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On list r-hued coloring of outer-1-planar graphs ☆

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ABSTRACT

Let L be list assignment of colors available for vertices of a graph G, an (L, r)-coloring of G is a proper coloring c such that for any vertex $v \in V(G)$, we have $c(v) \in L(v)$ and $|c(N(v))| \geq \min\{d(v), r\}$. The r-**hued list chromatic number** of G, denoted as $\chi_{L,r}(G)$, is the least integer k, such that for list assignment L satisfying $|L(v)| = k \ \forall \ v \in V(G)$, G has an (L, r)-coloring. A graph G is an outer-1-planar graph if G has a drawing on the plane such that all vertices of $V(G)$ are located on the outer face of this drawing, and each edge can cross at most one other edge. For any positive integer r, we completely determine the upper bound of list r-hued chromatic number for all outer-1-planar graphs. This extended a former result in [Discrete Mathematics and Theoretical Computer Science, 23:3 (2021)].

1. Introduction

We follow the notation and terms in Bondy and Murty [3], unless otherwise stated, and consider only simple graphs. Let $\mathbb{N}$ denote the set of all natural numbers. For any vertex $v \in V(G)$, the **neighborhood** of v in G, is $N_G(v) = \{u \in V(G) : u$ is adjacent to v in $G\}$, $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. For notational simplicity, we also write $d_G(v) = d(v)$.

There have been many different coloring of graphs under studies, such as edge coloring in Dailly et al. [6], strong edge-coloring in Debski and Sleszyska-Nowak [7], diagonal coloring in Huang et al. [10], total coloring in Zhang [16], among others. We mainly focus our study on list r-hued colorings of graphs. Let k and r be positive integers. A proper k-coloring of a graph G is a mapping $c : V(G) \to \{1, 2, 3, \ldots, k\}$ such that for any $uv \in E(G)$, $c(u) \neq c(v)$. An (k, r)-coloring of graph G is a proper k-coloring c of G such that every vertex $v \in V(G)$ satisfies the following **local** (k, r)-**coloring condition**:

(LC-1) $|c(N(v))| \geq \min\{d(v), r\}$.
The r-**hued chromatic number** $\chi_r(G)$ of G is the smallest integer k such that G has an (k, r)-coloring. Let L be a mapping which assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. A proper coloring c of G is an L-**coloring** if c satisfies the following list coloring condition:
(LC-2) $c(v) \in L(v)$, for any vertex $v \in V(G)$.
If $k \in N$, then a list L of a graph G is a k-list if $|L(v)| = k$ for any $v \in V(G)$. The **list chromatic number** $\chi_L(G)$ of G is the smallest integer k such that for any k-list L, G has an L-coloring. As in the classical graph coloring problems, the

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r-hued coloring of graphs also has a list coloring counterpart. For a given list L of a graph G, a proper coloring c is an (L, r)**-coloring** of G if c satisfies both (LC-1) and (LC-2). The **list r-hued chromatic number**, denoted by $\chi_{L,r}(G)$, which is also denoted by $ch_r(G)$ in the literature, is the smallest integer k such that for any k-list L of G, G has an (L, r)-coloring. The study of r-hued coloring of graphs was initiated in Lai et al. [12], in which only the case when $r = 2$ was investigated. In the early studies, 2-hued coloring is under the name of dynamic coloring, and the 2-hued chromatic number was called dynamic chromatic number of a graph, denoted by $\chi_d(G)$. Likewise, the list 2-hued chromatic number of G was also denoted as $ch_d(G)$.

There have been many studies on r-hued coloring and r-hued list coloring of graphs, as can be found in a recent survey [5]. One of the heavily studied area is to investigate best possible upper bounds for the r-hued chromatic number and list r-hued chromatic number of planar graphs and graphs that are structurally closed to be planar graphs. In [9], the author proved that $\chi_{L,3}(G) \le 6$ if G is a planar graph which is a near-triangulation, where a near-triangulation is a planar graph whose bounded faces are all 3-cycles. Extending a conjecture of Wegner, Song et al. proposed the following conjecture.

Conjecture 1.1 (Song et al. [13]). *Let r be a positive integer. If G is a planar graph, then $\chi_r(G) \le f(r)$, where*

$$f(r) = \begin{cases} r+3, & \text{if } 1 \le r \le 2; \\ r+5, & \text{if } 3 \le r \le 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 8. \end{cases}$$

It is believed that a list r-hued coloring version of Conjecture 1.1 with a similar upper bound would also be likely [4,13,14]. Utilizing the 4-color-theorem, Kim, Lee and Park, prove the following.

Theorem 1.2 (Kim et al. [11]). *Let G be a planar graph. Then $\chi_{L,2}(G) \le 5$.*

In [8], Eggleton introduced outer-1-planar graphs. A graph G is an **outer-1-planar** graph if G can be drawn on the plane in such a way that all vertices of $V(G)$ are located on the outer face (the exterior face) of the drawing and every edge of G can cross at most one other edge in the drawing. As indicated in Corollary 1 of [2],

$$\text{outer-1-planar graphs form a proper subset of all planar graphs.} \tag{1}$$

When $r = 3$, upper bounds of the list r-hued chromatic numbers for outer-1-planar graphs have been studied.

Theorem 1.3 (Zhang and Li [15]). *If G is an outer-1-planar graph, then $\chi_{L,3}(G) \le 6$.*

The current research is being motivated by the theorems mentioned above. To the best of our knowledge, there has been little former studies on the list r-hued chromatic numbers for outer-1-planar graphs when $r \ge 4$. The purpose of this study is to investigate upper bounds of the list r-hued chromatic numbers of outer-1-planar graphs. The following are the main results obtained in this research.

