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Supereulerian regular matroids without small cocircuits

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Abstract

A cycle of a matroid is a disjoint union of circuits. A matroid is supereulerian if it contains a spanning cycle. To answer an open problem of Bauer in 1985, Catlin proved in [J. Graph Theory 12 (1988) 29–44] that for sufficiently large n, every 2-edge-connected simple graph G with $n = |V(G)|$ and minimum degree $\delta(G) \geq \frac{n}{5}$ is supereulerian. In [Eur. J. Combinatorics, 33 (2012), 1765–1776], it is shown that for any connected simple regular matroid M, if every cocircuit D of M satisfies $|D| \geq \max\left\{\frac{r(M)-5}{5}, 6\right\}$, then M is supereulerian. We prove the following. (i) Let M be a connected simple regular matroid. If every cocircuit D of M satisfies $|D| \geq \max\left\{\frac{r(M)+1}{10}, 9\right\}$, then M is supereulerian. (ii) For any real number c with $0 < c < 1$, there exists an integer $f(c)$ such that if every cocircuit D of a connected simple cographic matroid M satisfies $|D| \geq \max\{c(r(M)+1), f(c)\}$, then M is supereulerian.

KEYWORDS

contractible restrictions, disjoint bases of matroids, fractional arboricity of a matroid, spanning cycles, strength of a matroid, supereulerian graphs

MATHEMATICAL SUBJECT CLASSIFICATION

05B35, 05C45

1 | INTRODUCTION

We investigate finite graphs and matroids, with loops and parallel edges or parallel elements permitted, and generally follow [3] for graphs and [23] for matroids for undefined terminologies and notations. A **circuit** in a graph is a nontrivial 2-regular connected graph and a **cycle** is an edge disjoint union of circuits. A connected cycle is a **closed trail** or an **eulerian** graph in [3]. A graph G is **supereulerian** if it contains a **spanning cycle**, a cycle of G that contains a spanning tree of G. Boesch, Suffel, and Tindell in [1] initiated the study of the supereulerian graph problem, which aims at characterizations of graphs with spanning cycles. Pulleyblank [24] indicated that the problem of determining if a graph is supereulerian, even when restricted to planar graphs, is NP-complete. Catlin's survey [5] and its updates [8, 17] are resourceful references for further literature on the supereulerian problem.

Following the notation in [23], $r_M, \mathcal{B}(M)$ and $\mathcal{C}(M)$ denote the rank function, the collections of bases and circuits of a matroid $M = (E, \mathcal{I})$, respectively. For a subset $X \subseteq E, M/X$ and $M|X$ denote the matroid contractions and matroid restrictions, respectively. We define a **cycle** of M to be a disjoint union of circuits in M, and use $\mathcal{C}_0(M)$ to denote the collection of all cycles of M. If a cycle $C \in \mathcal{C}_0(M)$ satisfies $r_M(C) = r_M(E)$, then C is a **spanning cycle** of M. A matroid M with a spanning cycle is called a **supereulerian matroid**. For a matroid M, the **girth** of M is

$$g(M) = \begin{cases} \min\{k : M \text{ has a circuit } C \text{ with} |C| = k\} : & \text{if } M \text{ has a circuit} \\ \infty & : \text{if } M \text{ has no circuits.} \end{cases}$$

If G is a graph, then we define the girth of a graph G to be $g(G) = g(M(G))$. The **cogirth** of a matroid M is $g(M^*)$, which is often denote by $g^*(M)$.

Let G be a connected graph and $M = M(G)$ be the cycle matroid of G. If $U, V \subseteq V(G)$, and W is a proper nonempty vertex subset of $V(G)$, then define $\overline{W} = V(G) - W$,

$$(U, V)_G = \{uv \in E(G) : u \in U \text{ and } v \in V\}, \text{ and } \quad \partial_G(W) = (W, \overline{W})_G.$$

Let $N_G(W) = \{u \in \overline{W} : \exists\, v \in W \text{ such that } uv \in E(G)\}$. We set $E_G(v) = \partial_G(\{v\}), N_G(v)$ for $N_G(\{v\})$, and define $N_G[v] = N_G(v) \cup \{v\}$. As in [3], let $\kappa(G), \kappa'(G), \Delta(G)$ and $\delta(G)$ denote the connectivity, the edge-connectivity, the maximum degree and the minimum degree of a graph G. The following is well-known.

$$\delta(G) \geq \kappa'(G) = g^*(M(G)). \tag{1}$$

Settling some open problems raised by Bauer in [2], the following has been proved.

Theorem 1.1. *Let G be a 2-edge-connected simple graph on $n = |V(G)|$ vertices.*

(i) (*Catlin, Theorem* 9 *of* [4]) *If $n > 16$ and $\delta(G) > \frac{n}{5} - 1$, then G is supereulerian.*
(ii) (*Theorem* 5 *of* [14]) *If $n \geq 31, g(G) > 4$ and $\delta(G) > \frac{n}{10}$, then G has a spanning cycle.*

Jaeger [12] showed that every 4-edge-connected graph is supereulerian. It has been observed in [16] that there exists an infinite family of non-supereulerian cographic matroids which can have arbitrarily large cogirth. Thus in general, large cogirth is not sufficient to assure that a regular matroid is supereulerian. It is natural (as seen in [11]) to replace the minimum degree of a graph by the cogirth of a matroid. Efforts extending Theorem 1.1 (i) to regular matroids have been done in [20].

Theorem 1.2 (Theorem 1.3 of [20]). *Let M be a connected simple regular matroid. If every cocircuit D of M satisfies* $|D| \geq \max\left\{\frac{r(M)-5}{5}, 6\right\}$, *then M is supereulerian.*

Theorems 1.1 and 1.2 motivate the current research. We prove the following Theorems 1.3 and 1.4. It is observed that Theorem 1.3 improves Theorem 1.2 when the rank of a matroid is sufficiently large.

Theorem 1.3. *Let M be a simple regular matroid. If*

$$g^*(M) \geq \max\left\{\frac{r(M)+1}{10}, 9\right\}, \tag{2}$$

then M is supereulerian.

Theorem 1.4. *Let M be a connected simple cographic matroid. For any real number c with $0 < c < 1$, there exists an integer $f(c)$ such that if every cocircuit D of M satisfies* $|D| \geq \max\{c(r(M)+1), f(c)\}$, *then M is supereulerian.*

Preliminaries will be displayed in the next section, including Seymour's decomposition theorem of regular matroids. Then we show that Theorem 1.3 is equivalent to a version with an additional girth requirement, which allows us to obtained a better bound in (2). The proof of the main results will be presented in the last two sections.

2 | PRELIMINARY

This section is a preparation for the mechanisms to be deployed in the arguments in each of the steps to prove the main results. We will use contraction and Seymour's decomposition theorem of regular matroids to proceed our inductive proofs. Certain properties of strength and fractional arboricity of matroids will also be utilized to deal with the cographic case in the proof.

2.1 | Contractible configurations

We start with contractions of graphs and matroids. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. Throughout this article, we adopt the convention to use X to denote both an edge subset of G and $G[X]$, the subgraph of G induced by the edges in X. Thus $V(X)$ is the set of vertices in G that are incident with edges in X. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the edge being contracted. If J is a subgraph of G, then we use G/J for $G/E(J)$, and define $G/\varnothing = G$.

Following [23], for a matroid M with a subset $X \subseteq E(M)$, we use $M - X$ to denote the restriction $M|(E - X)$.

