_1, a_2, x\}] \neq K_{1,3}$, we have $xa_1 \in E(G)$ or $xa_2 \in E(G)$. Without loss of generality, we assume that $xa_1 \in E(G)$. Thus $\mathcal{Q} - \{a_1b_1\} + \{xa_1\}$ is a $J$-cover of $G$. So we assume that $S_2 = \emptyset$ and $h_1 = 0$. Then, $C_1$ is a spanning $(S, J)$-alternating cycle of $G$ and $|S| \in \{2, 3\}$.

Notice that if $(S, J)$ is a split partition of a split graph $G$, then $S$ is a maximum clique of $G$. If $|S| = 2$, then $G = K_4 - \{e\}$, where $e \in E(K_4)$. By the definition of a split graph, $S = K_3$ and $J = K_1$, a contradiction. So $|S| = 3$. Since $G$ is $K_{1,3}$-free, the degree of each vertex in $J$ is 2. Thus $G = G_6$. □

**Corollary 3.4.** *Every 2-connected $K_{1,3}$-free split graph is fully cycle extendable.*

**Proof.** Corollary 3.4 follows directly from Lemma 2.1, Theorem 2.6, and Lemma 3.3. □

**Lemma 3.5.** *Let $G$ be a 2-connected $K_{1,3}$-free graph with a split partition $(S, J)$. If $\delta(G) \geq 3$ and $|V(G)| \geq 9$, then $G$ has a $J$-cover such that the length of each $(S, J)$-alternating path is 2.*

**Proof.** By Lemma 3.3, we may choose a $J$-cover $\mathcal{P} = \{P_1, P_2, \cdots, P_h\}$ so that $h$ is maximized. Let $S_1 = \cup_{i=1}^{h} V(P_i) \cap S$ and $S_2 = S - S_1$. Assume that the length of $P_i$ is $t_i$, and $t_1 \geq t_2 \geq \cdots \geq t_h$. By Lemma 3.2, $t_2 = \cdots = t_h = 2$, and $2 \leq t_1 \leq 6$. If $t_1 = 2$, then this lemma is true. We then assume that $t_1 \in \{4, 6\}$ and let $P_1 = a_1b_1a_2b_2\cdots b_sa_{s+1}$, where $s \in \{2, 3\}$, $a_1, \cdots, a_{s+1} \in S$ and $b_1, \cdots, b_s \in J$. Then $S_2 = \emptyset$. Otherwise, pick a vertex $w \in S_2$. Since $G[\{a_2, b_1, b_2, w\}] \neq K_{1,3}$, we have $wb_1 \in E(G)$ or $wb_2 \in E(G)$. Without loss of generality, we assume $wb_1 \in E(G)$. Then $\{a_1b_1w, P_1 - \{a_1, b_1\}, P_2, \cdots, P_h\}$ is a $J$-cover, contrary to the choice of $\mathcal{P}$.

Assume first that $t_1 = 6$. Then $s = 3$. As $|V(G)| \geq 9$, we have $h \geq 2$. As $t_2 = 2$, we set $P_2 = xyz$. Consider $G[\{a_3, b_2, b_3, x\}]$. By Lemma 3.2, $xb_3 \notin E(G)$. Thus $xb_2 \in E(G)$. By Lemma 3.2, $a_4y \notin E(G)$. Since $G[\{x, a_4, y, b_2\}] \neq K_{1,3}$, we have $b_2a_4 \in E(G)$. Similarly, $a_1b_2 \in E(G)$. By Lemma 3.2, $N_G(b_1) = \{a_1, a_2\}$, contrary to $\delta(G) \geq 3$.

Assume then that $t_1 = 4$ and $s = 2$. As $|V(G)| \geq 9$, $h \geq 3$. Let $P_i = x_iy_iz_i (i = 2, \cdots, h)$. Since $G[\{a_2, b_1, b_2, x_2\}] \neq K_{1,3}$, without loss of generality, we assume that $b_2x_2 \in E(G)$. By Lemma 3.2, for any $i \in \{3, \cdots, h\}$, $b_2x_i, b_2z_i \in E(G)$. As $G$ is $K_{1,3}$-free and $\delta(G) \geq 3$, by Lemma 3.2, $b_1a_3 \in E(G)$. Thus $N_G(y_2) = \{x_2, z_2\}$, a contradiction. □

**Theorem 3.6.** *Let $G$ be a 3-connected $K_{1,3}$-free split graph. Then $G$ is Hamilton-connected.*

**Proof.** By Theorem 3.1, we assume that $|V(G)| \geq 9$. Thus Theorem 3.6 follows directly from Lemma 2.3 and Lemma 3.5. □

## 4. $K_{1,4}$-free split graphs

By the structure property of $K_{1,4}$-free graphs, we have the following lemma.