Theorem 1.4. *Let r be an integer with $r \ge 2$ and let G be an outer-1-planar graph.*

(i) *If $r \ge 3$, then*

$$\chi_{L,r}(G) \le \begin{cases} 2r, & if \quad 3 \le r \le 6; \\ r+7, & if \quad r \ge 7. \end{cases} \tag{2}$$

(ii) *If $r = 2$, then $\chi_{L,2}(G) \le 5$. Moreover, this upper bound is sharp.*

The case when $r = 3$ is obtained by Theorem 1.3. The case when $r = 2$ can be derived from former results. We include them here for the sake of completeness.

Tools to be used in our arguments to prove Theorem 1.4 will be presented in the next section. The proof of the main result will be shown in Section 3.

2. Preliminaries

We in this section provide some former results and structural properties of outer-1-planar graphs, to be applied in our arguments for justifying the main results of the paper.

Lemma 2.1 (Akbari et al. [1]). *Let C_n be a cycle of order n, $n \ge 3$. Then*

$$\chi_{L,2}(C_n) = \begin{cases} 3, & if \quad n \equiv 0 (mod\ 3); \\ 5, & if \quad n = 5; \\ 4, & \text{otherwise.} \end{cases}$$

In the rest of the discussion, we call a graph H to be a **configuration** of a graph G if H is a subgraph of G and there exists a set $V' \subseteq V(H)$ such that for any $v \in V'$, either $d_H(v) = d_G(v)$ or $d_G(v)$ is bounded by prescribed bounds. In [15],

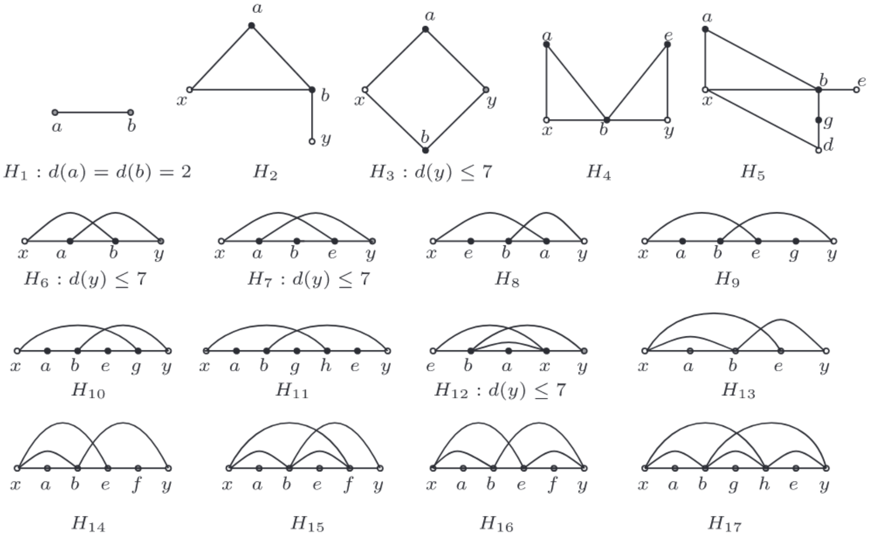

Fig. 1. Local structures of an outer-1-planar graph G with minimum degree at least 2.

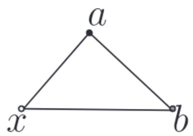

Fig. 2. S_1.

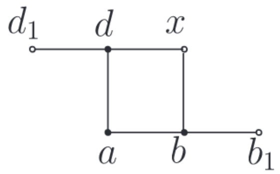

Fig. 3. S_2.

Zhang and Li investigated the local structure of outer-1-planar graphs. They discovered 17 graphs, which are depicted in Fig. 1, and showed that these 17 configurations are unavoidable local structures for an outer-1-planar graph. As indicated in Theorem 2.2 below, if G is an outer-1-planar graph, then G must contain one of these 17 configurations as a subgraph. For each graph H in the configurations presented in Figs. 1–3, as a subgraph of an outer-1-planar graph G, a black vertex in an H_i indicates that the degree of this vertex in H_i is exactly the degree of this vertex in G; a gray vertex in an H indicates the degree of this vertex in G has an upper bounded as given in the caption under H in Fig. 1; and a hollow vertex indicates that the degree of the vertex is not restricted in G.

Theorem 2.2 (Zhang and Li [15]). *Let G be a connected outer-1-planar graph. Then using the notation in Fig. 1, one of the following must hold.*

(a) *G has a vertex of degree 1.*
(b) *G contains two adjacent vertices of degree 2, as illustrated in H_1 of Fig. 1.*
(c) *G contains a triangle with a vertex of degree 2, and so G contains one of H_2, H_4, H_5, H_{12}, H_{13}, H_{14}, H_{15}, H_{16} and H_{17} in Fig. 1 as a configuration.*
(d) *G contains one of H_3, H_6, H_7, H_8, H_9, H_{10}, H_{11} in Fig. 1 as a configuration.*

Theorem 2.2 is a classification of the local structures of outer-1-planar graphs in Fig. 1. It reveals local structures an outer-1-planar graphs must have, which suggests possible ways for an inductive arguments.

Observation 2.3. Using the notation of the graphs as depicted in Fig. 1, we have the following observations.

(i) Each of the configurations in $\{H_2, H_4, H_5, H_{12}, H_{13}, H_{14}, H_{15}, H_{16}, H_{17}\}$ contains the structure S_1 (as seen in Fig. 2) with $3 \leq d_G(b) \leq 5$.
(ii) Each of H_7 and H_{10} contains a structure S_2 (as seen in Fig. 3).

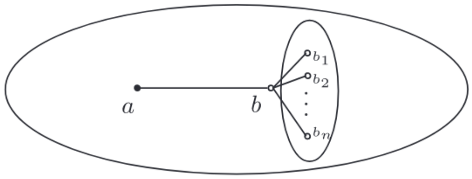

Fig. 4. G in the proof of Claim 3.1.