Let $\mathcal{F}$ denote a collection of matroids. A matroid N is a $\mathcal{F}$-**supereulerian contractible configuration**, or simply **contractible in** $\mathcal{F}$, if $N \in \mathcal{F}$ and for any $M \in \mathcal{F}$ containing N as a restriction,

$$M \text{ is supereulerian if and only if } M/N \text{ is supereulerian.} \tag{3}$$

Let $O(G)$ denote the set of all odd degree vertices in G. A graph H is **collapsible if for any subset** $R \subseteq V(H)$ **with** $|R| \equiv 0$ (**mod 2**), H **has a connected subgraph** Γ_R **such that** $O(\Gamma_R) = R$ **and** $V(\Gamma_R) = V(H)$. Catlin [4] showed that, if $\mathcal{G}$ denotes all graphic matroids, then the cycle matroids of collapsible graphs are contractible in $\mathcal{G}$. For a graph G with $H_1, H_2, ..., H_c$ being the maximal collapsible subgraphs of G, the contraction $G' = G/(H_1 \cup H_2 \cup \cdots \cup H_c)$ is the **reduction** of G. A graph G with $G = G'$ is a **reduced** graph. The following is a brief summary of some useful mechanisms.

Theorem 2.1. *Let G be a connected graph, and let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with 2 edge-disjoint spanning trees. Each of the following holds.*

(i) (*Catlin, Theorem* 3 *of* [4]) *If H is a collapsible subgraph of G, then G has a spanning cycle if and only if G/H has a spanning cycle.*
(ii) (*Catlin, Theorem* 5 *of* [4]) *A graph G is reduced if and only if G contains no nontrivial collapsible subgraphs.*
(iii) (*Catlin et al., Theorem* 1.3 *of* [7]) *If $F(G) \le 2$, then the reduction of G is in* $\{K_1, K_2\} \cup \{K_{2,t} : t \ge 1\}$.
(iv) (*Catlin et al., Lemma* 2.3 *of* [7]) *If $G \notin \{K_1, K_2\}$ is reduced, then* $F(G) = 2|V(G)| - |E(G)| - 2$.

Let $\tau(M)$ denote the maximum number of disjoint bases of M. If G is a connected graph, then $\tau(G) = \tau(M(G))$. Characterizations of matroids M with $\tau(M) \ge k$ have been obtained by Edmonds [10], extending the graphical results by Nash-Williams [22] and Tutte [26]. Catlin in [4] (also implied by Theorem 2.1(iii)) showed that if $\tau(G) \ge 2$, then $M(G)$ is contractible in $\mathcal{G}$. This has been extended to binary matroids.

Lemma 2.2. *Each of the following holds.*

(i) (*Theorem* 5.4 *of* [16]) *Let N be a binary matroid. If $\tau(N) \ge 2$, then for any binary matroid M that contains N as a restriction, M/N is supereulerian if and only if M is supereulerian.*
(ii) (*Theorem* 5.7 *of* [16]) *Let M be a binary matroid and let $N \in \mathcal{C}(M)$ with $|N| \le 3$. Then M/N is supereulerian if and only if M is supereulerian.*

Proof. An explanation for Lemma 2.2 (ii) might be needed as Theorem 5.7 of [16] only proves the case when $|N| = 3$. If $|N| = 2$, then $\tau(N) = 2$ and so Lemma 2.2(ii) follows from (i). If $|N| = 1$, then N consists of a loop e. Thus $M - e = M/N$ has a spanning cycle C if and only if C is a spanning cycle of M. Hence Lemma 2.2(ii) holds for $|N| = 1$ also. □

Definition 2.3. Let M be a binary matroid. A subset N is **eligible** in M if either $\tau(M|N) \geq 2$, or $N \in \mathcal{C}(M)$ with $|N| \leq 3$. A matroid M is eligible if M is spanned by an eligible subset. We run the following algorithm to a binary matroid M with $r(M) > 0$.

(i) Set $M_0 := M$.
(ii) For each $i = 0, 1, \ldots$, do the following.
(ii-1) If M_i has a subset N that is eligible in M_i with $r(N) < r(M_i)$, then set $M_{i+1} := M_i/N$ and continue.
(ii-2) If M_i has no eligible subsets or every eligible subset N in M_i satisfies $r(M_i) = r(N)$, then stop.

When the algorithm stops, the resulting matroid is a **binary reduction** of M.

For sets X and Y, the **symmetric difference** of X and Y is defined as $X \Delta Y = (X \cup Y) - (X \cap Y)$. It is known that for a binary matroid M, if C_1, C_2 are cycles of M, then $C_1 \triangle C_2$ is also a cycle of M.

Corollary 2.4. *Every eligible binary matroid is supereulerian.*

Proof. Let M be an eligible matroid. Then for some eligible subset $N \subseteq E(M), r(N) = r(M)$. If N is spanned by a 3-circuit C, then C is a spanning cycle of M, and so M is supereulerian. Assume that $\tau(M|N) \geq 2$. Since $r(M) = r(N)$, we have $\tau(M) \geq \tau(M|N) \geq 2$. Let B, B' be two disjoint bases of M. For each $e \in E(M) - B$, let $C_M(e, B)$ denote the fundamental circuit of e with respect to B, and

$$C = \triangle_{e \in E(M)-B} C_M(e, B).$$

As M is binary, C is a cycle of M. Since $B' \subseteq E(M) - B \subseteq C, M$ is supereulerian. □

Lemma 2.5. *Let M be a binary matroid and let M' denote a binary reduction of M. Each of the following holds.*

(i) *M is supereulerian if and only if M' is supereulerian.*
(ii) *Either M' is spanned by a subset N that is eligible in M, or $g(M') \geq 4$.*
(iii) $g^*(M') \geq g^*(M)$.

Proof. Suppose that M is supereulerian. Then it is known (see, e.g., Lemma 5.2 of [16]) that every contraction of M is also supereulerian. Assume that M' is supereulerian. Then a finite number of repeated applications of Lemma 2.2 would show that M is supereulerian. This proves (i).

By Definition 2.3, if M' is not spanned by a subset N that is eligible in M, then as M' is a binary reduction of M, M' does not have a circuit C with $|C| \leq 3$. It follows that $g(M') \geq 4$. As by definition, M' is a contraction of M, and so by Proposition 3.1.17 of [23], we have $g^*(M') \geq g^*(M)$. This proves (ii) and (iii). □

2.2 | Seymour's decomposition theorem of regular matroids

We follow [23] to adopt the definition of k-separations and k-connectedness of matroids, for an integer $k > 0$. Let M_1 and M_2 be two binary matroids with collections of cycles $\mathcal{C}_0(M_1)$ and $\mathcal{C}_0(M_2)$, respectively. Seymour ([25]) introduced a binary matroid $M_1 \triangle M_2$ with ground set $E = E(M_1) \triangle E(M_2)$ whose collection of cycles are $\{X_1 \triangle X_2 : X_1 \in \mathcal{C}_0(M_1) \text{ and } X_2 \in \mathcal{C}_0(M_2)\}$. (See Lemma 9.3.1 of [23]). Three special cases of $M_1 \triangle M_2$ are introduced by Seymour ([25]) as follows.

Definition 2.6. For $i \in \{1, 2\}$, let M_i be a binary matroid with $E_i = E(M_i)$.

(i) If $E_1 \cap E_2 = \varnothing$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **1-sum** of M_1 and M_2, denoted by $M_1 \oplus M_2$.
(ii) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{p\}$, say, and p is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **2-sum** of M_1 and M_2, denoted by $M_1 \oplus_2 M_2$.
(iii) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **3-sum** of M_1 and M_2, denoted by $M_1 \oplus_3 M_2$.

It follows from Definition 2.6 that for an integer $i \in \{2, 3\}$, if M is i-connected and $M = M_1 \oplus_i M_2$, then

$$r(M) = r(M_1) + r(M_2) - (i - 1). \tag{4}$$

The following is a summary of some of the useful properties related to the sums of binary matroids.