**Lemma 4.1.** *Let $G$ be a $K_{1,4}$-free split graph with a split partition $(S, J)$. Let $a, b, c \in J$ such that $N_G(a) \cap N_G(b) \cap N_G(c) \neq \emptyset$. Then for any three vertices $x, y, z \in J - \{a, b, c\}$, $N_G(x) \cap N_G(y) \cap N_G(z) = \emptyset$.*

**Lemma 4.2.** *Let $G$ be a 3-connected $K_{1,4}$-free split graph with a split partition $(S, J)$. Then $G$ has either a $J$-cover or a spanning $(S, J)$-alternating cycle.*

**Proof.** By Lemma 2.5, $G$ has a pseudo $J$-cover. Let $\mathcal{Q} = \{P_1, \ldots, P_{h_1}, C_1, \ldots, C_{h_2}\}$ be a pseudo $J$-cover such that
(i) $h_2$ is minimized;
(ii) subject to (i), $h_1$ is minimized.
We assume that $h_2 \geq 1$. Let $S_1 = V(\mathcal{Q}) \cap S$ and $S_2 = S - S_1$. Let $C_1 = a_1b_1a_2b_2 \cdots a_kb_ka_1$, where $a_1, \cdots, a_k \in S$ and $b_1, \cdots, b_k \in J$. By the choice of $\mathcal{Q}$, we have

**Claim 1.** For any $x \in J \cap V(C_i)$, $N_G(x) \cap (S_2 \cup End(\mathcal{Q})) = \emptyset$. Thus $N_G(x) \subseteq S_1 - End(\mathcal{Q})$.

**Claim 2.** $S_2 = \emptyset$.

Assume that $w \in S_2$. Consider $b_1$. Since $\delta(G) \geq 3$, there is a vertex $y \in S_1 - \{a_1, a_2\}$ such that $yb_1 \in E(G)$. As $y \notin End(\mathcal{Q})$, we assume that $N_{\mathcal{Q}}(y) = \{v_1, v_2\}$. As $G[\{y, b_1, v_1, v_2, w\}] \neq K_{1,4}$ and $b_1w \notin E(G)$, we have $\{v_1w, v_2w\} \cap E(G) \neq \emptyset$. Without loss of generality, we assume that $v_2w \in E(G)$. By Claim 1, $y \in P_{i_0}$. Thus $(P_{i_0} \cup C_1) - \{a_1b_1, yv_2\} + \{b_1y, v_2w\}$ are two alternating paths, contradicting the hypothesis that $h_2$ is smallest. Claim 2 holds.

**Claim 3.** $N_G(b_i) \subseteq \{a_1, \cdots, a_k\}$. Therefore, $h_2 = 1$.

Assume that $x \in N_G(b_1) - \{a_1, \cdots, a_k\}$. By Claims 1 and 2, $x \in V(C_i) \cup Inn(\mathcal{Q})$ for some $i \neq 1$. Let $y_1y_2xy_3y_4$ be a section in $\mathcal{Q}$ that contains $x$. Consider $b_2$ and $b_k$. As $G$ is $K_{1,4}$-free, $x \notin N_G(b_2) \cup N_G(b_k)$. As $d_G(b_2) \geq 3$, $N_G(b_2) \cap (S_1 - End(\mathcal{Q})) \neq \emptyset$. By Lemma 4.1, $N_G(b_2) \cap \{y_1, y_4\} \neq \emptyset$ and $N_G(b_k) \cap \{y_1, y_4\} \neq \emptyset$. Without loss of generality, we assume that $b_2y_4 \in E(G)$. By Claim 1, $y_4 \notin End(\mathcal{Q})$. If $b_2 \neq b_k$, then, as $G$ is $K_{1,4}$-free, we have $b_ky_4 \notin E(G)$ and so $b_ky_1 \in E(G)$. By Claim 1, $y_1 \notin End(\mathcal{Q})$, contrary to Lemma 4.1. So $b_2 = b_k$ and $C_1$ is a 4-cycle.

Assume that $y_1y_2xy_3y_4y_5$ is a section of some $P_{i_0} = y \cdots y_1y_2xy_3y_4y_5 \cdots y_s$. Then $yy_3 \notin E(G)$, otherwise, $P_{i_0} \cup C_1 - \{a_1b_1, xy_3\} + \{yy_3, xb_1\}$ is an alternating path, contradicting to the choice of $\mathcal{Q}$ with $h_2$ being minimized. Similarly, we have $yy_5 \notin E(G)$ as $b_2y_4 \in E(G)$. Thus $G[\{y_4, y_3, y_5, y, b_2\}] = K_{1,4}$, a contradiction. So $y_1y_2xy_3y_4$ (probably $y_1 = y_4$) is a section of some $C_{j_0}$. Thus $h_1 = 0$ (otherwise, let $w$ be an endpoint of $P_1$. By Claim 1, $G[\{x, y_2, y_3, b_1, w\}] = K_{1,4}$, a contradiction.)

If $y_1 \neq y_4$, then $|C_{j_0}| \geq 6$. Let $C_{j_0} = y_1y_2xy_3y_4y_5 \cdots y_sy_1$ (probably $y_5 = y_s$). Then $\{b_2, y_3, y_5\} \subseteq N_G(y_4)$. As $h_2$ is smallest and $xb_1, y_4b_2 \in E(G)$, $N_G(w) \cap \{a_1, a_2\} = \emptyset$ for $w \in \{y_2, y_3, y_5\}$. As $G$ is $K_{1,4}$-free, $y_5x \notin E(G)$. As $d_G(y_5) \geq 3$, by Lemma 4.1, $N_G(y_5) = \{y_4, y_6, y_1\}$. So $y_5 \neq y_1$ and $|C_{j_0}| \geq 8$. Since $G[\{y_1, y_2, y_s, y_5, a_1\}] \neq K_{1,4}$, we have $y_sa_1 \in E(G)$. Thus $C_1 \cup C_{j_0} - \{y_1y_s, y_4y_5, a_1b_2\} + \{y_4b_2, a_1y_s, y_5y_1\}$ is an alternating cycle, contrary to the hypothesis that $h_2$ is the smallest. So $y_1 = y_4$ and $C_{j_0} = y_1y_2xy_3y_1$ is a 4-cycle and $b_2y_1 \in E(G)$.