3. Proofs of Theorem 1.4

Throughout this section, if H is a graph and e is an edge with end vertices in $V(H)$, then we use $H+e$ to denote the graph with vertex set $V(H)$ and edge set $E(H)\cup\{e\}$. Let G be an outer-1-planar graph. By (1), every outer-1-planar graph is also a planar graph. It follows by Theorem 1.2 that $\chi_{L,2}(G)=ch_d(G)\le 5$. Since C_5 is an outer-1-planar graph, by Theorem 2.1, $\chi_{L,2}(C_5)=5$. Therefore, the bound $\chi_{L,2}(G)\le 5$ cannot be relaxed. This justifies Theorem 1.4(ii). By Theorem 1.3, it suffices to validate Theorem 1.4(i) for the cases when $r\ge 4$.

Throughout the section, we assume that $r\ge 4$ is an integer and let

$$\phi(r)=\begin{cases}2r, & \text{if } 3\le r\le 6;\\ r+7, & \text{if } r\ge 7.\end{cases}$$

For a graph G and an integer $i\in\mathbb{N}$, define

$$D_i(G)=\{v\in V(G): d_G(v)=i\}.$$

By the definition of outer-1-planar graphs, we observe the following:

$$\text{any subgraph of an outer-1-planar is outer-1-planar.} \tag{3}$$

If G is not connected, we may argue on each of its components. Therefore, in the rest of this section, we always assume that the graph G under consideration is a connected graph. By Lemma 2.1, Theorem 1.4(i) holds for cycles. In particular, Theorem 1.4(i) holds for all outer-1-planar graphs with at most 4 vertices. To prove Theorem 1.4(i), we argue by contradiction and assume that Theorem 1.4(i) has a counterexample. Therefore there exists an outer-1-planar graph G with $|V(G)|\ge 5$ such that

$$\chi_{L,r}(G)\ge\phi(r)+1 \text{ with } |V(G)|+|E(G)| \text{ being minimized.} \tag{4}$$

By (2.3) and (4), we observe that

$$\chi_{L,r}(G)\ge\phi(r)+1, \text{and for any proper subgraph } G' of G,\ \chi_{L,r}(G')\le\phi(r). \tag{5}$$

To prove Theorem 1.4(i), we shall make the following claims.

Claim 3.1. *If $\delta(G)=1$, then $\chi_{L,r}(G)\le\phi(r)$.*

Since $\delta(G)=1$, there exists a vertex $a\in V(G)$ with $d_G(a)=1$. Let b denote the only vertex in $N_G(a)$. Hence $ab\in E(G)$. (See Fig. 4 for an illustration.) By (5), there exists a $\phi(r)$-list of G, such that G does not exist an (L,r)-coloring.

Let $G'=G-\{a\}$ and define L' to be the restriction of L to $V(G')$. By (5) and as $|V(G')|<|V(G)|$, we conclude that G' has an (L',r)-coloring c. In the following, we shall extend the domain of c from $V(G')$ to $V(G)$ by choosing the color $c(a)$ and assigning it to the vertex a so that the extended c would be an (L,r)-coloring of G. This would lead to a contradiction to (4).

Suppose that $d_G(b)\ge r+1$. Then $d_{G'}(b)\ge r$. As c is an (L',r)-coloring of G', we have $|c(N_{G'}(b))|\ge\min\{d_{G'}(b),r\}=r$, and so every vertex $v\in V(G)-\{a\}$ already satisfies the local $(r+1,r)$-coloring condition (LC-1). Since $|L(a)|=\phi(r)>2$, we choose $c(a)\in L(a)\backslash\{c(b)\}$ and color a with $c(a)$. Thus the extended c is a proper coloring satisfying (LC-2), with every vertex of G fulfilling the requirement of (LC-1). Hence the extended c is an (L,r)-coloring of G.

Therefore, we assume that $d_G(b)\le r$. As c is an (L',r)-coloring of G', we have $|c(N_{G'}(b))|\ge\min\{d_{G'}(b),r\}=d_{G'}(b)$, and for any vertex $v\in V(G)-\{a,b\}$, (LC-1) is satisfied. As $|L(a)|=\phi(r)>r+1\ge|c(N_{G'}(b))\cup\{c(b)\}|$, we choose $c(a)\in L(a)\backslash\{c(b),c(N_{G'}(b))\}$, and color a with $c(a)$. Hence both a and b will also satisfy (LC-1) and (LC-2). Thus the extended c is an (L,r)-coloring of G. This justifies Claim 3.1.

Claim 3.2. *If G contains two vertices $a,b\in D_2(G)$ with $ab\in E(G)$ and $N_G(a)-\{b\}\ne N_G(b)-\{a\}$, then $\chi_{L,r}(G)\le\phi(r)$.*

As $a,b\in D_2(G)$ with $ab\in E(G)$, we denote $N_G(a)=\{b,x\}$ and $N_G(b)=\{a,y\}$. By the assumption of Claim 3.2, we conclude that $x\ne y$. By (5), there exists a $\phi(r)$-list of G, such that G does not have an (L,r)-coloring.

Let $G'=G-\{a,b\}$ and define L' to be the restriction of L to $V(G')$. By (5), G' have an (L',r)-coloring c. To find a contradiction to (4), we are to extend c by choosing colors $c(a)\in L(a)$ and $c(b)\in L(b)$ which would make the extended c be an (L,r)-coloring.