Lemma 2.7. *Let M_1 and M_2 be two binary matroids.*

(i) (*Proposition* 7.1.22(*i*) *of* [23]) $(M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^*$
(ii) (*Seymour*, [25]) $(M_1 \oplus_3 M_2)^* = M_1^* \triangle M_2^*$.
(iii) (*Proposition* 7.1.22(*ii*) *of* [23]) *$M_1 \oplus_2 M_2$ is connected if and only if both M_1 and M_2 are connected.*
(iv) (*Proposition* 7.1.22(*ii*) *and Lemma* 9.3.3 *of* [23]) *Let M_1 and M_2 be binary matroids with*

$$\min\{|E(M_1)|, |E(M_2)|\} \geq 7$$

and $E(M_1) \cap E(M_2) = Z$ such that $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. Then

$$\mathcal{C}(M_1 \oplus_3 M_2) = \mathcal{C}(M_1 - Z) \cup \mathcal{C}(M_2 - Z) \cup \{C_1 \triangle C_2 : C_1 \cap Z = C_2 \cap Z, C \in \mathcal{C}(M_1) \text{ and } D \in \mathcal{C}(M_2)\}.$$

(v) (*Lemma* 9.3.4 *of* [23]) *The 3-sum $M_1 \oplus_3 M_2$ has $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ as an exact 3-separation, with* $\min\{|E(M_1) - E(M_2)|, |E(M_2) - E(M_1)|\} > 3$.

(vi) (Theorem 2 of [27]) If $M = M_1 \oplus_3 M_2$ is simple and 3-connected, then M_1 and M_2 are connected.

Let R_{10} denote the vector matroid of the following matrix over $GF(2)$:

$$R_{10} = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

By definition, it is observed (Observation 2.6 of [20]) that

$$R_{10} \text{ is supereulerian}, \quad R_{10}^* \cong R_{10}, \quad \text{and} \quad \forall e \in E(R_{10}), \quad R_{10} - e \cong M(K_{3,3}). \tag{5}$$

Lemma 2.8 (Lemma 2.8 of [20]). *Let $i \in \{2, 3\}$ be an integer and M be an i-connected binary matroid such that M is an i-sum with one of the summands being isomorphic to either R_{10} or a graphic matroid or a cographic matroid. Then there exist binary matroids M_1 and M_2 with $M = M_1 \oplus_i M_2$ such that M_2 is isomorphic to either R_{10} or a graphic matroid or a cographic matroid satisfying*

$$r(M_2) \leq (r(M) - i + 1)/2, \text{ or equivalently}, r(M) \geq 2r(M_2) + i - 1. \tag{6}$$

The next result is a fundamental theorem of Seymour known as the decomposition theorem of regular matroids. The original version of Seymour's theorem in [25] is slightly different, and the version presented here is known (see (3.3) of [15]) to be equivalent to the original version.

Theorem 2.9 (Seymour [25]). *For a connected regular matroid M, one of the following must hold.*

(i) *M is graphic, cographic, or $M \cong R_{10}$.*
(ii) *M is 2-connected and $M = M_1 \oplus_2 M_2$ is a 2-sum of M_1 and M_2. Each of M_i ($i = 1, 2$) is isomorphic to a proper minor of M, where M_2 is either graphic, cographic, or isomorphic to R_{10}.*
(iii) *M is 3-connected and $M = M_1 \oplus_3 M_2$ is a nontrivial 3-sum of M_1 and M_2. Each of M_i ($i = 1, 2$) is isomorphic to a proper minor of M, where M_2 is either graphic or cographic.*

Corollary 2.10. *Let M be a connected regular matroid with cogirth $g^*(M)$ at least 5. Each of the following holds.*

(i) *M is not isomorphic to R_{10}.*
(ii) *M can not be expressed as a 2-sum of M_1 and R_{10}.*

Proof. As $g(R_{10}) = 4$, it follows from (5) that $g(R_{10}^*) = g(R_{10}) = 4 < 5$. Hence $M \ncong R_{10}$. Hence we must have (i). Suppose that M is 2-connected and $M = M_1 \oplus_2 M_2$ is a

2-sum of M_1 and M_2, with $M_2 = R_{10}$, and with $E(M_1) \cap E(P_{10}) = \{p\}$. By Lemma 2.7, $M^* = M_1^* \oplus_2 R_{10}^*$, and so $\mathcal{C}(R_{10}^* - p) \subset \mathcal{C}(M^*) = \mathcal{C}^*(M)$. However, by (5), $R_{10}^* - p \cong M(K_{3,3})$, and so $4 = g(R_{10}^* - p) \geq g(M^*) \geq 5$, a contradiction. This proves (ii). □

Proposition 2.11 (Proposition 5.5 of [16]). *Let M, M_1 and M_2 be binary matroids in which $M = M_1 \triangle M_2$ with $Z = E(M_1) \cap E(M_2)$ and such that one of the following holds.*

(i) *$Z = \{e_0\}$ and $M = M_1 \oplus_2 M_2$ is a 2-sum, or*
(ii) *$Z = \{e_1, e_2, e_3\}$ and $M = M_1 \oplus_3 M_2$ is a 3-sum,*

Suppose that $M_2 = M(G)$ is graphic and $G - Z$ contains a nontrivial collapsible subgraph L. Then $M/E(L)$ is supereulerian if and only if M is supereulerian.

2.3 | Fractional arboricity and strength of matroids

Our arguments depend on some of the former results in [6] as our tools, and we bring in some basics on fractional arboricity and strength of matroids. The notion of matroid strength is brought in by Cunningham in [9]. Some of the sensitivity discussions on matroid strength can be found in [18, 19]. Let M be a matroid with $r(M) > 0$ and let cl_M denote the closure operator of M. For any subset $X \subseteq E(M)$ with $r(X) > 0$, define

$$d_M(X) = \frac{|X|}{r(X)}.$$

The subscript M is often omitted when it is understood from the context. As in [6], we define

$$\eta(M) = \min_{X \subseteq E(M), r(X) < r(M)} d(M/X) \quad \text{and} \quad \gamma(M) = \max_{X \neq \emptyset} d(X), \tag{7}$$

respectively. The invariants $\eta(M)$ and $\gamma(M)$ are known as the **strength** and the **fractional arboricity** of a matroid M, respectively. Let $X \subseteq E(M)$ be a subset. As $r(X) = r(cl_M(X))$, it follows from (7) that

$$\eta(M/X) = \eta(M/cl_M(X)). \tag{8}$$

A summary of useful properties is displayed in Theorem 2.12 stated in the following.

Theorem 2.12. *Let p and q be integers with $p > q > 0$ and let M be a matroid with $r(M) > 0$. Each of the following holds.*

(i) *(Edmonds [10], see also Corollary 5 of [6]) $\gamma(M) \geq \frac{|E|}{r(M)} \geq \eta(M)$, and $\tau(M) = \lfloor \eta(M) \rfloor$.*

(ii) (*Corollary* 5 *of* [6]) *$E(M)$ has a nonempty subset X with $\eta(M|X) \geq \frac{p}{q}$ if and only if $\gamma(M) \geq \frac{p}{q}$.*

(iii) (*Theorem* 1 *of* [6]) *$\eta(M^*)(\gamma(M) - 1) = \gamma(M)$ and $\gamma(M^*)(\eta(M) - 1) = \eta(M)$.*

(iv) (*Lemma* 9 *of* [6]) *For any closed set $X \subset E(M)$ with $\gamma(X) < \gamma(M)$, $\eta(M) \leq \eta(M/X)$.*

(v) *If for some subset (possibly empty) $X \subseteq E(M)$, we have $\eta(M^*/X) \leq \frac{p}{q}$, then there exists a subset $N \subset E(M)$ such that $\eta(M|N) \geq \frac{p}{p-q}$.*

Proof. We only need to prove (v). As $M^*/X = (M - X)^*$, it follows by Theorem 2.12(iii) that

$$\begin{aligned}\gamma(M - X) &= \eta((M - X)^*)(\gamma(M - X) - 1) = \eta(M^*/X)(\gamma(M - X) - 1) \\ &\leq \frac{p}{q}(\gamma(M - X) - 1).\end{aligned}$$

By algebraic manipulation, we have $(p - q)\gamma(M - X) \geq p$ and so $\gamma(M - X) \geq \frac{p}{p-q}$. By Theorem 2.12(ii), we conclude that there exists a nonempty subset $N \subset E(M - X) \subseteq E(M)$ such that $\eta(M|N) = \eta((M - X)|N) \geq \frac{p}{p-q}$. □

3 | REDUCED TO REGULAR MATROIDS WITH GIRTH AT LEAST 4

The main purpose of this section is to show that Theorem 1.3 is equivalent to a seemingly weaker version that an additional girth condition is imposed, as stated in the following.