As $h_2$ is smallest, $N_G(y_2) \cap \{a_1, a_2\} = \emptyset$ and $N_G(y_3) \cap \{a_1, a_2\} = \emptyset$. Consider $y_1$. Using the above discussion, there is a 4-cycle $C_{j_1} = z_1z_2z_3z_4z_1$ with $z_1, z_3 \in S$, $z_2, z_4 \in J$ such that $y_2z_1, y_3z_3 \in E(G)$. As $h_2$ is smallest, $N_G(z_2) \cap \{a_1, a_2, x, y_1\} = \emptyset$ and $N_G(z_4) \cap \{a_1, a_2, x, y_1\} = \emptyset$. Consider $z_4$. Using the above discussion, there is a 4-cycle $C_{j_2} = u_1u_2u_3u_4u_1$ with $u_1, u_3 \in S$, $u_2, u_4 \in J$ such that $z_2u_1, z_4u_3 \in E(G)$. Thus $\{b_1, y_2, y_3\} \subseteq N_G(x)$ and $\{u_2, u_4, z_4\} \subseteq N_G(u_3)$, contrary to Lemma 4.1. Claim 3 holds.

As $G$ is $K_{1,4}$-free, by Claims 1 and 3, we have $h_1 = 0$, and so $V(G) = V(C_1)$. □

**Corollary 4.3.** *Every 3-connected $K_{1,4}$-free split graph is fully cycle extendable.*

**Proof.** Corollary 4.3 follows directly from Lemma 2.1, Theorem 2.6, and Lemma 4.2. □

**Theorem 4.4.** *Let $G$ be a 4-connected $K_{1,4}$-free split graph. Then $G$ is Hamilton-connected.*

**Proof.** Assume that $(S, J)$ is the split partition of $G$. As $G$ is 4-connected, the number of edges between $J$ and $S$ is at least $4|J|$. As $G$ is $K_{1,4}$-free, the number of edges between $S$ and $J$ is at most $3|S|$. So $4|J| \leq 3|S|$, and hence $|J| < |S|$. By Lemma 4.2, we may assume that $\mathcal{P} = \{P_1, P_2, \cdots, P_h\}$ is a $J$-cover of $G$. Let $S_1 = V(\mathcal{P}) \cap S$ and $S_2 = S - S_1$. Let $P_i = x_1^i y_1^i x_2^i y_2^i \cdots x_{k_i}^i y_{k_i}^i x_{k_i+1}^i$, where $y_1^i, \cdots, y_{k_i}^i \in J$, $x_1^i, \cdots, x_{k_i+1}^i \in S_1$. Assume that $G$ is not Hamilton-connected. Then there exist $u, v \in V(G)$ such that $G$ does not have a hamiltonian $(u, v)$-path. Thus $uv \notin E(\mathcal{P})$. By Lemma 2.2, either $\{u, v\} \cap Inn(\mathcal{P}) \neq \emptyset$ or $u, v$ are the endpoints of some $P_i \in \mathcal{P}$.

**Claim 1.** If $S_2 = \emptyset$, then $h \geq 2$.

As $S_2 = \emptyset$, we have $S = S_1$ and so $|S| = |J| + h$. Then $4|J| \leq 3|S| = 3(|J| + h)$, i.e., $|J| \leq 3h$. If $h = 1$, then $|J| \leq 3$ and so $n \leq 3 + 4 = 7$. Then the subgraph of $G$ induced by the edge set between $S$ and $J$ is isomorphic to $K_{3,4}$, and so $G$ is Hamilton-connected, a contradiction. Claim 1 holds.

**Claim 2.** $u, v$ cannot be the endpoints of some alternating path $P_i$.

Assume that $u, v$ are the endpoints of $P_1$ and $u = x_1^1$ and $v = x_{k_1+1}^1$. Then $N_G(y_i^1) \cap (End(\mathcal{P}) \cup S_2 - \{u, v\}) = \emptyset$ for each $i \in \{1, \cdots, k_1\}$ (otherwise, assume that $wy_i^1 \in E(G)$ for some $w \in End(\mathcal{P}) \cup S_2 - \{u, v\}$. Then $\mathcal{P}' = \mathcal{P} - \{x_i^1 y_i^1\} + \{wy_i^1\}$ is a $J$-cover of $G$ with $u, v \notin Inn(\mathcal{P}')$ and $u, v$ are not endpoints of some path of $\mathcal{P}'$. By Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path.)