Suppose first that $\min\{d_G(x), d_G(y)\} \geq r+1$. Then every vertex in $G - \{a, b\}$ satisfies both (CL-1) and (CL-2). As $|L(a)| = |L(b)| = \phi(r) \geq 8$, it is possible for us to choose

$$c(a) \in L(a) \setminus \{c(x), c(y)\}, \text{ and } c(b) \in L(b) \setminus \{c(x), c(y), c(a)\}.$$

Then we extend c by coloring a and b with $c(a)$ and $c(b)$, respectively. As $a, b \in D_2(G)$, under the extended coloring c, both a and b also satisfy (CL-1) and (CL-2). Thus in this case, we obtain an (L, r)-coloring of G.

Next we assume that both $\max\{d_G(x), d_G(y)\} \geq r+1$ and $\min\{d_G(x), d_G(y)\} \leq r$. By symmetry, we may assume that $d_G(x) \geq r+1$ and $d_G(y) \leq r$. This implies that every vertex in $G - \{a, b, y\}$ satisfies both (CL-1) and (CL-2), and y satisfies (CL-2). As $|L(a)\backslash\{c(x), c(y)\}| = \phi(r) - 2 \geq 8 - 2 = 6 > 0$, we can choose

$$c(a) \in L(a) \setminus \{c(x), c(y)\}.$$

As $d_{G'}(y) = d_G(y) - 1 \leq r-1$, we conclude that $|c(N_{G'}(y))| = d_G(y) - 1 \leq r-1$ and $|\{c(x), c(y), c(a), c(N_{G'}(y))\}| \leq 3 + r - 1 = r+2$, so $|L(b)\backslash\{c(x), c(y), c(a), c(N_{G'}(y))\}| \geq \phi(r) - (r+2) > 0$. Thus we can choose

$$c(b) \in L(b) \setminus \{c(x), c(y), c(a), c(N_{G'}(y))\}.$$

By the choice of $c(b)$, the vertex y satisfies (CL-1) under the extended coloring c. As $a, b \in D_2(G)$, under the extended coloring c, both a and b also satisfy (CL-1) and (CL-2). Thus in this case, we also obtain an (L, r)-coloring of G.

Finally, we assume that $\max\{d_G(x), d_G(y)\} \leq r$. Thus every vertex in $G - \{a, b, x, y\}$ satisfies both (CL-1) and (CL-2), and both x and y satisfy (CL-2). Since $d_{G'}(x) = d_G(x) - 1 \leq r-1$ and $|c(N_{G'}(x))\}| \leq r-1$, then we conclude that $|\{c(x), c(y), c(N_{G'}(x))\}| \leq 2 + r - 1 = r+1$, and $|L(a)\backslash\{c(x), c(y), c(N_{G'}(x))\}| \geq \phi(r) - (r+1) > 0$. Thus we can choose

$$c(a) \in L(a) \setminus \{c(x), c(y), c(N_{G'}(x))\}.$$

Similarly, we conclude that $|\{c(x), c(y), c(a), c(N_{G'}(y))\}| \leq 3 + r - 1 = r+2$, and $|L(b)\backslash\{c(x), c(y), c(a), c(N_{G'}(y))\}| \geq \phi(r) - (r+2) > 0$. Thus we can choose

$$c(b) \in L(b) \setminus \{c(x), c(y), c(a), c(N_{G'}(y))\}.$$

By the choices of $c(a)$ and $c(b)$, the vertices x and y satisfy (CL-1) under the extended coloring c. As $a, b \in D_2(G)$, under the extended coloring c, both a and b also satisfy (CL-1) and (CL-2). Thus in this case, we also obtain an (L, r)-coloring of G. This completes the proof for Claim 3.2.

In Claim 3.3 below, we follow the notation in Observation 2.3(i) for S_1 (see Fig. 2) in our arguments. Thus the vertices in S_1 are labeled as in Fig. 2.

Claim 3.3. *Suppose that G containing S_1 as a configuration with $3 \leq d_G(b) \leq 5$. Then $\chi_{L,r}(G) \leq \phi(r)$.*

Denote $N_G(b) = \{b_1, \ldots, b_l, a, x\} (1 \leq l \leq 3)$, let $G' = G - \{a\}$ and let L' denote the restriction of L to $V(G')$. By (4), G' has an (L', r)-coloring c.

Case 1. $r = 4$.

For $l = 3$, we have $N_G(b) = \{b_1, b_2, b_3, a, x\}$, $d_G(b) = 5$, and $d_{G'}(b) = 4 = r$, so $|c(N_{G'}(b))| = r = \min\{d_{G'}(b), r\}$. Thus every vertex in $G - \{a, x\}$ satisfies both (CL-1) and (CL-2), while the vertex x satisfies (CL-2) under the current coloring c. If $d_G(x) \geq 5 = r+1$, then x also satisfies (CL-1). In this case, as $|L(a)\backslash\{c(x), c(b)\}| \geq \phi(r) - 2 > 0$, we can choose

$$c(a) \in L(a) \setminus \{c(x), c(b)\}.$$

Thus the extended c becomes an (L, r)-coloring of G. If $d_G(x) \leq 4 = r$, then $|c(N_{G'}(x))| = d_G(x) - 1 \leq r - 1 = 3$. In this case, as $|L(a)\backslash\{c(x) \cup c(N_{G'}(x))\}| \geq \phi(r) - 4 > 0$, we can select

$$c(a) \in L(a) \setminus \{c(x) \cup c(N_{G'}(x))\}.$$

As $c(b) \in c(N_{G'}(x))$, $c(a) \neq c(b)$, which implies that the extended c is a proper coloring, satisfying both (CL-1) and (CL-2) for all vertices in G. Thus the extended c is an (L, r)-coloring of G in this case.