Theorem 3.1. *Let M be a simple regular matroid with $g(M) \geq 4$. If M satisfies (2), then M is supereulerian.*

The following Lemma 3.2 indicates that Theorem 1.3 and Theorem 3.1 are indeed equivalent to each other. Once we justify Lemma 3.2, the main goal of the article will be to prove Theorem 3.1.

Lemma 3.2. *Let M be a simple regular matroid that satisfies (2). The following are equivalent.*

(i) *M is supereulerian.*

(ii) *If, in addition, we have $g(M) \geq 4$, then M is supereulerian.*

Proof. It suffices to assume that (ii) holds to prove (i). Let M be a simple regular matroid that satisfies (2), and let M' be a binary reduction of M. If $\tau(M') \geq 2$ or if M' is spanned by a circuit, then by Lemma 2.2 or by definition, M' is supereulerian. It follows from Lemma 2.5(i) that M is supereulerian. Now assume that both $\tau(M') < 2$ and M' is not spanning by a circuit of size at most 3. By Lemma 2.5(ii), we have $g(M') \geq 4$. By Lemma 2.5(iii) and (2), we have

$$g^*(M') \geq g^*(M) \geq \max\left\{\frac{r(M)+1}{10}, 9\right\} \geq \max\left\{\frac{r(M')+1}{10}, 9\right\}.$$

Since Lemma 3.2(ii) holds, and since M' satisfies the hypothesis of Lemma 3.2(ii), we conclude that M' is supereulerian. It follows now by Lemma 2.5(i) that M is supereulerian. □

4 | SUPEREULERIAN GRAPHIC AND COGRAPHIC MATROIDS

In this section, we shall justify Theorem 1.4 and show that Theorem 3.1 holds for graphic and cographic matroids. Let G be a graph. For an integer $i \geq 0$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}, \text{ and } \quad d_i = |D_i(G)|. \tag{9}$$

By Theorem 2.1(i), collapsible graphs are supereulerian, and so Theorem 4.1(ii) implies Theorem 4.1(i).

Theorem 4.1. *Let G be a 4-edge-connected graph, and let $M = M(G)$ be the cycle matroid of G.*

(i) *(Jaeger [12] and Catlin [4]) The matroid $M(G)$ is supereulerian.*
(ii) *(Catlin [4]) The graph G is collapsible, and so M is contractible in $\mathcal{G}$.*

By (1), if M is a graphic matroid satisfying (2), then Theorem 4.1 indicates that M must be supereulerian. Hence

$$\text{Theorem 3.1 holds if } M \text{ is graphic.} \tag{10}$$

To discuss cographic matroids, we modify the idea presented in Section 3 of [20] to show a useful relationship involving the girth and the order of the graph. For vertices u, v in a graph H, we use $dist_H(u, v)$ to denote the length of a shortest path with u and v being the termini in the path, which is also known as the distance between u and v in H.

Lemma 4.2. *Let d_1, d_2, g, h be integers with $d_1 \geq d_2 \geq 2, g \geq 3$ and $h = \left\lfloor \frac{g-1}{2} \right\rfloor$. If G is a simple graph on n vertices with girth $g(G) \geq g$, $\Delta(G) = d_1$ and $\delta(G) = d_2$, then*

$$n \geq \begin{cases} 1 + \dfrac{d_1((d_2-1)^h - 1)}{2} & g = 2h+1, \\ 2 + \dfrac{(d_1+2)((d_2-1)^h - 1)}{2} & g = 2h. \end{cases}$$

Proof. Let $v_0 \in V(G)$ be a vertex with $d_G(v_0) = d_1$. For each integer $i \geq 0$, let $V_i = \{v \in V(G) : dist_G(v, v_0) = i\}$. Suppose first that $g = 2h+1$ is odd. As $d_1 = \delta(G)$ and $g(G) \geq g$,

$$n \geq \sum_{i=0}^{h}|V_i| \geq 1 + d_1\sum_{i=0}^{h-1}(d_2-1)^i = 1 + \frac{d_1((d_2-1)^h-1)}{2}.$$

Now assume that $g = 2h$ is even. Let $v_0' \in N_G(v_0)$ and define, $W_0 = \{v_0, v_0'\}$ and for each $i \geq 1$, $W_i = \{v \in V(G) : \min\{dist_G(v, v_0), dist_{G(v,v_0')}\} = i\}$. As $d_1 = \delta(G)$ and $g(G) \geq g$,

$$n \geq \sum_{i=0}^{t}|W_i| \geq 2 + (d_1+2)\sum_{i=0}^{h-1}(d_2-1)^i = 2 + \frac{(d_1+2)((d_2-1)^h-1)}{2}.$$

This proves the lemma. □

Definition 4.3 (Liu et al. [21]). Let c, δ and g be integers with $\min\{\delta, g\} \geq 3$ and $2 \leq c \leq \delta - 1$, and let $h = \left\lfloor \frac{g-1}{2} \right\rfloor$ Define

$$\nu(\delta, g, c) = \begin{cases} 1 + (\delta - c)\sum_{i=0}^{h-1}(\delta-1)^i & \text{if } g = 2h+1, \\ 2 + (2\delta - 2 - c)\sum_{i=0}^{h-1}(\delta-1)^i & \text{if } g = 2h. \end{cases} \tag{11}$$

Lemma 4.4 (Lemma 3.2 of [21]). *Let G be a simple connected graph and let X be a vertex cut of G and H be a connected component of $G - X$. If $\delta := \min\{d_G(v): v \in V(H)\} \geq 4$, $|X| \leq 3$, and the girth of H satisfies $g := g(H) \geq 4$, then*

$$|V(H)| \geq \nu(\delta, g, 3) = \begin{cases} 1 + (\delta-3)\sum_{i=0}^{h-1}(\delta-1)^i & \text{if } g = 2h+1, \\ 2 + (2\delta-5)\sum_{i=0}^{h-1}(\delta-1)^i & \text{if } g = 2h. \end{cases}$$

Proof. In the proof of Lemma 3.2 in [21] with $A = |V(H)|$, only the facts that X is a vertex cut of G with $|X| < \delta$ and that every vertex $v \in A$ satisfies $d_G(v) \geq \delta$ are used. Therefore, by Lemma 3.2 of [21], we have $|V(H)| \geq \nu(\delta, g, |X|)$. By Definition 4.3, we observe that if $c_1 \leq c_2$, then $\nu(\delta, g, c_1) \geq \nu(\delta, g, c_2)$. Hence as $|X| \in \{2, 3\}$, we have $|V(H)| \geq \nu(\delta, g, |X|) \geq \nu(\delta, g, 3)$. This completes the proof of the lemma. □

By definition, a matroid M has a spanning cycle if and only if every component of M has a spanning cycle. Thus to prove Theorems 1.3 and 3.1 for all simple regular matroids, it suffices to prove Theorems 1.3 and 3.1 for all connected simple regular matroids. We now assume that M is connected cographic matroid, and so there exists a connected graph G such that $M = M^*(G)$. As M is connected, M^* is also connected and so G is a 2-connected simple graph. By (1) and (2), $g(G) = g^*(M^*(G)) = g^*(M) \geq 4$. By the assumption of Theorem 3.1, we have $g(M) \geq 4$, and so $\delta(G) \geq \kappa'(G) = g(M) \geq 4$.