Consider $y_1^1$. Since $d_G(y_1^1) \geq 4$, there is $z \in Inn(\mathcal{P}) - \{x_2^1\}$ such that $zy_1^1 \in E(G)$. If $z \in V(P_1)$, then $h = 1$ and $S_2 = \emptyset$ (otherwise, there are three vertices $y_1^1, z', z'' \in N_{P_1}(z) \cap J$ such that $G[\{z, y_1^1, z', z'', w\}] = K_{1,4}$ for some $w \in S_2 \cup \{x_1^2\}$, a contradiction). If $z \notin V(P_1)$, then $h = 2$ and $S_2 = \emptyset$ (otherwise, assume without loss of generality that $z = x_i^2$. As $G[\{y_1^1, x_i^2, y_{i-1}^2, y_i^2, w\}] \neq K_{1,4}$ for any $w \in S_2 \cup \{x_1^3\}$, we have $wy_{i-1}^2 \in E(G)$ or $wy_i^2 \in E(G)$. We may assume that $wy_i^2 \in E(G)$. Then $\mathcal{P}' = \mathcal{P} - \{x_1^1 y_1^1, x_i^2 y_i^2\} + \{y_1^1 z, wy_i^2\}$ is a $J$-cover with $v \in End(\mathcal{P}')$ and $u \notin V(\mathcal{P}')$. By Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path, contrary to our assumption that $G$ has no hamiltonian $(u, v)$-path). By Claim 1, $h \geq 2$. Thus $h = 2$ and $z \in V(P_2)$. Assume that $z = x_i^2$. Similarly, consider $y_{k_1}^1$, we have there is some $z' \in V(P_2)$ such that $y_{k_1}^1 z' \in E(G)$.

If $y_1^1 \neq y_{k_1}^1$, then, by Lemma 4.1, $z' \in \{x_{i-1}^2, x_{i+1}^2\}$. Without loss of generality, we assume that $z' = x_{i+1}^2$. Then $x_1^2 y_i^2 \notin E(G)$ (otherwise, $\mathcal{P}' = \mathcal{P} - \{x_i^2 y_i^2, x_1^1 y_1^1\} + \{x_1^2 y_i^2, x_i^2 y_1^1\}$ is a $J$-cover of $G$ with $v \in End(\mathcal{P}')$ and $u \in S_2(\mathcal{P}')$. By Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path, a contradiction.) Similarly, $x_1^2 y_{i+1}^2 \notin E(G)$. Therefore, $G[\{x_{i+1}^2, y_i^2, y_{i+1}^2, y_{k_1}^1, x_1^2\}] = K_{1,4}$, a contradiction. So $y_1^1 = y_{k_1}^1$, and the length of $P_1$ is 2.

As $d_G(y_1^1) \geq 4$, there is a vertex $x_j^2 \in V(P_2)$ such that $y_1^1 x_j^2 \in E(G)$. We assume that $j > i$. By Lemma 2.2, $\{x_1^2 y_i^2, x_1^2 y_j^2, x_{k_2+1}^2 y_{i-1}^2, x_{k_2+1}^2 y_{j-1}^2\} \cap E(G) = \emptyset$. As $y_1^1 x_1^2 \notin E(G)$ and $G[\{x_j^2, y_1^1, y_j^2, y_{j-1}^2, x_1^2\}] \neq K_{1,4}$, we have $x_1^2 y_{j-1}^2 \in E(G)$. Similarly, $y_i^2 x_{k_2+1}^2 \in E(G)$. Let $\mathcal{P}' = \mathcal{P} - \{x_1^1 y_1^1, x_i^2 y_i^2, x_j^2 y_{j-1}^2\} + \{y_1^1 x_i^2, x_1^2 y_{j-1}^2, y_j^2 x_{k_2+1}^2\}$. Then $\mathcal{P}'$ is a $J$-cover with $u \notin V(\mathcal{P}')$ and $v \in End(\mathcal{P}')$. By Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path, contrary to our assumption. This completes the proof of Claim 2.

By Claim 2, $\{u, v\} \cap Inn(\mathcal{P}) \neq \emptyset$. Now we may assume that $\mathcal{P} = \{P_1, P_2, \ldots, P_h\}$ is a $J$-cover such that
(i) $|\{u, v\} \cap Inn(\mathcal{P})| > 1$ is minimized, and
(ii) subject to (i), $h$ is maximized.
Assume that $u = x_{i_0}^1$, where $1 < i_0 < k_1 + 1$. Thus we have the following.

**Claim 3.** (i) For any $w \in J$, $N_G(w) \cap S_2 = \emptyset$.
(ii) If there is $z \in S_1$ such that $|N_G(z) \cap J| = 3$, then $S_2 = \emptyset$ and $h \geq 2$.

**Claim 4.** If $v \neq x_1^1$, then $N_G(y_{i_0}^1) \subseteq \{v, x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$; if $v \neq x_{k_1+1}^1$, then $N_G(y_{i_0-1}^1) \subseteq \{v, x_1^1, x_2^1, \ldots, x_{i_0}^1\}$.