For $1 \leq l \leq 2$, $|N_{G'}[b]| = d_G(b) = l + 2 \leq 4 = r$. If $d_G(x) \geq r+1$, then x also satisfies (CL-1). In this case, as $|L(a) \setminus \{c(N_{G'}[b])\}| \geq \phi(r) - 4 > 0$, we can choose

$$c(a) \in L(a) \setminus \{c(N_{G'}[b])\},$$

to extend c. Thus c also is an (L, r)-coloring of G in this case. If $d_G(x) \leq 4 = r$, then $|N_{G'}(b) \cup N_{G'}(x)| \leq 6$. In this case, as $|L(a)\backslash\{c(N_{G'}(b)) \cup c(N_{G'}(x))\}| \geq \phi(r) - 6 > 0$, we can select

$$c(a) \in L(a) \setminus \{c(N_{G'}(b)) \cup c(N_{G'}(x))\},$$

to extend c. Thus c also is an (L, r)-coloring of G in this case.

Case 2. $r \geq 5$.

Since $d_G(b) = l + 2 \leq 5 \leq r (1 \leq l \leq 3)$, we have $|c(N_{G'}(b))| = d_G(b) - 1 \leq r - 1$. Thus every vertex in $G - \{a, b, x\}$ satisfies both (CL-1) and (CL-2) under the current coloring c, while the vertices b and x satisfy (CL-2).

If $d_G(x) \geq r+1$, then $|c(N_{G'}(x))| \geq \min\{d_G(x) - 1, r\} = r$, and so x satisfies (CL-1). Since $|L(a)\backslash\{c(N_{G'}[b])\}| \geq \phi(r) - 5 > 0$, we can choose

$$c(a) \in L(a) \setminus \{c(N_{G'}[b])\},$$

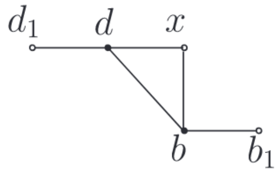

Fig. 5. $S_2 - a + bd$ in G'.

to extend c. The choice of $c(a)$ makes both the vertices a and b satisfy (CL-1) and (CL-2) under the extended coloring c, and so the extended c is an (L, r)-coloring of G.

If $d_G(x) \le r$, we have $|c(N_{G'}(x))| = d_{G'}(x) = d_G(x) - 1 \le r - 1$, and every vertex in $V(G) - \{a, b, x\}$ satisfy (CL-1) and (CL-2), and both b and x satisfy (CL-2). We have $|L(a) \backslash \{c(N_{G'}(x)) \cup c(N_{G'}(b))\}| \ge \phi(r) - (r - 1 + 4) > 0$. Hence we can assign to vertex a the color

$$c(a) \in L(a) \setminus \{c(N_{G'}(x)) \cup c(N_{G'}(b))\}.$$

Since $a \in N_G(x) \cup N_G(b)$, the choice of $c(a)$ makes the vertices a, b and x all satisfy (CL-1) and (CL-2). Once again the extended c is an (L, r)-coloring of G. This proves Claim 3.3.

In Claim 3.3 below, we follow the notation in Observation 2.3(ii) for S_2 (see Fig. 3) in our arguments. Thus the vertices in S_2 are labeled as in Fig. 3.

Claim 3.4. *Suppose that G containing S_2 as a configuration. Then $\chi_{L,r}(G) \le \phi(r)$.*

Let $G' = G - \{a\} + bd$ and let L' be the restriction of L to $V(G')$. Since G is outer-1-planar, there is a drawing of G on the plane on which the vertices in $V(G)$ are all located on the outer face of the drawing, in such way that every edge can cross at most one other edge. We call such a drawing of G an outer-1-planar drawing of G. Since G' can be viewed as the graph obtained from G by contracting the edge $e = ab$, this outer-1-planar drawing of G yields an outer-1-planar drawing of G', and so G' is also outer-1-planar. (See Fig. 5 for an illustration.)

By (5), $\chi_{L,r}(G') \le \phi(r)$, and so G' has an (L', r)-coloring c. Thus every vertex in $G - \{a, b, d\}$ satisfies both (CL-1) and (CL-2), and both vertices b and d satisfy (CL-2). Since $|L(a) \backslash \{c(b), c(b_1), c(d), c(d_1), c(x)\}| = \phi(r) - 5 \ge 8 - 5 > 0$, we can choose a color

$$c(a) \in L(a) \backslash \{c(b), c(b_1), c(d), c(d_1), c(x)\},$$

By the choice of $c(a)$, all the vertices in $\{a, b, d\}$ also satisfy (CL-1) and (CL-2) under the extended coloring c. Hence c can be extended to an (L, r)-coloring of G. This show that Claim 3.4 must hold.

By Theorem 2.2 and Observation 2.3, as an outer-1-planar graph G must have the configurations as shown in Theorem 2.2 and Observation 2.3. By Claims 3.1–3.4, we may assume that G must have one of the graphs in $\{H_i : i \in \{3, 6, 8, 9, 11\}\}$ as defined in Fig. 1 as a configuration of G. We shall show when any of these graphs occurs in G as a configuration, G always have an (L, r)-coloring, which leads to a contradiction. In the claims below, whenever we discuss an H in $\{H_i : i \in \{3, 6, 8, 9, 11\}\}$ as a configuration of G, we shall use the notation and vertex labels in Fig. 1 for the corresponding configuration.

Claim 3.5. *Suppose that G containing either an H_3 or an H_6 as a configuration. Then $\chi_{L,r}(G) \le \phi(r)$.*

Let $G' = G - \{a\}$ and let L' denote the restriction of L to $V(G')$. By (5), G' has an (L', r)-coloring c. Thus under this coloring c, every vertex in $G - \{a, x, y\}$ satisfies both (CL-1) and (CL-2), while the vertices x and y satisfy (CL-2).

Case 1. $4 \le r \le 7$.

