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On Critical Unicyclic Graphs with Cutwidth Four

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Abstract: The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph G on a line P_n with $n = |V(G)|$ vertices in such a way that the maximum number of overlapping edges (i.e., the congestion) is minimized. A graph G with a cutwidth of k is k-cutwidth critical if every proper subgraph of G has a cutwidth less than k and G is homeomorphically minimal. In this paper, we first verified some structural properties of k-cutwidth critical unicyclic graphs with $k > 1$. We then mainly investigated the critical unicyclic graph set $\mathcal{T}$ with a cutwidth of four that contains fifty elements, and obtained a forbidden subgraph characterization of 3-cutwidth unicyclic graphs.

Keywords: graph labeling; cutwidth; critical graph; unicyclic graph

MSC: 05C75; 05C78; 90C27

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1. Introduction

All graphs in this paper are finite, simple, and connected, with undefined notation following [1]. The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph G on a path P_n with $n = |V(G)|$ vertices in such a way that the maximum number of overlapping edges (i.e., the congestion) is minimized. As one of the most well-known optimization problems, it is also known as the minimum cut linear arrangement (or linear layout, optimal embedding, optimal labeling, etc.) problem [2]. Cutwidth has been extensively examined [2]. Computing cutwidth for general graphs is an NP-complete problem except for trees [3–6], and it remains NP-complete even if the input graph G is restricted to planar graphs with a maximum degree of three [7]. Hence, a number of studies have focused on polynomial-time approximation algorithms for general graphs and polynomial-time algorithms for some special graphs to solve their cutwidth [2,8]. Relatively little work has been conducted on detecting special graph classes whose cutwidths can be computed polynomially [2] and critical graph classes with cutwidths of $k \geq 1$. Let $\mathcal{T}_k(*)$ be the set of critical graphs with the graph parameter $* = k$. From [9], $|\mathcal{T}_1(c(G))| = 1$, $|\mathcal{T}_2(c(G))| = 2$, $|\mathcal{T}_3(c(G))| = 5$ (see Figure 1). For critical graphs with cutwidth $k \geq 4$, $|\mathcal{T}_k(c(G))|$ has been unknown except that $|\mathcal{T}_4(c(T))| = 18$, as reported by [10], where T is a tree (see Figure 2). Similar studies have been conducted for the treewidth, pathwidth, and branchwidth of a graph G (abbreviated by $tw(G)$, $pw(G)$, and $bw(G)$, respectively). A graph G is said to be k-treewidth (pathwidth, branchwidth) critical if $tw(G)$ $(pw(G), bw(G)) = k$ but $tw(G')$ $(pw(G'), bw(G')) < k$ for any minor G' of G. From [11–13], $|\mathcal{T}_3(tw(G))| = |\mathcal{T}_3(bw(G))| = 1$, $|\mathcal{T}_4(tw(G))| = |\mathcal{T}_4(bw(G))| = 4$, $|\mathcal{T}_3(pw(G))| = 110$. As shown in [14], the critical graphs for parameters with a similar nature are worthy of further study, and the number of these critical graphs for a given value of the parameter would be finite and have yet to be characterized. The cutwidth problem for graphs and a class of optimal embedding (or layout) problems have significant applications in VLSI layouts [15,16], network reliability [17], automatic graph drawing [18], information retrieval [19], urban drainage network design [20], and other domains. In particular, the cutwidth is related to a basic parameter, called the congestion, in designing microchip circuits [2,21,22]. Herein, a graph G may be viewed as a model of the wiring diagram of an electronic circuit with the vertices representing components and the edges representing wires connecting them. When a circuit is laid out in a certain architecture (i.e., a path P_n), the maximum number of overlap wires is the congestion, which is one of major parameters in the determination of electronic performance. This motivates the enthusiasm for studying the cutwidth problem in graph theory practically. Theoretically, it appears to be closely related with other graph parameters such as pathwidth, treewidth, linear width, bandwidth, and modified bandwidth [2,8,23,24], among others. For instance, for any graph G with vertices of degree bounded by an integer $d \geq 1$, $pw(G) \leq c(G) \leq d \cdot pw(G)$, where $c(G)$ and $pw(G)$ are cutwidth value and pathwidth value, respectively. In this paper, we mainly study the critical unicyclic graph set $\mathcal{T}$ with a cutwidth of four that contains fifty elements.

For an integer $n > 0$, define $\mathcal{S}_n = \{1, 2, ..., n\}$. A labeling of a graph $G = (V(G), E(G))$ with $|V(G)| = n$ is a bijection $f : V(G) \to \mathcal{S}_n$, viewed as an embedding of G into a path P_n with vertices in $\mathcal{S}_n$, where consecutive integers are the adjacent vertices. The cutwidth of G with respect to f is

$$c(G, f) = \max_{1 \leq j < n} |\{uv \in E(G) : f(u) \leq j < f(v)\}|, \tag{1}$$

which is also the congestion of the labeling. The cutwidth of G is defined by

$$c(G) = \min_{f} c(G, f), \tag{2}$$

where the minimum is taken over all labelings f. If $k = c(G, f)$, then f, and the embedding induced by f is called a k-cutwidth embedding of G. A labeling f attaining the minimum in (2) is an optimal labeling. For each i with $i \in \mathcal{S}_n$, let $u_i = f^{-1}(i)$ and $S_j = \{u_1, u_2, ..., u_j\}$. Define $\nabla_f(S_j) = \{u_i u_h \in E : i \leq j < h\}$ is called the (edge) cut at $[j, j+1]$ with respect to f. From (2), we then have

$$c(G, f) = \max_{1 \leq j < n} |\nabla_f(S_j)|. \tag{3}$$

An f-max cut of G is a $\nabla_f(S_j)$, achieving the maximum in (3).

For a graph G and integer $i \geq 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, where $d_G(v)$ is the degree of vertex $v \in V(G)$. Any vertex in $D_1(G)$ is called a pendent vertex in G. For each $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For $V' \subset V(G)$ and $V' \neq \emptyset$, $G[V']$ is the subgraph of G induced by V'. If H, H' are subgraphs of G and $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by X, $H \cup H' = G[E(H) \cup E(H')]$ and $H \cup X = G[E(H) \cup X]$. Specially, if $X = \{e\}$, then we write $G + e$ instead of $G \cup \{e\}$. If G has a vertex $v \in D_2(G)$ with $N_G(v) = \{v_1, v_2\}$ and $v_1 v_2 \notin E(G)$, then $G - v + v_1 v_2$, and the graph obtained from $G - v$ by adding a new edge $v_1 v_2$, is called a series reduction of G. A graph G is homeomorphically minimal if G does not have any series reductions. Two graphs G' and G'' are homeomorphic if both of them can be obtained from the same graph G by inserting new vertices of degree two into its edges. A graph G is said to be k-cutwidth critical if G is homeomorphically minimal with $c(G) = k$ such that every proper subgraph G' of G satisfies $c(G') < k$. The basic properties of cutwidth follow immediately from this definition.

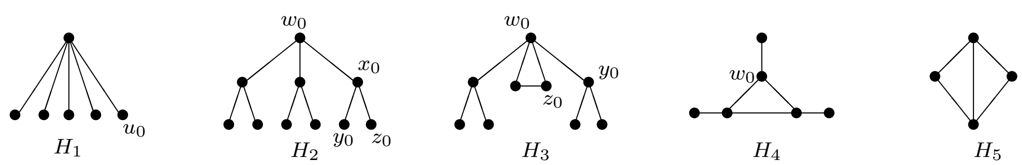

Figure 1. The 3-cutwidth critical graphs.