Lemma 4.5. *Let G be a connected simple graph with $\delta(G) \geq 4$, $m = |E(G)|$, $n = |V(G)|$ and $g = g(G)$.*

(i) *Suppose that* $g \geq 4$. *For any real number* c *with* $0 < c < 1$, *there exists a positive integer* $f(c)$ *such that if* $g \geq \max\{c(m-n+2), f(c)\}$, *then* G *is* 4-*regular.*
(ii) *If* $g \geq \max\left\{\frac{m-n+2}{10}, 9\right\}$, *then* G *is* 4-*regular.*

Proof. Let $d_1 = \Delta(G)$, and $v_0 \in V(G)$ such that $d_G(v_0) = d_1$. With the definition of $D_i(G)$ in (9), we define $R = \bigcup_{i\geq 5} D_i(G)$ and $r = |R|$. Thus as $\delta(G) \geq 4$, we have

$$2m = 2|E(G)| = \sum_{v\in V(G)} d_G(v) = \sum_{i\geq 4} i|D_i(G)| = 4n + \sum_{i\geq 5}(i-4)|D_i(G)| \geq 4n + r.$$

This, together with the assumption that $g \geq c(m-n+2)$, implies that

$$2n \leq \frac{2g}{c} - r - 4. \tag{12}$$

For a fixed real number c with $0 < c < 1$, define a function $\phi_c(x) = 3^x - \frac{4x}{c} - \frac{2}{c} + 5$. As $\phi_c'(x) = 3^x \ln(3) - \frac{4}{c} > 0$ for any values of x, it follows that $\phi_c(x)$ is an increasing function on the interval $\left(\log_3\left(\frac{4}{c\ln(3)}\right), \infty\right)$ with $\lim_{x\to\infty}\phi_c(x) = \infty$, and so there exists a smallest positive integer x_0 such that for any $x \geq x_0$,

$$3^x \geq \frac{4x}{c} + \frac{2}{c} - 5. \tag{13}$$

With a similar argument, we conclude that there exists also a smallest positive integer x_0' such that for any $x \geq x_0'$,

$$3^{x+1} \geq \frac{2x}{c} - 7. \tag{14}$$

Now let $f(c) = \max\{2x_0 + 1, 2x_0'\}$, $g \geq f(c)$, and $h = \left\lfloor \frac{g-1}{2} \right\rfloor$. Then $h \geq x_0$. Suppose $g = 2h + 1$. If G is not 4-regular, then $d_1 = \Delta(G) \geq 5$ and $r > 0$. By Lemma 4.2, G has at least $1 + d_1\left(\frac{(\delta-1)^h - 1}{\delta - 2}\right) \geq 1 + \frac{d_1}{2}(3^h - 1)$ vertices of distance at most h from v_0. Thus by (12), $2 + d_1(3^h - 1) \leq 2n \leq \frac{2}{c}(2h+1) - r - 4$. Since $d_1 \geq 5$ and $r > 0$, we have $3^h < \frac{4h}{5c} + \frac{2}{5c} - \frac{1}{5}$. On the other hand, as $h \geq x_0$, it follows by (13) that $3^h \geq \frac{4h}{5c} + \frac{2}{5c} - \frac{1}{5}$, a contradiction. Hence G must be 4-regular in this case.

Assume next that $g = 2h + 2$. If G is not 4-regular, then we have $d_1 \geq 5$ and $r > 0$. By 4.2, G has at least $2 + (d_1 + 2)\left(\frac{(\delta-1)^{h-1} - 1}{\delta - 2}\right) \geq 2 + \frac{d_1+2}{2}(3^{h-1} - 1)$ vertices of distance at most h from v_0. Thus by (12), $4 + (d_1 + 2)(3^{h-1} - 1) \leq 2n \leq \frac{4h}{c} - r - 4$. It follows from $d_1 \geq 5$ and $r > 0$ that $3^{h-1} < \frac{4}{7c}h - \frac{1}{7}$. As $h \geq x_0'$, it follows by (14) that $3^{h-1} \geq \frac{4}{7c}h - \frac{1}{7}$, a contradiction. Hence G must be 4-regular in any case. This proves (i).

We choose $c_0 = \frac{1}{10}$ in the arguments above. As $\phi_{c_0}(4) = 3^4 - 32 - 4 + \frac{1}{5} > 0$, we have $x_0 \leq 4$. With a similar argument, we also have $x_0' \leq 4$. Hence $f(c_0) = \max\{2x_0 + 1, 2x_0'\} \leq 9$. This completes the proof for (ii). □

An **acyclic** k**-partition** of a graph G is a collection of mutually disjoint nonempty subsets $V_1, V_2, ..., V_k$ of $V(G)$ with $V(G) = \bigcup_{i=1}^{k} V_i$ such that for each i with $1 \le i \le k$, $G[V_i]$ is acyclic. Define the **vertex arboricity** of a graph G to be

$$a(G) = \min\{k : V(G) \text{ has an acyclic } k\text{-partition}\}.$$

Kronk and Mitchem in [13] proved a useful relationship between $a(G)$ and the maximum degree $\Delta(G)$ of a graph G. It is also known that $a(G)$ can be used to study if $M^*(G)$ is supereulerian.

Theorem 4.6 (Kronk and Mitchem [13]). *If G is connected, not complete and $a(G) = k \ge 3$, then $\Delta(G) \ge 2k - 1$.*

Lemma 4.7 (Lemma 3.2 of [20]). *Let G be a graph and $M = M(G)$. The following are equivalent.*

(i) *$M^*(G)$ is supereulerian.*
(ii) *There eixsts a 2-partition $\{V_1, V_2\}$ of $V(G)$ such that each of $G[V_1]$ and $G[V_2]$ is a forest.*
(iii) $a(G) \le 2$.

Theorem 4.6 and Lemma 4.7 are now applied to prove the Theorem 4.8, another main result in this subsection.

Theorem 4.8. *Let G be a 2-connected graph with $m = |E(G)|, n = |V(G)|$ and $g(G) \ge 4$, and $M = M^*(G)$ be the cocycle matroid of G. Each of the following holds.*

(i) *For any real number c with $0 < c < 1$, there exists a positive integer $f(c)$ such that if $g^*(M) \ge \max\{c(r(M) + 1), f(c)\}$, then M is supereulerian.*
(ii) *If $g^*(M) \ge \max\left\{\frac{m-n+2}{10}, 9\right\}$, then M is supereulerian.*

Proof. As $M = M^*(G)$, we have $g(G) = g(M^*) = g^*(M)$. By Lemma 4.7, we only need to show that $a(G) \le 2$. By (1), $\kappa'(G) = g(M) \ge 4$, and so $\delta(G) \ge 4$. By Lemma 4.5(i), there exists a positive integer $f(c)$ such that if

$$g(G) = g^*(M) \ge \max\{c(r(M) + 1), f(c)\} = \max\{c(m - n + 2), f(c)\},$$

then G is 4-regular. Therefore, we assume that G is simple 2-connected 4-regular graph with girth $g \ge 4$. This implies that $|V(G)| \ge 5$. As $g(G) \ge 4$, G is not a complete graph. If $a(G) \ge 3$, then by Theorem 4.6, we must have $\Delta(G) \ge 5$, contrary to the assumption that G is 4-regular. Hence $a(G) \le 2$ and so Theorem 4.8 (i) follows from Lemma 4.7.

Now assume that $g^*(M) \ge \max\left\{\frac{m-n+2}{10}, 9\right\}$. We adopt the same arguments above but quoting Lemma 4.5(ii) instead of Lemma 4.5(i). This leads to the conclusion that $a(G) \le 2$, and so by Lemma 4.7, M is supereulerian. □

Proof of Theorem 1.4. Let M be a connected simple cographic matroid. For a fixed real number c with $0 < c < 1$, let $f(c)$ be the integer as determined by Theorem 4.8(i).