If $\min\{d_G(x), d_G(y)\} \ge r + 1$, then both vertices x and y also satisfy (CL-1); As $|L(a)| = \phi(r) \ge 8$, if G contains H_3 as a configuration, then $|L(a) \backslash \{c(x), c(y)\}| > 0$; and if G contains H_6 as a configuration, then we always have $|L(a) \backslash \{c(x), c(y), c(b)\}| > 0$. Thus we can choose

$$c(a) \in \begin{cases} L(a) \setminus \{c(x), c(y)\}, & \text{if } G \text{ contains} H_3 \text{ as a configuration,} \\ L(a) \setminus \{c(x), c(y), c(b)\}, & \text{if } G \text{contains } H_6 \text{as a configuration.} \end{cases}$$

to color the vertex a. The choice of $c(a)$ makes a satisfy (CL-1) and (CL-2), and all other vertices satisfy (CL-1). Hence the extended c becomes an (L, r)-coloring of G.

Next we assume that $\max\{d_G(x), d_G(y)\} \ge r + 1$ and $\min\{d_G(x), d_G(y)\} \le r$. we may further assume that $d_G(x) \ge r + 1$ and $d_G(y) \le r$, as the proof for the case when $d_G(y) \ge r + 1$ and $d_G(x) \le r$ is similar. Thus x satisfies (CL-1) and $|c(N_{G'}(y))| = d_{G'}(y) = d_G(y) - 1 \le r - 1$. As $|L(a)| = \phi(r) = 2r$ and $r \ge 4$, we always have $|L(a) \backslash (\{c(x), c(y), c(b)\} \cup c(N_{G'}(y)))| > 0$. Hence we can find an color

$$c(a) \in L(a) \setminus (\{c(x), c(y), c(b)\} \cup c(N_{G'}(y))),$$

to color the vertex a. By the choice of $c(a)$ and as $d_G(x) \geq r+1$, and as $b \in N_{H_6}(y)$ when G contains H_6 as a configuration, we conclude that the vertices a, x, y and b (when G contains H_6 as a configuration) all satisfy (CL-1) and (CL-2), and so the extended c is an (L,r)-coloring of G.

Finally we assume that $\max\{d_G(x), d_G(y)\} \leq r$, and so $\max\{|c(N_{G'}(x))|, |c(N_{G'}(y))|\} \leq r-1$. Since in H_3 or in H_6, we have $b \in N_{G'}(x) \cup N_{G'}(y)$. Hence $|c(N_{G'}(x)) \cup c(N_{G'}(y))| \leq |c(N_{G'}(x))| + |c(N_{G'}(y))| - 1 = 2r-3$. As $4 \leq r \leq 7$, we have $\phi(r) = 2r$ and so

$$|L(a)\backslash(\{c(x), c(y)\} \cup c(N_{G'}(x)) \cup c(N_{G'}(y)))| \geq \phi(r) - (2r-3+2) = 2r - (2r-1) > 0.$$

Hence we can pick a color $c(a) \in L(a)\backslash(\{c(x), c(y)\} \cup c(N_{G'}(x)) \cup c(N_{G'}(y)))$ to color a. The choice of $c(a)$ makes the vertices a, b, x and y all satisfy (CL-1) and (CL-2) under the extended coloring c. Therefore, in Case 1, we can always extend c to an (L,r)-coloring of G.

Case 2. $r \geq 8$.

Since H_3 or H_6 is a configuration of G, we have $d_G(y) \leq 7$. Note that in either H_3 or H_6, $b \in N_{G'}(y)$. Assume first that $d_G(x) \geq r+1$. Then x also satisfies (CL-1). As $|L(a)\backslash(\{c(x), c(y)\} \cup c(N_{G'}(y)))| \geq \phi(r) - (2+r-1) > 0$, we can extend c by coloring a with a color $c(a) \in L(a) \setminus (\{c(x), c(y)\} \cup c(N_{G'}(y)))$. Thus under the extended coloring c, the vertices a, b and y satisfy (CL-1) and (CL-2), and so we find an (L,r)-coloring of G. Hence we may assume that $d_G(x) \leq r$, and so $|c(N_{G'}(x))| = d_G(x) - 1$. As $|c(N_{G'}(y))| = d_{G'}(y) = d_G(y) - 1 \leq 6$ and as b is in the configuration H_3 or H_6 in G, we have $b \in N_{G'}(x) \cup N_{G'}(y)$, implying that

$$|c(N_{G'}(x)) \cup c(N_{G'}(y))| \leq |c(N_{G'}(x))| + |c(N_{G'}(y))| - 1 = r-1+6-1 = r+4,$$

and so $|L(a)\backslash(\{c(x), c(y)\} \cup c(N_{G'}(x)) \cup c(N_{G'}(y)))| \geq \phi(r) - (r+6) = (r+7) - (r+6) > 0$. Hence

$$c(a) \in L(a) \setminus (\{c(x), c(y)\} \cup c(N_{G'}(x)) \cup c(N_{G'}(y))).$$

By this choice of $c(a)$, each of the vertices a, x and y satisfies (CL-1) and (CL-2) under the extended c. Hence c can be extended to an (L,r)-coloring of G. This proves Claim 3.5.

Claim 3.6. *Suppose that G containing an H_8 as a configuration. Then $\chi_{L,r}(G) \leq \phi(r)$.*

We adopt the notation in Fig. 1 for the configuration H_8. Let $G' = G - \{b, e\}$ and let L' denote the restriction of L to $V(G')$. By (5), G' has an (L', r)-coloring c. Thus under this coloring c, every vertex in $G - \{a, b, e, x, y\}$ satisfies both (CL-1) and (CL-2), and every vertex in $G - \{b, e\}$ satisfies (CL-2).