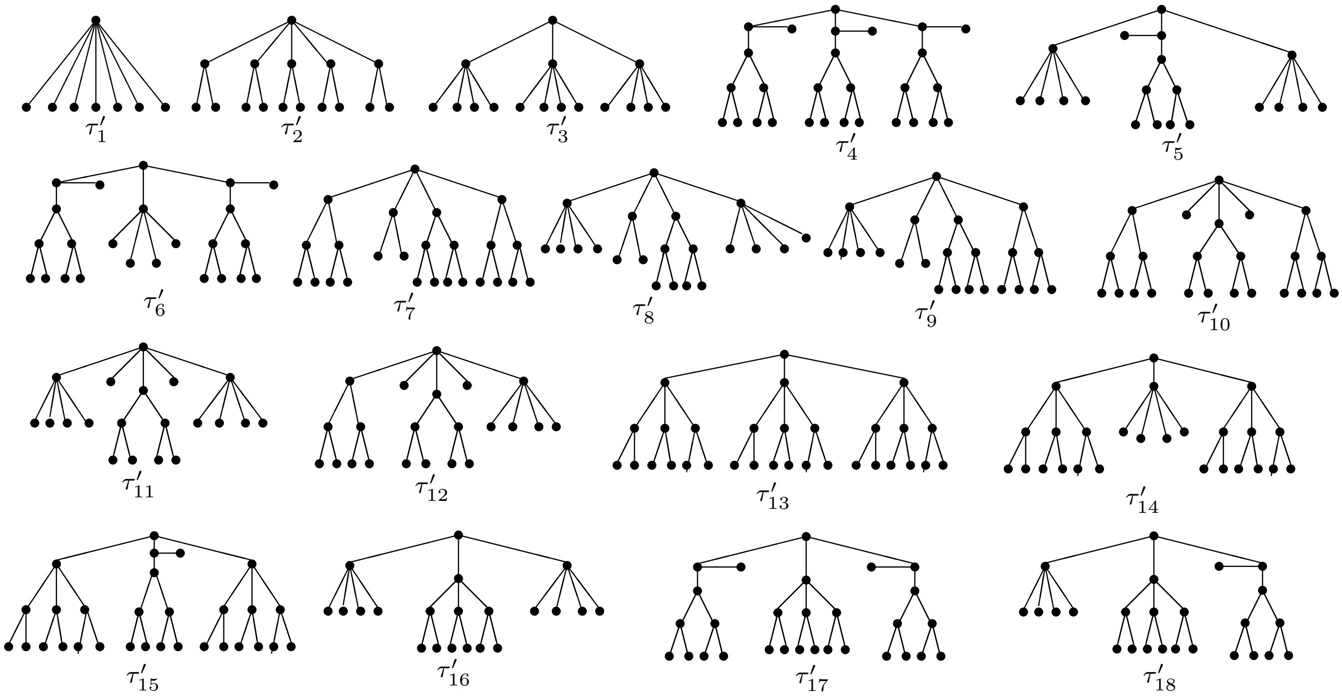

Figure 2. All elements of the 4-cutwidth critical tree set $\mathcal{T}_4$.

Lemma 1. *For graphs G and G′, each of the following holds. (1) If G′ is a subgraph of G, then* $c(G') \leq c(G)$*. (2) If G′ is homeomorphic to G , then* $c(G') = c(G)$*.*

Lemma 2. *The unique 1-cutwidth critical graph is* K_2*. The only 2-cutwidth critical graphs are* K_3 *and* $K_{1,3}$*. All 3-cutwidth critical graphs are* H_1, H_2, H_3, H_4*, and* H_5 *in Figure 1.*

Lemma 3 ([10])**.** *A tree T is 4-cutwidth critical if and only if* $G \in \mathcal{T}_4$*, where* $\mathcal{T}_4 = \{\tau'_i : 1 \leq i \leq 18\}$*, as depicted in Figure 2.*

A connected graph G with $|E(G)| = |V(G)|$ is called a *unicyclic graph*. The purpose of this paper is to characterize critical unicyclic graphs with a cutwidth of four and to present a forbidden subgraph characterization for unicyclic graphs with a cutwidth of three. Let $\mathcal{T} = \{\tau_i : 1 \leq i \leq 50\}$ be the collection of the critical unicyclic graphs depicted in Figure 3. The main results of this paper are the following:

Theorem 1. *A unicyclic graph G is 4-cutwidth critical if and only if* $G \in \mathcal{T}$*.*

Corollary 1. *A unicyclic graph G has a cutwidth of three if and only if it does not contain any subgraph homeomorphic to any member in* $\mathcal{T}$*.*

The rest of this paper is as follows. Section 2 presents some preliminary results. The proof of Theorem 1 is given in Section 3 by a series of lemmas. We give a short remark in Section 4.

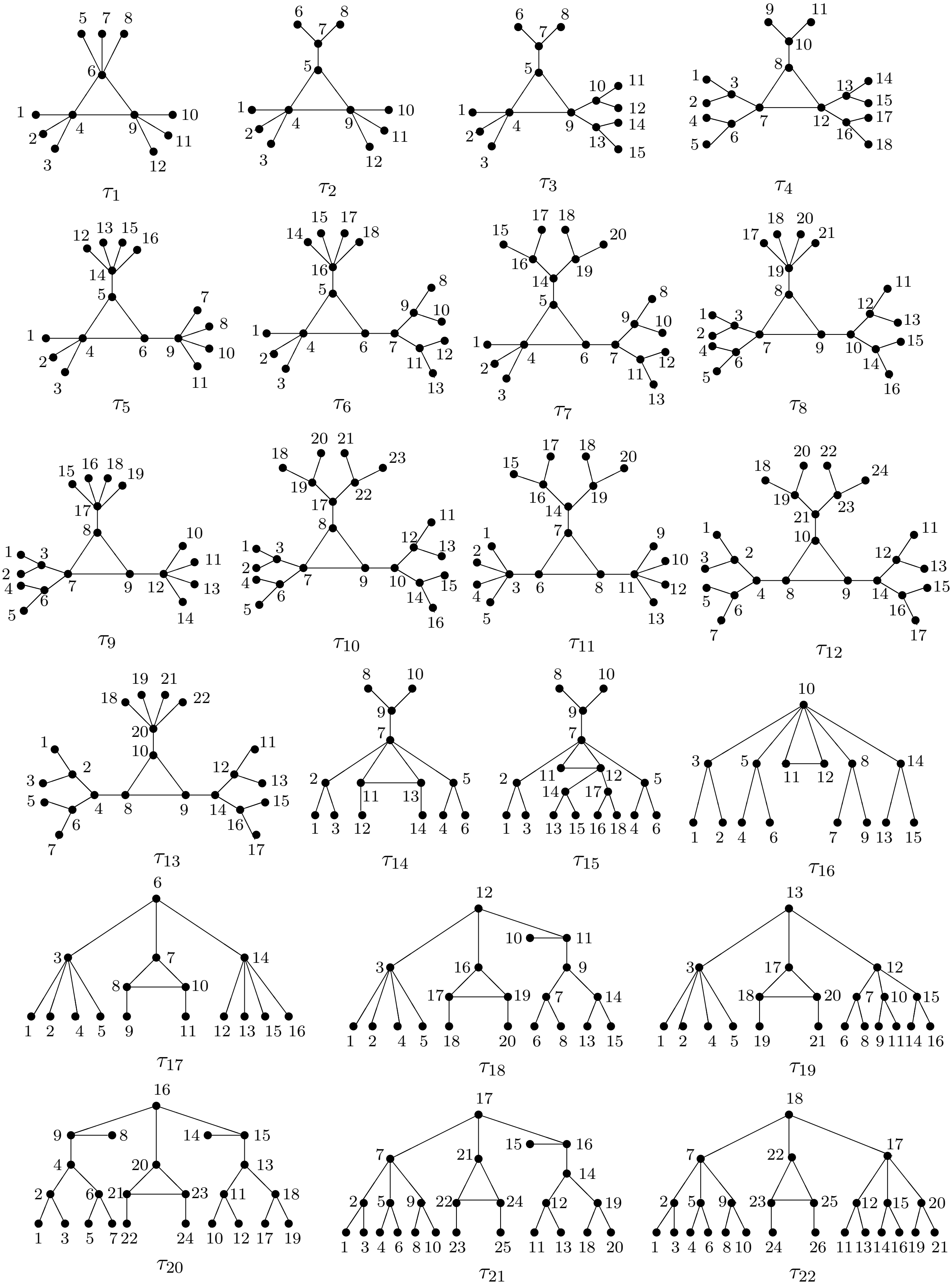

Figure 3. *Cont*.