By symmetry, we only prove $N_G(y_{i_0}^1) \subseteq \{v, x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$ if $v \neq x_1^1$. Assume that $z \in N_G(y_{i_0}^1) - \{v, x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$. By the choice of $\mathcal{P}$, $x_1^1 y_{i_0}^1 \notin E(G)$. By Claim 3(i), $z \notin S_2$. Furthermore, $z \notin End(\mathcal{P}) - \{v\}$ (otherwise, $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0}^1\} + \{zy_{i_0}^1\}$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}')| < |\{u, v\} \cap Inn(\mathcal{P})|$, contrary to the choice of $\mathcal{P}$). Thus $z \in Inn(\mathcal{P}) - \{x_{i_0}^1, x_{i_0+1}^1\}$. By Claim 3(ii), $S_2 = \emptyset$ and $h \geq 2$. Without loss of generality, assume that $v \neq x_1^2$. Thus, $y_{i_0}^1 x_1^2 \notin E(G)$.

Assume that $z = x_j^2 \in V(P_2)$. Consider $G[\{x_j^2, x_1^1, y_{i_0}^1, y_{j-1}^2, y_j^2\}]$. We have either $y_{j-1}^2 x_1^1 \in E(G)$ or $y_j^2 x_1^1 \in E(G)$. Without loss of generality, we assume that $x_1^1 y_{j-1}^2 \in E(G)$. Then $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0}^1, x_j^2 y_{j-1}^2\} + \{x_1^1 y_{j-1}^2, y_{i_0}^1 x_j^2\}$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}')| < |\{u, v\} \cap Inn(\mathcal{P})|$, contrary to the choice of $\mathcal{P}$. So $N_G(y_{i_0}^1) \subseteq V(P_1)$ and $z = x_{j_0}^1$ for some $j_0 \in \{2, \cdots, i_0 - 1\}$.

As $G[\{x_{j_0}^1, y_{j_0-1}^1, y_{j_0}^1, y_{i_0}^1, x_1^2\}] \neq K_{1,4}$, we have either $y_{j_0-1}^1 x_1^2 \in E(G)$ or $y_{j_0}^1 x_1^2 \in E(G)$. If $y_{j_0-1}^1 x_1^2 \in E(G)$, we set $\mathcal{P}' = \mathcal{P} - \{y_{j_0-1}^1 x_{j_0}^1, x_{i_0}^1 y_{i_0}^1\} + \{x_1^2 y_{j_0-1}^1, x_{j_0}^1 y_{i_0}^1\}$; if $y_{j_0}^1 x_1^2 \in E(G)$, we set $\mathcal{P}' = \mathcal{P} - \{x_{j_0}^1 y_{j_0}^1, x_{i_0}^1 y_{i_0}^1\} + \{x_1^2 y_{j_0}^1, x_{j_0}^1 y_{i_0}^1\}$. Then $\mathcal{P}'$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}')| < |\{u, v\} \cap Inn(\mathcal{P})|$, contrary to the choice of $\mathcal{P}$. Claim 4 holds.

**Claim 5.** $v \in S_1$.

Assume that $v \notin S_1$. Then $v \in J \cup S_2$. By Claims 3 and 4, $N_G(y_{i_0}^1) \subseteq \{x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$ and $N_G(y_{i_0-1}^1) \subseteq \{x_1^1, x_2^1, \cdots, x_{i_0}^1\}$. As $\delta(G) \geq 4$, there exist $z_1 \in \{x_{i_0+2}^1, x_{i_0+3}^1, \cdots, x_{k_1}^1\}$ and $z_2 \in \{x_2^1, x_3^1, \cdots, x_{i_0-2}^1\}$ such that $z_1 y_{i_0}^1, z_2 y_{i_0-1}^1 \in E(G)$. This contradicts Lemma 4.1. So Claim 5 holds.

**Claim 6.** $v \in Inn(\mathcal{P})$.

Assume that $v \notin Inn(\mathcal{P})$. Then $v \in End(\mathcal{P})$. Without loss of generality, we assume that $v \neq x_1^1$. By Claim 4, $N_G(y_{i_0}^1) \subseteq \{v, x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$.

**Claim 6.1.** $N_G(y_{i_0}^1) = \{v, x_{i_0}^1, x_{i_0+1}^1, x_{k_1+1}^1\}$.

Otherwise, there is a vertex $x_j^1 \in \{x_{i_0+2}^1, \cdots, x_{k_1}^1\}$ such that $y_{i_0}^1 x_j^1 \in E(G)$. By Claim 3(ii), $S_2 = \emptyset$ and $h \geq 2$. By symmetry, we assume that $v \neq x_1^t$, where $t \in \{1, 2 \cdots, h\}$. If $x_1^t y_{i_0-1}^1 \in E(G)$, we let $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0-1}^1\} + \{x_1^t y_{i_0-1}^1\}$. Then $\mathcal{P}'$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}')| < |\{u, v\} \cap Inn(\mathcal{P})|$, contrary to the choice of $\mathcal{P}$. So $x_1^t y_{i_0-1}^1 \in E(G)$. Similarly, $x_1^t y_{i_0}^1, x_1^t y_{j-1}^1, x_1^1 y_{i_0}^1, x_1^1 y_{j-1}^1 \notin E(G)$. As $G[\{x_j^1, y_{i_0}^1, y_{j-1}^1, y_j^1, x_1^t\}] \neq K_{1,4}$, we have $x_1^t y_j^1 \in E(G)$ for each $t \in \{1, \cdots, h\}$.