Again we start the discussion by assuming $\min\{d_G(x), d_G(y)\} \geq r+1$. Then x and y satisfy (CL-1) also. As $|L(b)| = |L(e)| = \phi(r) > \max\{3, 4\} = 4$, we choose the colors

$$c(e) \in L(e) \setminus \{c(x), c(y), c(a)\}, \text{ and } c(b) \in L(b) \setminus \{c(x), c(y), c(a), c(e)\},$$

for the vertices e and b, respectively, to extend c. These choices of $c(b)$ and $c(e)$ make all the vertices a, b, e, x, y satisfy (CL-1) and (CL-2), and so the extended c is an (L,r)-coloring of G.

Next we assume that $\max\{d_G(x), d_G(y)\} \geq r+1$ and $\min\{d_G(x), d_G(y)\} \leq r$. We may further assume that $d_G(x) \geq r+1$ and $d_G(y) \leq r$, as the arguments for the case when $d_G(y) \geq r+1$ and $d_G(x) \leq r$ are similar. Thus x satisfies (CL-1) and $|c(N_{G'}(y))| \leq r-1$. As $|L(b)| = |L(e)| = \phi(r) > \max\{r+1, 4\} = r+1$, we can select

$$c(b) \in L(b) \setminus \{c(x), c(y), c(N_{G'}(y))\}, \text{ and } c(e) \in L(e) \setminus \{c(x), c(y), c(a), c(b)\},$$

as the colors for the vertices b and e, respectively. Under the extended c, all vertices in G satisfy both (CL-1) and (CL-2), and so the extended c becomes an (L,r)-coloring of G.

Finally we assume that $\max\{d_G(x), d_G(y)\} \leq r$, and so $\max\{|c(N_{G'}(x))|, |c(N_{G'}(y))|\} \leq r-1$. As $|L(b)| = |L(e)| = \phi(r) > \max\{r+1, r+2\} = r+2$, we can select

$$c(e) \in L(e) \setminus (\{c(x), c(y)\} \cup c(N_{G'}(x))), \text{ and } c(b) \in L(b) \setminus (\{c(x), c(y), c(e)\} \cup c(N_{G'}(y))),$$

to color vertices e and b, respectively. By these choices of $c(b)$ and $c(e)$, the extended c is an (L,r)-coloring of G, which completes the proof for Claim 3.6.

Claim 3.7. *Suppose that G containing an H_9 as a configuration. Then $\chi_{L,r}(G) \leq \phi(r)$.*

We again use the notation in Fig. 1 for the configuration H_9. Let $G' = G - \{a, b\}$ and let L' denote the restriction of L to $V(G')$. By (5), G' has an (L', r)-coloring c. Thus under this coloring c, every vertex in $G - \{a, b, e, x, y\}$ satisfies both (CL-1) and (CL-2), and every vertex in $G - \{a, b\}$ satisfies (CL-2).

Assume first that $\min\{d_G(x), d_G(y)\} \geq r+1$. Then both x and y satisfy (CL-1). As $|L(a)| = |L(b)| = \phi(r) > 4$, we can choose

$$c(b) \in L(b) \setminus \{c(x), c(y), c(g), c(e)\}, \text{ and } c(a) \in L(a) \setminus \{c(x), c(y), c(b), c(e)\},$$

to color the vertices b and a, respectively. Thus the choices of $c(a)$ and $c(b)$ make the extended c an (L,r)-coloring of G.

Next we assume that $\max\{d_G(x), d_G(y)\} \geq r+1$ and $\min\{d_G(x), d_G(y)\} \leq r$. If $d_G(x) \geq r+1$ and $d_G(y) \leq r$, then x satisfies (CL-1) and $|c(N_{G'}(y))| = d_G(y) - 1 \leq r-1$. As $|L(a)| = |L(b)| = \phi(r) > \max\{r+2, 4\}$, we choose

$$c(b) \in L(b) \setminus (\{c(x), c(y), c(e)\} \cup c(N_{G'}(y))), \text{ and } c(a) \in L(a) \setminus \{c(x), c(y), c(b), c(e)\},$$

to color the vertices b and a, respectively, to extend the coloring c. If $d_G(x) \le r$ and $d_G(y) \ge r+1$, then y satisfies (CL-1) and $|c(N_{G'}(x))| = d_G(x) - 1 \le r - 1$. As $|L(a)| = |L(b)| = \phi(r) > \max\{r+2, 4\} = r+2$, we choose

$$c(b) \in L(b) \setminus \{c(x), c(y), c(g), c(e)\}, \text{ and } c(a) \in L(a) \setminus (\{c(x), c(y), c(b)\} \cup c(N_{G'}(x))),$$

to color the vertices b and a, respectively, to extend the coloring c. It follows that in either case, under the extended c, all vertices in G satisfy (CL-1) and (CL-2), and so it is an (L, r)-coloring of G.

We then may assume that $\max\{d_G(x), d_G(y)\} \le r$, and so $\max\{|c(N_{G'}(x))|, |c(N_{G'}(y))|\} \le r - 1$. As $|L(b)| = |L(a)| = \phi(r) > r+2$, we can select

$$c(b) \in L(b) \setminus (\{c(x), c(y), c(g)\} \cup c(N_{G'}(y)))\}, \text{ and } c(a) \in L(a) \setminus (\{c(x), c(y), c(e)\} \cup c(N_{G'}(x))),$$

to color vertices b and a, respectively. By these choices of $c(b)$ and $c(a)$, the extended c is an (L, r)-coloring of G, which completes the proof for Claim 3.7.