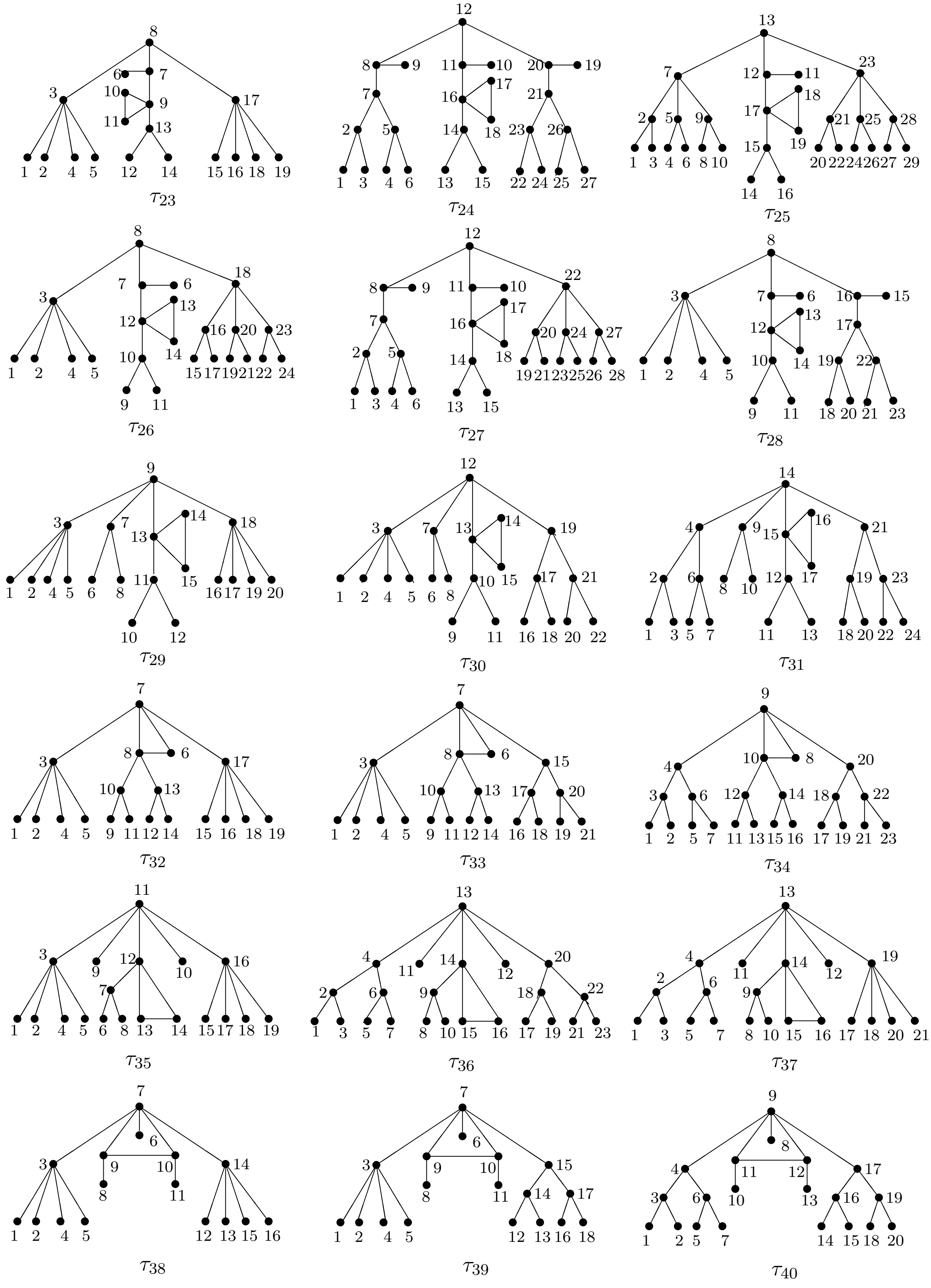

Figure 3. *Cont.*

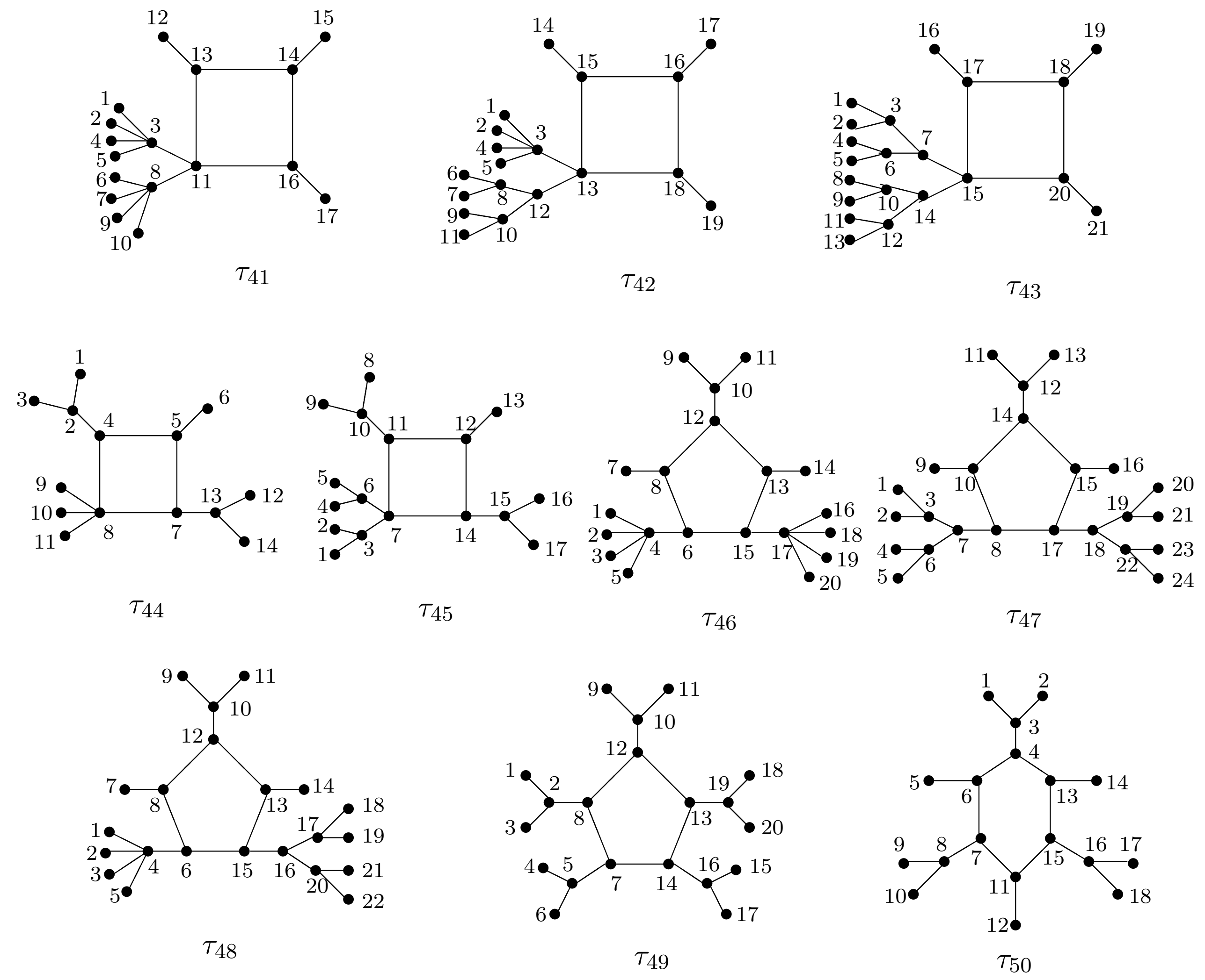

Figure 3. All 4-cutwidth critical unicyclic graphs with optimal labelings.