Assume that $g^*(M) \geq \max\{c(r(M)+1), f(c)\}$.

Let M' be a binary reduction of M. If M' is spanned by an eligible subset, then by Corollary 2.4, M' is supereulerian, and so by Lemma 2.5(i), M is supereulerian. Hence we assume that M' is not spanned by an eligible subset. By Lemma 2.5, $g(M') \geq 4$ and $g^*(M) \geq g^*(M) \geq \max\{c(r(M)+1), f(c)\} \geq \max\{c(r(M')+1), f(c)\}$. It follows by Theorem 4.8(i) that M' is supereulerian. By Lemma 2.5(i), M is supereulerian. This completes the proof of Theorem 1.4. □

5 | PROOF OF THEOREM 3.1

We argue by contradiction and assume that there exists a counterexample to Theorem 3.1. Among all the counterexamples, we choose a regular matroid M that satisfies (2) but violates the conclusion of Theorem 3.1 so that

$$M \text{ is not supereulerian with } |E(M)| \text{ is minimized.} \tag{15}$$

Thus by (15), (5), (10) and by Theorem 4.8, we may assume that M is connected, and M is neither graphic nor cographic, and $M \ncong R_{10}$. This, together with Lemma 2.8, Theorem 2.9 and Corollary 2.10, leads to the conclusion that for some $i \in \{2, 3\}$, and some proper minors M_1 and M_2 of M, we have

$$M = M_1 \oplus_i M_2, \text{ such that } M_2 \text{ is either graphic or cographic satisfying (6).} \tag{16}$$

Therefore, we will proceed our proof arguments according to the cases when M_2 is graphic and when M_2 is cographic.

5.1 | Proof of Theorem 3.1 when M_2 is graphic

Let G be a connected graph with $M_2 = M(G)$ as the cycle matroid of G, and $Z = E(M_1) \cap E(M_2)$ such that either $Z = \{p\}$ for some element p that is neither a loop nor a coloop in each of M_1 and M_2, or $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. Then $Z \subseteq E(M_2) = E(G)$. Define $V(Z)$ to be the set of vertices in G that are incident with edges in Z. Note that we have $\mathcal{C}^*(M_2/Z) = \mathcal{C}(M_2^* - Z) \subset \mathcal{C}(M^*) = \mathcal{C}^*(M)$. As M_2 is graphic with $M_2 = M(G)$, we have

$$\delta(G/Z) \geq \kappa'(G/Z) = g^*(G/Z) \geq g^*(M_2/Z) \geq g^*(M). \tag{17}$$

If M satisfies (2), then it follows from (17) and (1) that, for any $v \in V(G) - V(Z)$, $d_G(v) \geq \delta(G) \geq \kappa'(G) = g^*(M) \geq 9$. To prove Theorem 3.1 when M_2 is graphic, we need the following lemmas.

Lemma 5.1. *Let G be a 2-connected graph with an edge subset $Z \subseteq E(G)$ such that for any $v \in V(G) - V(Z)$, $d_G(v) \geq 9$. Each of the following holds.*

(i) *If $Z = \{e\}$, then $G - e$ contains a nontrivial collapsible subgraph.*

(ii) *If* $Z = \{e_1, e_2, e_3\}$ *is a circuit of* G, *then* $G - Z$ *contains a nontrivial collapsible subgraph.*

Proof. Let $H = G - Z$ and $n = |V(H)|$. As Z is either an edge or a 3-circuit, we have $|V(Z)| \le 3$. For integers $i \ge 1$, define $D_i(H)$ and $d_i = |D_i(H)|$ as in (9), with H replacing G in (9). We start arguing by contradiction to assume that H has no nontrivial collapsible subgraphs. By Theorem 2.1(ii), H is reduced. Since for any $v \in V(H) = V(G) - V(Z)$, we have $d_H(v) \ge d_G(v) - |V(Z)| \ge 9 - 3 = 6$, and so $n \ge 7$, implying that $H \notin \{K_1, K_2\}$. By Theorem 2.1(iii), we have $F(H) \ge 2$. By Theorem 2.1(iv),

$$4n - 2|E(H)| - 4 = 2F(H) \ge 4, \text{ and so } 2|E(H)| \le 4n - 8.$$

Suppose first that $Z = \{e\}$, and so $|V(Z)| \le 2$. Then $4n - 8 \ge 2|E(H)| = \sum_{v \in V(H)} d_H(v) \ge \sum_{v \in V(H)} (d_G(v) - 2) = 7(n - 2)$, implying $n \le 2$, a contradiction. Now assume that Z is a 3-circuit in G, and so $|V(Z)| = 3$. In this case, we have $4n - 8 \ge 2|E(H)| = \sum_{v \in V(H)} d_H(v) \ge \sum_{v \in V(H)} (d_G(v) - 3) = 6(n - 3)$. Once again we are led to $n \le 5$, contrary to the fact that $n \ge 7$. These contradictions indicate that the assumption that H does not have a nontrivial collapsible subgraph is false. This proves the lemma. □

Lemma 5.2. *Let* $i \in \{2, 3\}$ *be an integer,* M *be a binary matroid such that for some binary matroids* M_1 *and* M_2, *we have* $M = M_1 \oplus_i M_2$ *with* $Z = E(M_1) \cap E(M_2)$. *Suppose that for some connected graph* G, $M_2 = M(G)$ *is the graphic matroid of* G. *Let* $H = G - Z$ *and* $J_1, J_2, \ldots, J_s$ *be the maximal nontrivial collapsible subgraphs of* H, $J = \bigcup_{i=1}^{s} E(J_i)$. *If* M/J *is supereulerian, then* M *is supereulerian.*

Proof. We argue by induction on s. If $s \le 1$, then Lemma 5.2 follows from Proposition 2.11. Inductively, we assume t hat $s \ge 2$ and Lemma 5.2 holds for smaller values of s. Let $N = M/(\bigcup_{i=1}^{s-1} E(J_i))$. Then N is supereulerian. By Proposition 2.11, $N/E(J_s) = M/J$ is supereulerian, and so by Proposition 2.11 again, N is supereulerian. Now by the inductive assumption, that N is supereulerian implies that M is supereulerian. This proves Lemma 5.2. □

The next result is the main conclusion of this subsection. It implies that, under the assumptions of (15) and (16),

$$\text{Theorem 3.1 holds if } M_2 \text{ is a graphic matroid.} \tag{18}$$

Theorem 5.3. *Let* $i \in \{2, 3\}$ *be an integer and let* M *be an* i*-connected regular matroid with* $g(M) = g \ge 4$ *such that for some proper minors* M_1 *and* M_2, *we have* $M = M_1 \oplus_i M_2$, *and such that* M_2 *is graphic. Then* M *is supereulerian if both of the following hold.*

(i) M *satisfies* (2).
(ii) *Every simple regular matroid* M' *satisfying* (2) *with* $|E(M')| < |E(M)|$ *is supereulerian.*

Proof. Let M be a regular matroid satisfying the hypotheses of Theorem 5.3. By Lemma 2.7 (iii) and (vi), M_2 is connected. Since M_2 is graphic and connected, by Proposition 4.1.7 of [23], there is a 2-connected graph G with $M_2 = M(G)$. Set $Z = E(M_1) \cap E(M_2)$ and let $H = G - Z$.

Then by Lemma 5.1, H has a nontrivial collapsible subgraph. Let J be the disjoint union of all nontrivial collapsible subgraphs of H, and let $N = M/E(J)$. If $g(N) \geq 4$, then define $M' = N$. If N has a circuit of size at most 3, then let M' be a binary reduction of N. Thus M' is always defined.