We first claim that $v = x_{k_1+1}^1$. Otherwise, we assume that $v = x_{k_t+1}^t$ for some $t \neq 1$. By Claim 4, $N_G(y_{i_0-1}^1) \subseteq \{x_{k_t+1}^t, x_1^1, x_2^1, \cdots, x_{i_0}^1\}$. By Lemma 4.1, $N_G(y_{i_0-1}^1) = \{x_1^1, x_{i_0-1}^1, x_{i_0}^1, x_{k_t+1}^t\}$. Consider $\mathcal{P}' = \mathcal{P} - \{y_{i_0-1}^1 x_{i_0}^1\} + \{y_{i_0-1}^1 x_{k_t+1}^t\}$. Then $\mathcal{P}'$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}')| = |\{u, v\} \cap Inn(\mathcal{P})| = 1$, $u = x_{i_0}^1 \in End(\mathcal{P}')$ and $v \in Inn(\mathcal{P}')$. Consider $y_{k_t}^t$ in $\mathcal{P}'$. By Claim 4 and Lemma 4.1, we have $N_G(y_{k_t}^t) = \{x_{k_t}^t, x_{k_t+1}^t, x_1^t, x_{i_0}^1\}$. Let $\mathcal{P}'' = \mathcal{P} - \{y_{i_0-1}^1 x_{i_0}^1, x_{i_0}^1 y_{i_0}^1, y_{k_t}^t x_{k_t+1}^t\} + \{x_{i_0}^1 y_{k_t}^t, y_{i_0-1}^1 x_{k_t+1}^t, y_{i_0}^1 x_1^t\}$. Then $\mathcal{P}''$ is a $J$-cover with $|\{u, v\} \cap Inn(\mathcal{P}'')| < |\{u, v\} \cap Inn(\mathcal{P})| = 1$, contrary to the choice of $\mathcal{P}$. So $v = x_{k_1+1}^1$.

We next claim that $N_G(y_{i_0-1}^1) = \{x_1^1, x_{i_0-1}^1, x_{i_0}^1, x_{k_1+1}^1\}$. By Lemma 2.2 and Claim 2, $N_G(y_{i_0-1}^1) \cap (End(\mathcal{P}) - \{x_1^1, v\}) = \emptyset$. By Lemma 4.1, we have $N_G(y_{i_0-1}^1) \subseteq \{x_1^1, x_{i_0-1}^1, x_{i_0}^1, x_{i_0+1}^1, x_{j-1}^1, x_{j+1}^1, x_{k_1+1}^1\}$. If $y_{i_0-1}^1 x_{i_0+1}^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0-1}^1, x_{i_0+1}^1 y_{i_0}^1, x_j^1 y_j^1\} + \{y_{i_0}^1 x_j^1, y_{i_0-1}^1 x_{i_0+1}^1, x_1^1 y_j^1\}$; if $y_{i_0-1}^1 x_{j-1}^1 \in E(G)$, then $y_{j-2}^1 x_1^2 \in E(G)$ as $G[\{x_{j-1}^1, y_{i_0-1}^1, y_{j-1}^1, y_{j-2}^1, x_1^2\}] \neq K_{1,4}$, let $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0-1}^1, x_{j-1}^1 y_{j-2}^1\} + \{y_{i_0-1}^1 x_{j-1}^1, y_{j-2}^1 x_1^2\}$; if $y_{i_0-1}^1 x_{j+1}^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0-1}^1, x_{j+1}^1 y_j^1\} + \{x_1^2 y_j^1, x_{j+1}^1 y_{i_0-1}^1\}$. Then $u, v \in End(\mathcal{P}')$. By Claim 2 and Lemma 2.2, $G$ has a hamiltonian $(u, v)$-path, a contradiction. Therefore, $N_G(y_{i_0-1}^1) = \{x_1^1, x_{i_0-1}^1, x_{i_0}^1, x_{k_1+1}^1\}$.

Notice that if $y_{i_0}^1 x_{k_1+1}^1 \in E(G)$, then $x_1^2 y_{k_1}^1 \in E(G)$ since $G[\{x_{k_1+1}^1, y_{k_1}^1, y_{i_0-1}^1, y_{i_0}^1, x_1^2\}] \neq K_{1,4}$. Thus $\mathcal{P}' = \mathcal{P} - \{y_{i_0-1}^1 x_{i_0}^1, y_{k_1}^1 x_{k_1+1}^1\} + \{y_{k_1}^1 x_1^2, y_{i_0-1}^1 x_{k_1+1}^1\}$ is a $J$-cover with $u, v \in End(\mathcal{P}')$. Then $G$ has a hamiltonian $(u, v)$-path, a contradiction. It implies that $N_G(y_{i_0}^1) \subseteq \{x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1}^1\}$. As $\delta(G) \geq 4$, by Lemma 4.1, $y_{j_0}^1 x_{j+1}^1 \in E(G)$. Thus $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0}^1, y_j^1 x_{j+1}^1\} + \{y_{i_0}^1 x_{j+1}^1, y_j^1 x_1^2\}$ is a $J$-cover of $G$ with $u, v \in End(\mathcal{P}')$. So $G$ has a hamiltonian $(u, v)$-path, a contradiction. Hence, $N_G(y_{i_0}^1) = \{v, x_{i_0}^1, x_{i_0+1}^1, x_{k_1+1}^1\}$. Claim 6.1 holds.