2. Preliminary Results

Throughout this section, for any integer $n > 1$, we always use P_n to denote the path with $V(P_n) = \mathcal{S}_n$ such that for all $1 \leq i < n$, i and $i+1$ are adjacent vertices in P_n. In addition, because $K_{1,2k-1}$ is k-cutwidth critical, as demonstrated by [10], we can let $d_G(v) \leq 2k-2$ for each $v \in V(G)$ in this paper.

The following observation is immediate from Lemma 1:

$$\text{if } v \in V(G), \text{ then } c(G-v) \leq c(G). \tag{4}$$

Definition 1.

(i) Let $r \geq 0$ be an integer, and v be a vertex of graph G with $d_G(v) > r$. For $v_1, v_2, ..., v_r \in N_G(v)$, define $G(v; v_1, v_2, ..., v_r)$ as the component of $G - \{vv_1, vv_2, ..., vv_r\}$ that contains v (see an illustration in Figure 4a).

(ii) Let G, H be two disjoint graphs with $u \in V(G)$ and $v \in V(H)$. To identify u and v, denoted as $G \oplus_{u,v} H$, is to replace u, v with a single vertex z (i.e., $u = v = z$) incident to all the edges which are incident to u and v, where z is called the identified vertex.

(iii) Let G_1, G_2 and G_3 be disjoint graphs $D_3(K_{1,3}) = \{u_0\}$ and $D_1(K_{1,3}) = \{u_1, u_2, u_3\}$. For each $j \in \mathcal{S}_3$, pick $v_j \in V(G_j)$. Define $K_{1,3} \circ (G_1, G_2, G_3)$ as the graph obtained from the disjoint union G_1, G_2, G_3 and $K_{1,3}$ by identifying u_j with v_j (again denoted as v_j) for each $j \in \mathcal{S}_3$ (see Figure 4b).

(iv) If $|V(G)| \geq 2$*, then define* $\mathcal{M}(G) = \{G - uv : uv \in E(G)$ *and* uv *is not a cut edge*$\} \cup \{G - v : v \in D_1(G)\}$ *to be the family of all proper maximal subgraphs of* G*.*

Definition 2. *For a graph* G *with* $|V(G)| = n$*, suppose that* $v_0 \in V(G)$ *is a vertex with* $N_G(v_0) = \{v_1, v_2, ..., v_p\}$*,* v_0v_1 *and* v_0v_2 *are two cut edges of* G*,* $H_1' = G(v_0; v_2, v_3, ..., v_p) - v_0$*,* $H_2' = G(v_0; v_1, v_2)$ *and* $H_3' = G(v_0; v_1, v_3, ..., v_p) - v_0$*. For* $1 \leq i \leq 3$*, let* $f_i : V(H_i') \mapsto \mathcal{S}_{|V(H_i')|}$ *be an optimal labeling of* H_i'*, and let a labeling* $f : V(G) \mapsto \mathcal{S}_n$ *of* G *be as follows: for* $v \in V(G)$*,*

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(H_1'), \\ f_2(v) + |V(H_1')| & \text{if } v \in V(H_2'), \\ f_3(v) + |V(H_1')| + |V(H_2')| & \text{if } v \in V(H_3'). \end{cases} \tag{5}$$

Then the labeling f *is called a labeling by the order* (f_1, f_2, f_3) *or* $(V(H_1'), V(H_2'), V(H_3'))$ *of* G*. For example, let* $G = K_{1,3} \circ (G_1, G_2, G_3)$ *with* $v_0 = u_0$ *in Figure 4b; then,* $N_G(u_0) = \{v_1, v_2, v_3\}$*,* $H_1' = G(u_0; v_2, v_3) - u_0 = G_1, H_2' = G(u_0; v_1, v_3) = G_2 + u_0v_2$*, and* $H_3' = G(u_0; v_1, v_2) - u_0 = G_3$*. If* f_i *is an optimal labeling of* H_i' *for* $1 \leq i \leq 3$*, and for* $v \in V(G)$*, define*

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1), \\ f_2(v) + |V(G_1)| & \text{if } v \in V(G_2) \cup \{v_0\}, \\ f_3(v) + |V(G_1)| + |V(G_2)| + 1 & \text{if } v \in V(G_3), \end{cases}$$

then, f *is a labeling of the order* (f_1, f_2, f_3) *of* G*.*

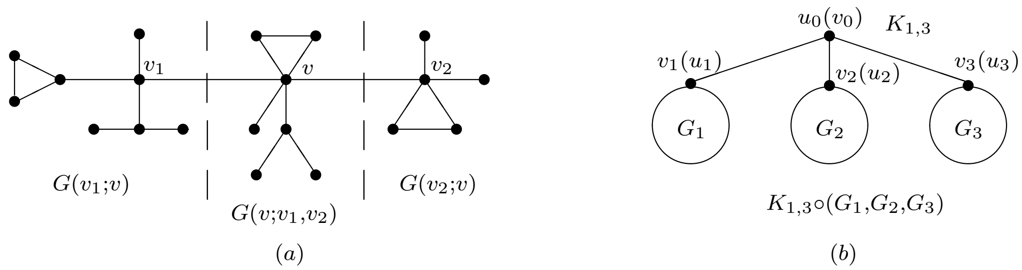

Figure 4. (**a**,**b**) Illustrations of Definitions 1 (i) and (iii).

Theorem 2. *For any* $v \in D_{\geq 3}(G)$*, if there always are two vertices* v_1, v_2 *in* $N_G(v)$ *such that* vv_1, vv_2 *are cut edges in* G*, then* $c(G) \leq k$ *if and only if* $c(G(v; v_1, v_2)) \leq k - 1$*.*

Proof. We first provide a claim.

Claim 1. *Let* $v_1'v_2'$ *be a cut edge in* G *and* V_1, V_2 *the vertex sets of two components of* $G - v_1'v_2'$*. Then, there exists an optimal labeling* f^* *such that the vertices in each of* V_1 *and* V_2 *are labeled consecutively.*

In fact, if f is an optimal labeling of G with $f(v_1') < f(v_2')$, then we can construct a labeling f^* as follows. First, label the vertices of V_1 in the same order as f, and then label the vertices of V_2 in the same order as f. Because the edges in $G[V_1]$ and those in $G[V_2]$ are not overlapped, it follows that $c(G, f^*) \leq c(G, f)$. Thus, f^* is also an optimal labeling of G.