Claim 3.8. *Suppose that G containing an H_{11} as a configuration. Then $\chi_{L,r}(G) \le \phi(r)$.*

We continue adopting the notation in Fig. 1 for the configuration H_{11}. Let $G' = G - \{e, g, h\}$ and let L' denote the restriction of L to $V(G')$. By (5), G' has an (L', r)-coloring c. Thus under this coloring c, every vertex in $G - \{b, e, g, h, x, y\}$ satisfies both (CL-1) and (CL-2), and every vertex in $G - \{e, g, h\}$ satisfies (CL-2).

Assume first that $\min\{d_G(x), d_G(y)\} \ge r+1$. Then both x and y satisfy (CL-1). As $|L(e)| = |L(g)| = |L(h)| = \phi(r) \ge 8$, it is possible to extend c by choosing $c(g) \in L(g) \setminus \{c(x), c(y), c(a), c(b)\}$, $c(h) \in L(h) \setminus \{c(x), c(y), c(b), c(g)\}$ and then $c(e) \in L(e) \setminus \{c(x), c(y), c(g), c(h)\}$. By these choices, the extended c becomes an (L, r)-coloring of G.

Next we assume that $\max\{d_G(x), d_G(y)\} \ge r+1$ and $\min\{d_G(x), d_G(y)\} \le r$. If $d_G(x) \ge r+1$ and $d_G(y) \le r$, then x satisfies (CL-1) and $|c(N_{G'}(y))| = d_G(y) - 1 \le r - 1$. In this case, as $|L(h)| = |L(e)| = |L(g)| = \phi(r) > \max\{r+2, 4, 5\} = r+2$, we choose $c(g) \in L(g) \setminus \{c(x), c(y), c(b), c(a)\}$, $c(e) \in L(e) \setminus (\{c(x), c(y), c(g)\} \cup c(N_{G'}(y)))$ and then $c(h) \in L(h) \setminus \{c(x), c(y), c(g), c(b), c(e)\}$. Since in H_{11}, we have $b \in N_{G'}(y)$, and so in this case the extended c will be an (L, r)-coloring of G.

If $d_G(x) \le r$ and $d_G(y) \ge r+1$, then y satisfies (CL-1) and $|c(N_{G'}(x))| = d_G(y) - 1 \le r - 1$. In this case, as $|L(h)| = |L(e)| = |L(g)| = \phi(r) > \max\{r+2, 6, 3\} = r+2$, we choose $c(h) \in L(h) \setminus (\{c(x), c(y), c(b)\} \cup c(N_{G'}(x)))$, $c(e) \in L(e) \setminus \{c(x), c(y), c(h)\}$ and then $c(g) \in L(g) \setminus \{c(x), c(y), c(a), c(b), c(h), c(e)\}$. Since in H_{11}, we have $a \in N_{G'}(x)$, and so in this case the extended c will ne an (L, r)-coloring of G.

Therefore, we may assume that $\max\{d_G(x), d_G(y)\} \le r$, and so $\max\{|c(N_{G'}(x))|, |c(N_{G'}\$y))|\} \le r - 1$. As $|L(h)| = |L(e)| = |L(g)| = \phi(r) > \max\{r+2, 6\} = r+2$, we can extend c by selecting $c(h) \in L(h) \setminus (\{c(x), c(y), c(b)\} \cup c(N_{G'}(x)))$, $c(e) \in L(e) \setminus (\{c(x), c(y), c(h)\} \cup c(N_{G'}(y)))$ and then $c(g) \in L(g) \setminus \{c(x), c(y), c(a), c(b), c(h), c(e)\}$. Since in H_{11}, we have $a \in N_{G'}(x)$ and $b \in N_{G'}(y)$, these choices of $c(h)$, $c(e)$ and $c(g)$ make the extended c an (L, r)-coloring of G, which completes the proof for Claim 3.8.

By the choice of L and (5), G does not have an (L, r)-coloring. By Observation 2.3 and Claims 3.1–3.4, we may assume that $\delta(G) \ge 2$ and G does not have a member in $\{H_2, H_4, H_5, H_7, H_{10}\} \cup \{H_i : 12 \le i \le 17\}$ as a configuration. This, together with Theorem 2.2, implies that G must have an $H \in \{H_i : i \in \{3, 6, 8, 9, 11\}\}$ as a configuration. By Claims 3.5–3.8, when G contains an $H \in \{H_i : i \in \{3, 6, 8, 9, 11\}\}$, then G must have an (L, r)-coloring, contrary to (5). This completes the proof of Theorem 1.4(i).

4. Concluding remarks

By the analysis of unavoidable configurations of outer-1-planar graphs as stated in Theorem 2.2, we utilize an inductive argument to prove that any outer-1-planar graph G has a list r-hued chromatic number at most $\phi(r)$. However, it is not straight forward to find examples to show that for sufficiently large r, $\phi(r)$ would be the best possible. In general, it is of interest to investigating the best possible upper bounds of the r-hued chromatic numbers and the list r-hued chromatic numbers for an bigger graph family that properly contain all outer-1-planar graphs. Let $\ell \ge 1$ be an integer. Define a graph G to be an **outer-ℓ-planar** graph if G can be drawn on the plane in such a way that all vertices of $V(G)$ are located on the outer face (the exterior face) of the drawing and every edge of G can cross at most ℓ other edges in the drawing. Thus it is of interest to determine two functions $f_\ell(r)$ and $f_{L,\ell}(r)$ such that $f_\ell(r)$ (or $f_{L,\ell}(r)$, respectively) is the smallest integer such that every outer-ℓ-planar graph G satisfies $\chi_r(G) \le f_\ell(r)$ (or $\chi_{L,r}(G) \le f_{L,\ell}(r)$, respectively). In this paper, it is known that $f_{L,1}(r) \le \phi(r)$. Future researches are needed to be conducted to better understand these functions $f_\ell(r)$ and $f_{L,\ell}(r)$.

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Further reading

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