As $d_G(y_{i_0}^1) \geq 4$ and $v \in End(\mathcal{P}) - \{x_1^1\}$, we have $h \geq 2$. Without loss of generality, we assume that $v = x_1^2$. Thus $v \neq x_{k_1+1}^1$. By applying the discussion in Claim 6.1 on $y_{i_0-1}^1$, we have $N_G(y_{i_0-1}^1) = \{x_1^1, x_{i_0}^1, x_{i_0-1}^1, x_1^2\}$. Since $G[\{x_1^2, y_{i_0-1}^1, y_{i_0}^1, y_1^2, x_{k_2+1}^2\}] \neq K_{1,4}$, we have $y_1^2 x_{k_2+1}^2 \in E(G)$.

If $h \geq 3$, then $y_1^2 x_1^3 \in E(G)$ since $G[\{x_1^2, y_{i_0-1}^1, y_{i_0}^1, y_1^2, x_1^3\}] \neq K_{1,4}$. Consider $\mathcal{P}' = \mathcal{P} - \{y_{i_0-1}^1 x_{i_0}^1\} + \{y_{i_0-1}^1 x_1^2\}$. Then $\mathcal{P}'$ is a $J$-cover of $G$ with $x_1^2 \in Inn(\mathcal{P}')$ and $x_i^1 \in End(\mathcal{P}')$. By applying the discussion in Claim 6.1 on $y_1^2$ in $\mathcal{P}'$, we have $N_G(y_1^2) = \{x_1^2, x_2^2, x_{k_2+1}^2, x_{i_0}^1\}$. So $x_{i_0}^1 y_1^2 \in E(G)$. Thus $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0}^1, x_1^2 y_1^2\} + \{x_1^2 y_{i_0}^1, y_1^2 x_1^3\}$ is a $J$-cover of $G$ with $u, v \in End(\mathcal{P}')$. So $G$ has a hamiltonian $(u, v)$-path, a contradiction. So $h = 2$.

Consider $y_1^1$. By Lemma 4.1, $N_G(y_1^1) \subseteq \{x_1^1, x_2^1, x_{i_0-1}^1, x_{i_0+1}^1, x_{k_1+1}^1, x_2^2, x_{k_2+1}^2\}$. If $y_1^1 x_{k_1+1}^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{y_{i_0-1}^1 x_{i_0}^1, x_1^1 y_1^1\} + \{x_{k_1+1}^1 y_1^1, y_{i_0-1}^1 x_1^1\}$; if $y_1^1 x_{i_0+1}^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_{i_0+1}^1 y_{i_0}^1, x_1^1 y_1^1, y_{i_0-1}^1 x_{i_0}^1\} + \{y_{i_0}^1 x_{k_1+1}^1, x_{i_0+1}^1 y_1^1, y_{i_0-1}^1 x_1^1\}$; if $y_1^1 x_{k_2+1}^2 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_1^1 y_1^1, y_{i_0-1}^1 x_{i_0}^1\} + \{y_1^1 x_{k_2+1}^2, y_{i_0-1}^1 x_1^1\}$; if $y_1^1 x_2^2 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{y_1^2 x_2^2, x_1^1 y_1^1, y_{i_0-1}^1 x_{i_0}^1\} + \{y_1^2 x_{k_2+1}^2, x_2^2 y_1^1, y_{i_0-1}^1 x_1^1\}$. Then $\mathcal{P}'$ is a $J$-cover of $G$ with $u, v \in End(\mathcal{P}')$. Thus $G$ has a hamiltonian $(u, v)$-path, a contradiction. Therefore, $N_G(y_1^1) = \{x_1^1, x_2^1, x_{i_0-1}^1\}$, contrary to the hypothesis that $\delta(G) \geq 4$. This completes the proof of Claim 6.