Now, by using this observation, we proceed to prove Theorem 2. From the assumption that vv_1, vv_2 are cut edges of G, let V_0, V_1, V_2 be the vertex sets of three components of $G - \{vv_1, vv_2\}$, where $v_0 \in V_0, v_1 \in V_1, v_2 \in V_2$. Then, there exists an optimal labeling f^* such that each of the vertices of V_0, V_1, V_2 are labeled consecutively. If $c(G, f^*) \leq k$, then, because the edges vv_1 and vv_2 give a congestion of one to $G[V_0]$, we have $c(G[V_0], f^*) \leq k - 1$. Thus, $c(G) \leq k$ implies $c(G(v; v_1, v_2)) \leq k - 1$. Conversely, if $c(G[V_0], f^*) \leq k - 1$ and V_1, V_2 contain no vertices in $D_{\geq 3}(G)$, then $G[V_1]$ and $G[V_2]$ are two paths and so have a congestion of one. It follows that $c(G, f^*) \leq k$. If V_1 (or V_2) contains a vertex $v' \in D_{\geq 3}(G)$, then there must be two cut edges $v'v'_1, v'v'_2$ in G by assumption. In this way, $V(G)$ can be further decomposed into a sequence $V_1, V_2, ..., V_r$ such that $G[V_i]$ and $G[V_{i+1}]$ are connected by a cut edge $(1 \leq i < r)$. From $c(G(v; v_1, v_2)) \leq k-1$, $c(G[V_i], f^*) \leq k-1$ for $1 \leq i \leq r$. Hence, $c(G, f^*) \leq k$, resulting in $c(G) \leq k$. $\square$

Corollary 2. *With the notation in Theorem 2, for graph G, if there exists a vertex $v \in D_{\geq 3}(G)$ such that $c(G(v; v_i, v_j)) \geq k-1$ holds for any $v_i, v_j \in N_G(v)$, then $c(G) \geq k$, where vv_i, vv_j are cut edges in G.*

Lemma 4. *Let graph G be k-cutwidth critical with $D_1(G) \neq \emptyset$, and $P_l = u_0u_1...u_l$ be a path with length l. Then, $c(G \oplus_{v_0,u_0} P_l) = k$ for $v_0 \in V(G)$.*

Proof. Let v_0z be a pendant edge of G, where z is a pendant vertex. We subdivide the edge v_0z into a path P with length l and denote the resulting graph with G'. Then, by Lemma 1, $c(G \oplus_{v_0,u_0} P_l) = c(G') = c(G) = k$. $\square$

Theorem 3. *With the notation of Definition 1 (iii), let at least one of $\{G_1, G_2, G_3\}$, say G_2, be $(k-1)$-cutwidth critical with $D_1(G_2) \neq \emptyset$. Then $c(K_{1,3} \circ (G_1, G_2, G_3)) = k$.*

Proof. Let $G = K_{1,3} \circ (G_1, G_2, G_3)$. If $d_G(v_j) = 2$ for $j \in \mathcal{S}_3$; then, the series reduction can be implemented without affecting $c(G) = k$. Because u_0v_2 is a pendent edge of the subgraph $G_2 \oplus_{u_2,v_2} u_0v_2$ and G_2 is $(k-1)$-cutwidth critical with $D_1(G_2) \neq \emptyset$, $c(G_2 \oplus_{u_2,v_2} u_0v_2) = k-1$ by Lemma 4.

As $G - \{u_0v_1, u_0v_3\}$ has components $G_1, G_2 \oplus_{u_2,v_2} u_0v_2$ and G_3 with cutwidth $k-1$, similar to that of (5), an optimal labeling $f : V(G) \mapsto \mathcal{S}_n$ obtained by the order $(V(G_1), V(G_2 \oplus_{u_2,v_2} u_0v_2), V(G_3))$ satisfies $c(G, f) \leq (k-1) + 1 = k$. So, $c(G) \leq k$ by (2). On the other hand, it is routine to verify that $c(G) \geq k$ using Corollary 2 because $c(G(u_0; v_i, v_j)) = k-1$ for any $v_i, v_j \in N_G(u_0)$. Thus, $c(G) = k$, and the proof is complete. $\square$

Corollary 3. *With the notation of Definition 1 (iii), for each $j \in \mathcal{S}_3$, if G_j is $(k-1)$-cutwidth critical with $v_j \in D_1(G_j)$, then $K_{1,3} \circ (G_1, G_2, G_3)$ is k-cutwidth critical.*

Proof. Let $G = K_{1,3} \circ (G_1, G_2, G_3)$. Because $d_G(v_j) = 2$ for each $j \in \mathcal{S}_3$, three series reductions are carried out first. Furthermore, we still let $N_G(u_0) = \{v_1, v_2, v_3\}$ for convenience. Thus, $G(u_0; v_1, v_2) = G_3$, $G(u_0; v_2, v_3) = G_1$ and $G(u_0; v_1, v_3) = G_2$.

First, by assumption and Theorem 3, $c(G) = k$.

Second, we prove that G is k-cutwidth critical. It remains to be shown that, for any $G' \in \mathcal{M}(G)$, $c(G') \leq k-1$. Because any G' is obtained by deleting a pendent edge xy or an nonpendent edge $xy \in E(C_t)$ in G, $xy \notin \{u_0v_1, u_0v_2, u_0v_3\}$. Without loss of generality, let $xy \in E(G_2)$. By assumption that G_j is $(k-1)$-cutwidth critical for each $j \in \mathcal{S}_3$, we have $c(G_1 - u_0v_1) \leq k-2$, $c(G_2 - xy) \leq k-2$ and $c(G_3 - u_0v_3) \leq k-2$. Thus, similar to (5), a labeling $f' : V(G') \mapsto \mathcal{S}_{|V(G')|}$ by the order $(V(G_1 - u_0v_1), V(G_2 - xy), V(G_3 - u_0v_3))$ is obtained and $c(G', f') = k-1$. So, $c(G') \leq k-1$ by (2), and G is k-cutwidth critical. $\square$

3. Proof of Theorem 1

We verify our main results by using a series of lemmas. Throughout this section, G denotes an unicyclic graph and $C_t = v_1v_2...v_tv_1$ denotes the unique cycle of G with $t \geq 3$. Furthermore, we have a convention that a graph H is designated to be homeomorphic to a subgraph of G if H can be obtained by deleting vertices or edges and some series reductions of G and $c(H) = c(G)$. Because a cutwidth critical graph is homeomorphically minimal, if G is k-cutwidth critical then

$$d_G(v_i) \geq 3 \text{ for } v_i \in V(C_t), \tag{6}$$

unless v_i is a special vertex.

Let $x_0, w_0 \in D_3(H_2), y_0, z_0 \in D_1(H_2), u_0 \in D_1(K_{1,5})$ (see H_2 in Figure 1). Furthermore, let $F_1 = H_2 - y_0 - z_0$, $F_2 = H_2 - \{x_0, y_0, z_0\}$, $\mathcal{F}_1 = \{K_2, K_{1,3}, K_{1,5}, F_1, F_2\}$ (see F_1, F_2 in Figure 5a), and $\mathcal{F}_2 = \{K_{1,5} \oplus_{u_0,u_0'} K_{1,5}', K_{1,5} \oplus_{u_0,x_0} F_1, F_1 \oplus_{x_0,x_0'} F_1'\}$ (see Figure 5b), where $K_{1,5}'$, F_1' are copies of $K_{1,5}$ and F_1, $u_0' \in D_1(K_{1,5}')$, $x_0' \in D_3(F_1')$ are copies of u_0 and x_0, respectively. For an integer $p > 1$, we call *a star* $K_{1,p}$ *centered at vertex* x if $d_{K_{1,p}}(x) = p$.