If M' is eligible, then by Corollary 2.4, M' is supereulerian. By Lemma 2.5(i), M is supereulerian, and we are done. Hence we assume that M' is not an eligible matroid, and so by Lemma 2.5(ii), $g(M') \geq 4$. By Lemma 2.5(iii) and (2), M' satisfies (2) as well:

$$g^*(M') \geq g^*(M) \geq \max\left\{\frac{r(M)+1}{10}, 9\right\} \geq \max\left\{\frac{r(M')+1}{10}, 9\right\}.$$

Since J contains a nontrivial collapsible subgraph of $H, |E(J)| > 0$ and so $|E(M)| > |E(M/E(J))| = |E(N)| \geq |E(M')|$. It follows by the assumption in Theorem 5.3(ii) that M' is supereulerian. Thus whether $M' = N$ or M' is a binary reduction of N, by Lemma 2.5 (i), N is supereulerian, it follows by Lemma 5.2 that M is supereulerian. This proves the theorem. □

5.2 | Proof of Theorem 3.1 when M_2 is cographic

By Theorem 5.3, we conclude that Theorem 3.1 holds if M_2 is graphic. Hence we assume that M_2 is cographic, and so there exists a connected graph G such that $M_2 = M^*(G)$ is the cographic matroid of G. Throughout this subsection, we let $n = |V(G)|$ and $m = |E(G)|$. To deal with this case, we modify an idea used in [20] to investigate the structural properties of graphs with large girth and with some specific 3-edge-cuts.

5.2.1 | The graph family $\mathcal{F}(g)$

We will introduce a family of graphs which plays an important role in the proof arguments to deal with the case when M_2 is cographic. Throughout this subsection, let $g \geq 4$ be an integer and $h = \left\lfloor \frac{g}{2} \right\rfloor$.

Definition 5.4. Let $\mathcal{F}(g)$ be the family of all 2-connected simple graphs such that $G \in \mathcal{F}(g)$ if and only if each of the following holds.

(F1) There exists a 3-edge-cut Z of G (called a **special-cocircuit** of G) such that $|E(G-Z)| > 3$.
(F2) If G has an edge cut X with $|X| \leq 3$, then $X \cap Z \neq \emptyset$,
(F3) $g(G - Z) \geq g$.

Suppose that $G \in \mathcal{F}(g)$ with a special-cocircuit Z. For any $v \in V(G) - V(Z)$, $E_G(v)$ is an edge cut of G disjoint from Z. It follows by Definition 5.4(F2) that $|E_G(v)| \geq 4$. Hence we have the following.

$$\text{If } G \in \mathcal{F}(g), \text{ then for any cocircuit } D \text{ of } G - V(Z), \text{ we have } |D| \geq 4. \tag{19}$$

Lemma 5.5. *Let $G \in \mathcal{F}(g)$ with $m = |E(G)|$ and $n = |V(G)| \geq 7$, and with Z being the special-cocircuit of G. If $g \geq max\{\frac{m-n+2}{5}, 9\}$, then each of the following holds.*

(i) $\eta(G/Z) \leq 2$.
(ii) *$M^*(G) - Z$ contains a nonempty eligible subset.*

Proof. Assume that (i) holds. Then $\eta(G/Z)) \leq 2$. By Theorem 2.12(v) with $p = 2$ and $q = 1$, there exists a subset $N \subset E(M^*(G) - Z)$ such that $\eta((M^*(G) - Z)|N) \geq 2$. It follows by Theorem 2.12(i) that $\tau((M^*(G) - Z)|N) = \lfloor \eta((M^*(G) - Z)|N) \rfloor \geq 2$. By Definition 2.3, $M^*(G) - Z$ has a nonempty eligible subset N and so (ii) holds.

It remains to prove (i). Let G_1 and G_2 denote the two connected components of $G - Z$. For each $i \in \{1, 2\}$, let $n_i = |V(G_i)|$ and $U_i = V(Z) \cap V(G_i)$. By symmetry, we assume that $n_2 \geq n_1$. If $n_2 = |U_2|$, then $n \leq n_1 + n_2 \leq |V(Z)| \leq 6$, contrary to the assumption that $n \geq 7$. Hence $n_2 > |U_2|$ and so U_2 is a vertex cut of G with $|U_2| \leq 3$. Since G is 2-connected, we have $|U_1|, |U_2| \in \{2, 3\}$.

Let $M = M(G)$ be the cycle matroid of M and let $Z' = cl_M(Z)$. By contradiction and by (8), we assume

$$\eta(G/Z') = \eta(G/Z) > 2. \tag{20}$$

Let $t = |Z'| \geq |Z| = 3$. By (20) and Theorem 2.12(i),

$$2 < \eta(G/Z') \leq \frac{|E(G/Z')|}{|V(G/Z')| - 1} \leq \frac{m - t}{(n-3) - 1}.$$

Algebraic manipulations lead to

$$m - n \geq n - (7 - t) \geq n - 4.$$

Let $d = \min\{d_G(v) : v \in V(G_2)\}$. By (19), $d \geq 4$. By Definition 4.3, if $d \geq d'$, then $\nu(d, g, c) \geq \nu(d', g, c)$. Hence applying Lemma 4.4 to the graph G with vertex cut U_2 of G and component G_2 of $G - U_2$, we conclude that $|V(G_2)| \geq \nu(4, g, 3)$. It follows that

$$n - 4 \geq n - (|U_1| + |U_2|) \geq |V(G_2)| \geq \nu(4, g, 3) = \begin{cases} \dfrac{3^h + 1}{2} & \text{if } g = 2h + 1, \\ \dfrac{3^{h+1} + 1}{2} & \text{if } g = 2h. \end{cases} \tag{21}$$

As $g \geq 9$, by (21), we have $n - 4 \geq \frac{3^4 + 1}{2} = 41$. Since $h = \left\lfloor \frac{g}{2} \right\rfloor \geq \left\lfloor \frac{m-n+2}{5} \right\rfloor$, we have

$$m - n \geq n - 4 > \frac{3^{\lfloor \frac{m-n+2}{5} \rfloor} + 1}{2}.$$

Solving the inequality $m - n > \frac{3^{\lfloor \frac{m-n+2}{5} \rfloor} + 1}{2}$, we have $m - n \leq 12$. As $12 \geq n - 4$, we have $n \leq 16$, contrary to the fact that $n \geq 45$. This contradiction implies that the assumption (20) must be false, and so the lemma is proved. □

Lemma 5.6. *Let $G \in \mathcal{F}(g)$ with $m = |E(G)|$ and $n = |V(G)|$, and with Z being the special-cocircuit of G. If $g \geq 4$ and $m \geq 7$, then $n \geq 9$.*

Proof. As $|Z| = 3$ and G is simple, Z must be acyclic, and so $4 = |Z| + 1 \leq |V(Z)| \leq 6$. Thus by $m \geq 7$, there must be a vertex $v \in V(G) - V(Z)$. Let G_1 and G_2 be the two components of $G - Z$ such that $v \in V(G_1)$. By (19), $d_G(v) \geq 4$. Since $|V(Z) \cap V(G_1)| \leq |Z| = 3$ and since Z is a cocircuit of G, we have $d_{G_1}(v) = d_G(v) \geq 4$ and so there must be a vertex $w \in N_G[v] - V(Z)$. Since $g(G_2) \geq g(G) \geq g \geq 4$, $N_{G_1}(v) \cap N_{G_1}(w) = \varnothing$ and so $n = |V(G)| \geq |V(G_1)| + |V(Z) \cap V(G_2)| \geq |N_{G_1}(v)| + |N_{G_1}(w)| + 1 \geq 9$. This justifies the lemma. □

5.2.2 | Proof of Theorem 3.1 when M_2 is cographic

We start by validating two lemmas.