By Claim 6, $v \in Inn(\mathcal{P})$. Consider $y_{i_0}^1$. Then $N_G(y_{i_0}^1) \cap (End(\mathcal{P}) - \{x_{k_1+1}^1\}) = \emptyset$. As $d_G(y_{i_0}^1) \geq 4$, there is $z \in Inn(\mathcal{P}) - \{x_{i_0}^1, x_{i_0+1}^1\}$ such that $zy_{i_0}^1 \in E(G)$. By Claim 3(ii), $S_2 = \emptyset$ and $h \geq 2$. If $z \notin V(P_1)$, we assume that $z = x_j^2 \in V(P_2)$, where $1 < j < k_2 + 1$. Since $G[\{x_j^2, y_{j-1}^2, y_j^2, y_{i_0}^1, x_1^1\}] \neq K_{1,4}$, we have either $x_1^1 y_{j-1}^2 \in E(G)$ or $x_1^1 y_j^2 \in E(G)$. Without loss of generality, we assume that $x_1^1 y_j^2 \in E(G)$. Then $\mathcal{P}' = \mathcal{P} - \{x_{i_0}^1 y_{i_0}^1, x_j^2 y_j^2\} + \{y_{i_0}^1 x_j^2, x_1^1 y_j^2\}$ is a $J$-cover of $G$ such that $u \in End(\mathcal{P}')$ and $v \in Inn(\mathcal{P}')$, contrary to the choice of $\mathcal{P}$. So $z \in V(P_1)$. If $z = x_j^1$, where $1 < j < i_0$, then either $x_1^2 y_j^1 \in E(G)$ or $x_1^2 y_{j-1}^1 \in E(G)$ since $G[\{x_j^1, y_j^1, y_{j-1}^1, y_{i_0}^1, x_1^2\}] \neq K_{1,4}$. If $x_1^2 v_{j-1}^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_j^1 y_{j-1}^1, x_{j_0}^1 y_{j_0}^1\} + \{x_1^2 y_{j-1}^1, x_j y_{j_0}^1\}$; if $x_1^2 y_j^1 \in E(G)$, let $\mathcal{P}' = \mathcal{P} - \{x_j^1 y_j^1, x_{j_0}^1 y_{j_0}^1\} + \{x_1^2 y_j^1, x_j y_{j_0}^1\}$. Then $u \in End(\mathcal{P}')$ and $v \in Inn(\mathcal{P}')$, contrary to the choice of $\mathcal{P}$. So $N_G(y_{j_0}^1) \subseteq \{x_{i_0}^1, x_{i_0+1}^1, \cdots, x_{k_1+1}^1\}$. Similarly, $N_G(y_{j_0-1}^1) \subseteq \{x_1^1, x_2^1, \cdots, x_{i_0-1}^1\}$. As $\delta(G) \geq 4$, there exist $z_1 \in \{x_{i_0+2}^1, x_{i_0+3}^1, \cdots, x_{k_1}^1\}$ and $z_2 \in \{x_2^1, x_3^1, \cdots, x_{i_0-2}^1\}$ such that $z_1 y_{i_0}^1, z_2 y_{i_0-1}^1 \in E(G)$. This contradicts Lemma 4.1. The proof of Theorem 4.4 is done. □

## 5. Concluding remarks

In view of Theorem 2.6, we may restate Conjecture 1.4 in the following seemingly stronger version.

**Conjecture 5.1.** *Let $r \geq 2$ be an integer. Every $r$-connected $K_{1,r+1}$-free split graph is fully cycle extendable.*

It is natural to consider the Hamilton-connected version of the conjecture above. We consider the following example. Let $H$ be a copy of the complete bipartite graph $K_{r-1,r} (r \geq 3)$ with the bipartition $(X, Y)$, where $X = \{x_1, x_2, \cdots, x_{r-1}\}$

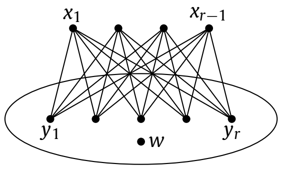


**Fig. 2.** $r$-connected $K_{1,r+1}$-free non-Hamilton-connected graph $G$.

and $Y = \{y_1, y_2, \cdots, y_r\}$. Let $K$ be a copy of $K_{r+1}$ with $V(K) = \{y_1, y_2, \cdots, y_r, w\}$. Then $G = H \cup K$ (see Fig. 2) is an $r$-connected $K_{1,r+1}$-free graph, but $G$ is not Hamilton-connected (there is no hamiltonian $(y_1, y_r)$-path). This, together with Theorems 3.6 and 4.4, motivates the following conjecture.

**Conjecture 5.2.** *Let $r \geq 3$ be an integer. Every $r$-connected $K_{1,r}$-free split graph is Hamilton-connected.*

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## References

[1] R.E.L. Aldred, J. Fujisawa, Y. Egawa, K. Ota, A. Saito, The existence of a 2-factor in $K_{1,n}$-free graphs with large connectivity and large edge-connectivity, J. Graph Theory 68 (2011) 77–89.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[3] R.E. Burkazd, P.L. Hammer, A note on Hamiltonian split graphs, J. Comb. Theory, Ser. B 28 (2) (1980) 245–248.
[4] G. Dai, Z. Zhang, H. Broersma, X. Zhang, The Hamiltonian properties in $K_{1,r}$-free split graphs, Discrete Math. 345 (2022) 112826.
[5] S. Foldes, P.L. Hammer, Split graphs having Dilworth number two, Can. J. Math. 24 (3) (1977) 666–672.
[6] S. Foldes, P.L. Hammer, Split graphs, in: Proceedings of the 8th 320 South-Eastern Conference on Combinatorics, Graph Theory and Computing, 1977, pp. 311–315.
[7] G.R.T. Hendry, Extending cycles in graphs, Discrete Math. 85 (1990) 59–72.
[8] D. Kratsch, J. Lehel, H. Muller, Toughness, hamiltonicity and split graphs, Discrete Math. 150 (1) (1996) 231–245.
[9] R. Merris, Split graphs, Eur. J. Comb. 24 (2003) 413–430.
[10] P. Renjith, N. Sadagopan, The Hamiltonian cycle in $K_{1,r}$-free split graphs – a dichotomy, Int. J. Found. Comput. Sci. 33 (2022) 1–32.
[11] N.D. Tan, L.X. Hung, On the Burkard-Hammer condition for Hamiltonian split graphs, Discrete Math. 296 (1) (2005) 59–72.
[12] Z.S. Wu, Hamilton connectivity of $K_{1,3}$-free graphs, J. Math. Res. Exposition 9 (1989) 447–451.