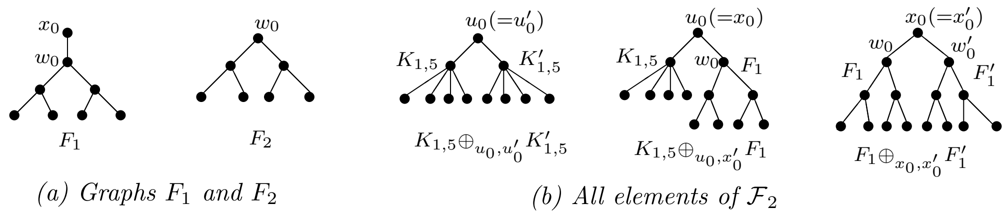

Figure 5. Two elements of $\mathcal{F}_1$ and set $\mathcal{F}_2$.

Lemma 5. *Each member of set $\mathcal{T}$ is 4-cutwidth critical in Figure 3.*

Proof. For a unicyclic graph G, let $C_t = v_1 v_2 ... v_t$ be the unique cycle of G. Then, $G - E(C_t)$ is a forest of t subtrees $T_1, T_2, ..., T_t$ where T_i is called the v_i-branch leading from v_i. We first consider the case of $t = 3$ in which $G - E(C_3)$ has three subtrees T_1, T_2, T_3. For an optimal labeling f of G, suppose that $f(v_1) < f(v_2) < f(v_3)$; then, the number set $\mathcal{S}_n$ is divided into three intervals $I_1 = [1, f(v_1)], I_2 = (f(v_1), f(v_3)), I_3 = [f(v_3), n]$. Subtrees T_1, T_2, T_3 are then embedded into I_1, I_2, I_3 in different manners. We have the following classifications of 4-cutwidth critical unicyclic graphs.

(1) Type 3A (including τ_1 to τ_4): T_1 is embedded in I_1 with a congestion of three, T_2 is embedded in I_2 with a congestion of four, and T_3 is embedded in I_3 with a congestion of three. Herein, T_1 and T_3 are the star $K_{1,3}$ with center v_i or the two stars $K_{1,3}$ with an identifying leaf at v_i (i=1,3) (see F_2 in Figure 5a). Let $\tilde{T}_i$ denote T_i combining with the two edges in C_3 incident with v_i. Then, $\tilde{T}_1$ and $\tilde{T}_3$ are the 3-cutwidth critical tree $H_1 = K_{1,5}$ or the 3-cutwidth critical tree H_2 with a central edge (i.e., similar to $w_0 x_0$ in H_2) contracted. As to T_2 embedded in I_2 with a congestion of four, the cycle C_3 yields a congestion of two in this interval, and we have to choose T_2 as a 2-cutwidth critical tree, namely, a $K_{1,3}$ such that either $d_G(v_2) = 3$ or $d_G(v_2) = 5$. For this type of construction, the maximum congestion is four, that is, $c(G) = 4$. Furthermore, for any edge $e \in E(G)$, if $e \in E(C_3)$, then the deletion of e reduces the congestion two of cycle-edge in I_2 by one. Hence, T_2 embedded in I_2 has a congestion of three, and so $c(G - e) < 4$. If $e \notin E(C_3)$, for τ_1 with $T_j = K_{1,3}$ and $d_G(v_j) = 5$ for each $1 \leq j \leq 3$, we can let $e \in E(T_2)$. Because $T_2 - e = K_{1,3} - e$ has a congestion of one, we can embed T_1 in I_1, $T_2 - e$ in $(f(v_1), f(v_3) - 1)$ and T_3 in $[f(v_3) - 1, n - 1]$. Thus, $c(\tau_1 - e) = 3$. For τ_2, τ_3, and τ_4, because $d_G(v_2) = 3$, we can let $e \in E(T_1)$; then, $T_1 - e$ in I_1 has a congestion of two. Thus, we embed $T_2 - v_2$ in I_1 such that I_1 has a congestion of three (for example, T_1 is an F_2 at v_1 for τ_4), and the same is true for the case of $e \in E(T_3)$. Hence, $c(G - e) < 4$. Thus, G is 4-cutwidth critical.

(2) Type 3B (including τ_5 to τ_{13}): $f(v_1), f(v_2), f(v_3)$ are consecutive and $I_2 = \{f(v_2)\}$, T_1 is embedded in I_1 with a congestion of three, T_2 is embedded in $I_2 \cup I_3$ with a congestion of four, T_3 is embedded in I_3 with a congestion of three. Herein, we denote the subtree of H_2 obtained by deleting two leaves in the same branch (say y_0, z_0 in Figure 1) with F_1 , and denote the subtree of H_2 obtained by deleting three vertices in the same branch (say x_0, y_0, z_0 in Figure 1) with F_2 (see F_1, F_2 in Figure 5a). Then, T_1 is a star $K_{1,3}$, $K_{1,5}$, F_1, F_2, or T_2 and T_3 is a star $K_{1,5}$ or F_1. Note that if $T_i = K_{1,3}$ then $\tilde{T}_i = H_1$; if $T_i = F_1$ then $\tilde{T}_i = H_2$, where H_1 and H_2 are 3-cutwidth critical. Because T_2 and T_3 are embedded in I_3 consecutively and an edge of T_2 incident with v_2 strides over all edges of T_3, we see that the overlapped edges of T_2 and T_3 give rise to a congestion of four in the embedding. Hence, $c(G) = 4$. Furthermore, for any edge $e \in E(G)$, if $e \in E(C_3)$, then $G - e$ is a tree made up with H_1 and H_2, which has a cutwidth of three. Thus, $c(G - e) < 4$. Otherwise, we may assume $e \in E(T_2)$ (we

may change the order of T_1, T_2, T_3 if necessary). Then, $c(T_2 - e) = 2$, and so the embedding of T_2 and T_3 in I_3 gives a congestion of three by making $\min\{f(v) : v \in V(T_2 - e - v_2)\} = f(v_3) + 1$ and $\max\{f(v) : v \in V(T_2 - e - v_2)\} = \min\{f(v) : v \in V(T_3 - v_3)\} - 1$. Thus, we have $c(G - e) < 4$. Hence, G is 4-cutwidth critical.

(3) Type 3C (including τ_{14}, τ_{15} and τ_{38} to τ_{40}): T_2 and T_3 are K_2, T_1 is decomposed and embedded into different intervals. For τ_{14} and τ_{15}, T_1 is an H_2, and it is decomposed into two stars $K_{1,3}$ embedded in I_1 and one star $K_{1,3}$ embedded in I_2. The star $K_{1,3}$ in I_2 and the two cycle edges give rise to the congestion of four in I_2. For τ_{38}, T_1 is decomposed into two stars $K_{1,5}$ and K_2, where a star $K_{1,5}$ and a K_2 are embedded in I_1, and a star $K_{1,5}$ is embedded in I_3. Additionally, τ_{39} and τ_{40} are similar. Similar to the previous cases, it can be shown that G is 4-cutwidth critical.