Lemma 5.7. *Let g be an integer and G be a graph with $m = |E(G)|, n = |V(G)|$ and with a distinguished edge $e_0 = u_0v_0$ such that $M(G)$ is a connected matroid. Let $Z = \{e_0\}$ and suppose that $G - Z$ is simple with $|E(G - Z)| \geq 2$, and for any circuit C of $G - Z, |E(C)| \geq g$. If for any $v \in V(G) - V(Z), d_G(v) \geq 4$, and $g \geq \max\left\{\frac{m-n+2}{5}, 9\right\}$, then each of the following holds.*

(i) $\eta(M(G)/e_0) \leq 2$.
(ii) $M^*(G) - e_0$ *contains a nonempty eligible subset.*

Proof. Suppose (i) holds. By Theorem 2.12 (v) with $p = 2$ and $q = 1$, and by Theorem 2.12 (i), there exists a nonempty subset $N \subseteq E(M^* - e_0)$ with $\tau((M^*(G) - Z)|N) = \lfloor \eta((M^*(G) - Z)|N) \rfloor \geq 2$. By Definition 2.3, N is a nonempty eligible subset of $M^*(G) - Z$, and so (ii) holds.

We argue by contradiction to prove (i) and assume that $\eta(M(G)/e_0) > 2$. Let $Z' = E(G[V(Z)])$. By Theorem 2.12(i) and (8),

$$2 < \eta(M(G)/e_0) = \eta(G/Z') \leq \frac{|E(G/Z')|}{|V(G/Z')| - 1} \leq \frac{m-1}{(n-1)-1},$$

so $m - n \geq n - 2$.

Let J be the graph obtained from G by replacing e_0 by a path $u_0z_0v_0$ with a new vertex z_0. Then $\{u_0, v_0\}$ is a vertex cut of J, with G' being a component of $J - \{u_0, v_0\}$. Applying

Lemma 4.4 to J with vertex cut $\{u_0, v_0\}$ and G' being a component of $J - \{u_0, v_0\} = G - \{u_0, v_0\}$, we have

$$n > |V(G')| \geq \nu(4, g, 3) = \begin{cases} \dfrac{3^h + 1}{2} & \text{if } g = 2h + 1, \\ \dfrac{3^{h+1} + 1}{2} & \text{if } g = 2h + 2. \end{cases} \tag{22}$$

As $g \geq 10$, by (22), we have $n > \frac{3^5 + 1}{2} = 122$. By (22) and by $h = \lfloor \frac{g}{2} \rfloor \geq \lfloor \frac{m-n+2}{10} \rfloor$, we have

$$m - n \geq n - 2 > \frac{3^{\lfloor \frac{m-n+2}{10} \rfloor} + 1}{2}.$$

Hence $m - n \leq 37$ and $n \leq 39$, contrary to the fact that $n > 122$. Hence we must have $\eta(M(G)/e_0) \leq 2$. This justifies (i). □

Lemma 5.8. *Let M be a binary matroid such that for some binary matorids M_1 and M_2, we have $M = M_1 \triangle M_2$. Then for any $X \subseteq E(M_2) - E(M_1)$, $M/X = M_1 \triangle (M_2/X)$.*

Proof. As the cycles of a binary matroid uniquely determine the matroid, it suffices to show that $\mathcal{C}_0(M/X) = \mathcal{C}_0(M_1 \triangle (M_2/X))$. It is known (see, e.g., Lemma 5.3 of [16]) that

$$\mathcal{C}_0(M/X) = \{C - X : C \in \mathcal{C}_0(M)\}. \tag{23}$$

By the definition of $M_1 \triangle M_2$, by (23) (with M replaced by M_2), and by the fact that $X \subseteq E(M_2) - E(M_1)$, we have

$$\begin{aligned} \mathcal{C}_0(M_1 \triangle (M_2/X)) &= \left\{C_1 \triangle C_2' : C_1 \in \mathcal{C}_0(M_1) \text{ and } C_2' \in \mathcal{C}_0(M_2/X)\right\} \\ &= \{C_1 \triangle (C_2 - X) : C_1 \in \mathcal{C}_0(M_1) \text{ and } C_2 \in \mathcal{C}_0(M_2)\} \\ &= \{(C_1 \triangle C_2) - X : C_1 \in \mathcal{C}_0(M_1) \text{ and } C_2 \in \mathcal{C}_0(M_2)\} \\ &= \mathcal{C}_0(M/X). \end{aligned}$$

This proves the lemma. □

The next theorem is the main result of this subsection. This theorem, together with (5), (10), Theorem 4.8, and (18), implies the validity of Theorem 3.1. By Lemma 3.2, Theorem 1.3 is established.

Theorem 5.9. *Let $i \in \{2, 3\}$ be an integer and M be an i-connected simple regular matroid with $g(M) = g \geq 4$, and M_1 and M_2 be proper minors of M such that $M = M_1 \oplus_i M_2$ and such that M_2 is cographic. Then M is supereulerian if both of the following hold.*

(i) *M satisfies (2).*
(ii) *If M' is a regular simple matroid M' satisfying (2) with $|E(M')| < |E(M)|$, then M' is supereulerian.*

Proof. As M is i-connected, by Lemma 2.7(iii) and (vi), M_2, as well as M_2^*, is also connected. As M_2^* is cographic and connected, there exists a 2-connected graph G such that $M_2^* = M(G)$ is the cycle matroid of G. Let $Z = E(M_1) \cap E(M_2)$. By Definition 2.6, for each $i \in \{2, 3\}$, either $i = 2$, $|Z| = 1$ and the only element in Z is not a loop nor a coloop of M_1 and M_2 with $\min\{|E(M_1)|, |E(M_2)|\} \geq 1$; or $i = 3$, $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$ with $\min\{|E(M_1)|, |E(M_2)|\} \geq 7$. By Lemma 5.6, $|V(G)| \geq 7$. □

Claim 5.10. $M_2 - Z = M^*(G) - Z$ contains a nonempty eligible subset.

Since $\mathcal{C}^*(M_2/Z) = \mathcal{C}\left(M_2^* - Z\right) \subset \mathcal{C}(M^*) = \mathcal{C}^*(M)$, it follows that

$$g^*(M_2/Z) \geq g^*(M) \geq \max\left\{\frac{r(M)+1}{10}, 9\right\}.$$

By (16) and Lemma 2.8, for each $i \in \{2, 3\}$, we have $r(M) \geq 2r(M_2) + i - 1$. Let $m = |E(G)|$, $n = |V(G)|$. Then

$$\begin{aligned} g(G-Z) &\geq \max\left\{\frac{r(M)+1}{10}, 9\right\} \geq \max\left\{\frac{2(m-n)+i+2}{10}, 9\right\} \\ &\geq \max\left\{\frac{m-n+2}{5}, 9\right\}. \end{aligned} \tag{24}$$

If $i = 3$, then as $g(M_2 - Z) \geq g(M) = g \geq 4$, it follows that $G \in \mathcal{F}(g)$ with Z being the special-cocircuit of G. Hence (19) holds and so for any $v \in V(G) - V(Z)$, $d_G(v) \geq 4$. By (24), by the fact that $|V(G)| \geq 7$ and by Lemma 5.5(ii), $M_2 - Z = M^*(G) - Z$ contains a nonempty eligible subset. Assume that $i = 2$, and so there exists an edge $e_0 = u_0v_0 \in E(G)$ such that $Z = \{e_0\}$. By (24) and Lemma 5.7(ii), $M_2 - Z = M^*(G) - Z$ also contains a nonempty eligible subset. This justifies Claim 5.10.

Let M' be a binary reduction of M. By Claim 5.10, $|E(M')| < |E(M)|$. If M' is spanned by an eligible subset, then by Corollary 2.4, M' is supereulerian, and so by Lemma 2.5, M is supereulerian. Hence we may assume that M' is not spanned by an eligible subset, and so by Lemma 2.5, $g(M') \geq 4$ and M' satisfies (2). It follows by the assumption in Theorem 5.9(ii) that M' is supereulerian. Then by Lemma 2.5, M is supereulerian. This completes the proof of the theorem.

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Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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