(4) Type 3D (including τ_{16} to τ_{37}): This type of unicyclic graphs are obtained from 4-cutwidth critical trees by making the following local transformations: the star $K_{1,3}$ is transformed into a triangle K_3 (for example, H_2 is transformed into H_3, see Figure 1) and the star $H_1 = K_{1,5}$ is transformed into a 'sun' H_4. Because these local transformations do not change the congestion of two of $K_{1,3}$ or the congestion of three of $K_{1,5}$, this part of the proof is based on Lemma 3. Let $\tau'_1 - \tau'_{18}$ denote the 4-cutwidth critical trees in Lemma 3 (see Figure 2). Then, for τ_{16} to τ_{37} in Figure 3, we have the following correspondences: τ_{16} is from τ'_2, τ_{17} is from τ'_3, τ_{18} is from τ'_3, τ_{19} is from τ'_{16}, τ_{20} is from τ'_6, τ_{21} is from τ'_{18}, τ_{22} is from τ'_{14}, τ_{23} is from τ'_5, τ_{24} is from τ'_4, τ_{25} is from τ'_{15}, τ_{26} is from τ'_{18}, τ_{27} is from τ'_{17}, τ_{28} is from τ'_6, τ_{29} is from τ'_8, τ_{30} is from τ'_9, τ_{31} is from τ'_7, τ_{32} is from τ'_5 (or τ'_{11}), τ_{33} is from τ'_{12} ,τ_{34} is from τ'_{10} , τ_{35} is from τ'_{11}, τ_{36} is from τ'_{10}, and τ_{37} is from τ'_{12}. Thus, each of $\{\tau_{16}, \tau_{17}, ..., \tau_{37}\}$ is 4-cutwidth critical.

For $t \geq 4$, we have the similar arguments as follows.

(5) Type 4A (including τ_{41} to τ_{43}): Similar to above, for each $G \in \{\tau_{41}, \tau_{42}, \tau_{43}\}$, $c(G) = 4$. To show that G is 4-cutwidth critical, we take $e \in E(G)$. If $e \in E(C_4)$, then $G - e$ is a tree made up with T_1 in $\mathcal{F}_2$ and $T_i = K_2$ for $2 \leq i \leq 4$, which has a cutwidth of three. Thus, $c(G - e) < 4$. If $e \in E(T_1)$, then $T_1 - e$ has a congestion of three, and so $c(G - e) < 4$. If $e \in E(T_i)$ with $T_i = K_2$ for $2 \leq i \leq 4$, then $G - e$ is a proper subgraph of one of τ_{38} to τ_{40}, and so, $c(G - e) < 4$.

(6) Type 4B (including τ_{44} to τ_{45}): Similar to the previous cases, the cutwidth of $G \in \{\tau_{44}, \tau_{45}\}$ is four. Now let $e \in E(G)$. If $e \in E(C_4)$ then $G - e$ is a tree made up with $T_i \in \{K_2, K_{1,3}, F_2\}$ for $1 \leq i \leq 4$, which has a cutwidth of three. Thus, $c(G - e) < 4$. For $e \in E(T_i)$ with any $1 \leq i \leq 4$, we can always find a labeling f' of $G - e$ such that $c(G - e, f') \leq 3$. So, $c(G - e) \leq 3$ leading to that G is 4-cutwidth critical.

(7) Type 5A (including τ_{46} to τ_{48}): Similar to that of Type 4A, omitted here.

(8) Type 5B (including τ_{49} only): The labeling f of τ_{49} in Figure 3 implies that $c(\tau_{49}) \leq 4$. $\tau_{49} - E(C_5)$ has five subtrees T_1, T_2, T_3, T_4, T_5 each of which is a star $K_{1,3}$, and each $v_i \in V(C_5)$ $(1 \leq i \leq 5)$ is a pendent vertex of T_i correspondingly. Without loss of generality, for an optimal labeling f of τ_{49}, let $f(v_i) = j_i$ and $j_1 < j_2 < j_3 < j_4 < j_5$. Clearly, $|\nabla_f(S_{j_i})| \geq 2$ for $1 \leq i \leq 4$. Because T_i has a congestion of two for each $1 \leq i \leq 5$ and $t = 5$, there is at least a vertex $x \in V(T_2) \cup V(T_3)$ (or $x \in V(T_3) \cup V(T_4)$) with $f(x) = j$ such that $|\nabla_f(S_j)| = 4$. From (3), $c(\tau_{49}, f) = 4$, resulting in $c(\tau_{49}) = 4$ by (2). On the other hand, τ_{49} has two maximal proper subgraphs where one is obtained by deleting a pendent edge $e \notin V(C_5)$, the other is obtained by deleting any cycle edge $e \in V(C_5)$. For each maximal proper subgraph $G - e$, we can always find a labeling f' such that $c(G - e, f') \leq 3$ easily. Thus, $c(G - e) \leq 3$, leading to the finding that G is 4-cutwidth critical.

(9) Type 6A (including τ_{50} only): Similar to that of Type 5B, omitted here. This completes the proof. □

Lemma 6. *Let G be a 4-cutwidth critical graph with unique cycle C_t and $t \geq 4$. Then each of the following holds.*
(i) If, for each $v_i \in V(C_t)$ $(1 \leq i \leq t)$, each member in $\mathcal{F}_2$ is not an induced subgraph of T_i, then $T_i \in \mathcal{F}_1$.

(ii) If there is at least a vertex $v_i \in V(C_t)$ such that one of $\mathcal{F}_2$ is an induced subgraph of T_i, then $T_i \in \mathcal{F}_2$.

Proof. (i) From (6), $d_G(v_i) \geq 3$. First, from the assumption that G is 4-cutwidth critical, it follows that $c(G - xy) = 3$ for any $xy \in E(G)$ and $c(T_i) \leq 3$ for each $v_i \in V(C_t)$. For the edge xy, there are three cases to consider.

Case 1 $x \in V(C_t), y \notin V(C_t)$. In this case, $C_t \subset G - xy$. So, by the minimality of H_3 and H_4 (see H_3, H_4 in Figure 1), either H_3 or H_4 is a subgraph of $G - xy$ resulting in that either F_2 or K_2 is contained in some T_i, say T_1.

Case 2 $x \notin V(C_t), y \notin V(C_t)$. Similar to that of Case 1, we can conclude that F_2 or K_2 is also contained in some T_i, say T_1.

Case 3 $x \in V(C_t), y \in V(C_t)$. Clearly, $G - xy$ is a 3-cutwidth tree. So, by the minimality of H_1 and H_2 in Figure 1, either H_1 or H_2 is a subgraph of $G - xy$ leading to the conclusion that either $K_{1,5}$ or F_1 is contained in some T_i with $v_i \neq v_1$.

In addition, because $K_{1,3}$ is a proper subgraph of any of $\{K_{1,5}, F_1, F_2\}$ and $t \geq 4$, we can conclude that there is at least a vertex $v_i \in V(C_t)$ such that $T_i = K_{1,3}$ with $d_G(v_i) = 3$. Otherwise, $T_i \in \mathcal{F}_1 \setminus \{K_{1,3}\}$ for every $i \in \mathcal{S}_t$. In this case, we can verify that either $c(G) = 3$ (contradicting $c(G) = 4$) or one of $\{\tau_{44}, ..., \tau_{50}\}$ is homeomorphic to a subgraph of G contradicting the minimality of G. Thus, $T_i \in \mathcal{F}_1$ for $i \in \mathcal{S}_t$.

(ii) Assume that one member of $\mathcal{F}_2$ is a subgraph of some T_{i_0} and $v_{i_0} = u_0$ (or x_0) by homeomorphism. Because $t \geq 4$ and $d_G(v_i) \geq 3$ for each $i \neq i_0$ by (6), one of $\{\tau_{41}, \tau_{42}, \tau_{43}\}$ must be either a subgraph of G or homeomorphic to a subgraph of G, contrary to the minimality of G. Hence, $T_{i_0} \in \mathcal{F}_2$. Thus, by Lemma 5, G is 4-cutwidth critical with $T_{i_0} \in \mathcal{F}_2$ for $v_{i_0} \in V(C_t)$ if and only if $G \in \{\tau_{41}, \tau_{42}, \tau_{43}\}$. This completes the proof. □

Lemma 7. *Let G be a 4-cutwidth critical graph with cycle C_t. Then, $t \leq 6$.*

Proof. This is a proof by contradiction. Assume that $t \geq 7$; then, $T_i \notin \mathcal{F}_2$ for each $i \in \mathcal{S}_t$. This is because, otherwise, one of $\{\tau_{41}, \tau_{42}, \tau_{43}\}$ is homeomorphic to a subgraph G' of G in which the cycle C_4 is subdivided into C_t. So, $c(G') = 4$, contradicting the conclusion that G is 4-cutwidth critical. Thus, $T_i \in \mathcal{F}_1$ by Lemma 6.

For each $i \in \mathcal{S}_t$, if $T_i = K_2$, then direct computation yields that $c(G) = 3$. This implies that at least a $T_i \in \mathcal{F}_1 \setminus \{K_2\}$. In addition, because $c(\tau_1) = 4$, there are at most two vertices $v_{i_1}, v_{i_2} \in V(C_t)$ such that $T_{i_1} = K_{1,3}$ centered at v_{i_1} and $T_{i_2} = K_{1,3}$ centered at v_{i_2}. In the sequel, let $f : V(G) \to \mathcal{S}_n$ be an optimal 4-cutwidth labeling with $f(v_1) = \min\{f(v_i) : v_i \in V(C_t)\}$ and $f(v_h) = \max\{f(v_i) : v_i \in V(C_t)\}$ for some $2 \leq h \leq t$, and embed T_1 into the interval $[1, f(v_1)]$, T_i $(i \neq 1, h)$ into the interval $(f(v_1), f(v_h))$ and T_h into the interval $[f(v_h), n]$.

Case 1 $\min\{d_G(v_1), d_G(v_h)\} \geq 4$, i.e., $T_1 = K_{1,3}$ is centered at v_1 or F_2 with $d_G(v_1) = 4$, and $T_h = K_{1,3}$ is centered at v_h or F_2 with $d_G(v_h) = 4$ because $T_i \in \mathcal{F}_1$ for each $i \in \mathcal{S}_t$. Thus, the congestions of T_1 and T_h are at most three under f. Because G is 4-cutwidth critical and each cycle edge v_iv_{i+1} of C_t has a congestion of two, the subtree T_i for $i \neq 1, h$ must be 1- or 2-cutwidth critical, namely, $T_i = K_2$ or $K_{1,3}$. If each $T_i = K_2$, then $c(G) = 3$ by direct computation. So, there are at least a vertex v_{i_0} $(i_0 \neq 1, h)$ such that $T_{i_0} = K_{1,3}$ in G, which results in the conclusion that one of $\{\tau_2, \tau_3, \tau_4\}$ is homeomorphic to a subgraph of G; this is a contradiction. Hence, this case is not possible.

Case 2 $\max\{d_G(v_1), d_G(v_h)\} \geq 4$ and $\min\{d_G(v_1), d_G(v_h)\} = 3$, say $d_G(v_1) \geq 4$, $d_G(v_h) = 3$.

From the minimality of G, in this case, T_1 is either a $K_{1,3}$ centered at v_1 or F_2, $T_h = K_{1,3}$ not centered at v_h. For each $i \neq 1, h$, if $T_i = K_2$ then $c(G) = 3$ by the direct computation, contrary to $c(G) = 4$. So, there is at least a T_i except T_1 and T_h such that $T_i \in \{K_{1,3}, K_{1,5}, F_1\}$. This results in the conclusion that one of $\{\tau_{44}, \tau_{45}\}$ must be homeomorphic to a subgraph of G, contrary to the minimality of G. For example, if $t = 7$ and $h = 4$, τ_{44} must be homeomorphic to Figure 6a (or Figure 6b), while τ_{45} is homeomorphic to Figure 6c. So, this case is impossible.

Case 3 $\max\{d_G(v_1), d_G(v_h)\} = \min\{d_G(v_1), d_G(v_h)\} = 3$, i.e., $d_G(v_i) = 3$ for each $1 \le i \le t$. By Lemma 6 and the minimality of G, $T_i \in \mathcal{F}_1 \backslash \{F_2\}$ for each $1 \le i \le t$. Because τ_{11}, τ_{12}, and τ_{13} are 4-cutwidth critical, at most two subtrees of $G - E(C_t)$, say T_1 and T_h, are in $\{K_{1,5}, F_1\}$. So, similar to Cases 1 and 2, either one of $\{\tau_{46}, ..., \tau_{50}\}$ is homeomorphic to a subgraph of G or $c(G) < 4$ (see seven typical cases in Figure 7 each of whose cutwidth is three by homemorphism), contradicting the conclusion that G is 4-cutwidth critical. So, this case is also impossible.

To sum up, we have $t \le 6$. This completes the proof. □

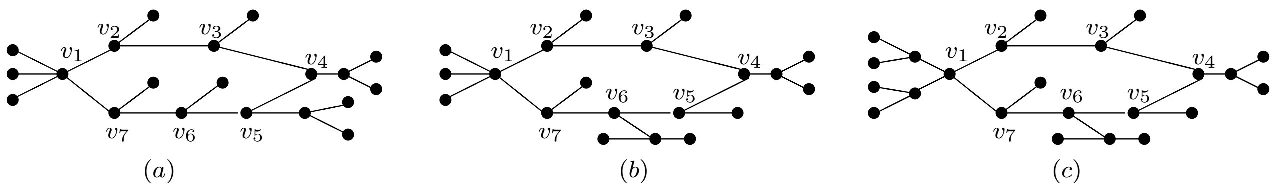

Figure 6. (**a**–**c**) Three examples of Case 2 with the proof of Lemma 7.

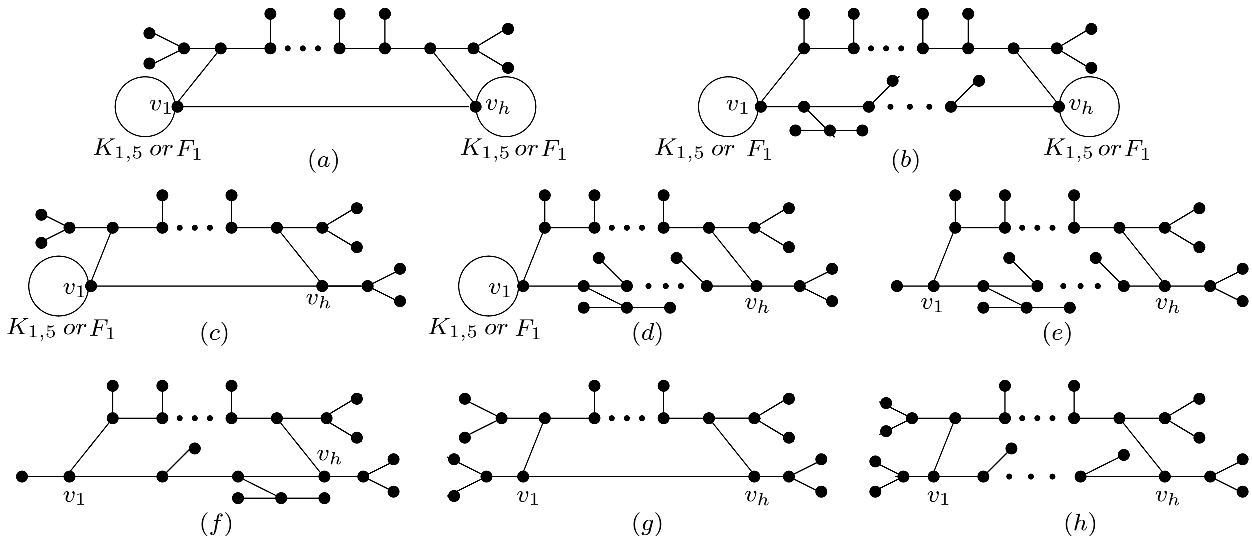

Figure 7. (**a**–**h**) Seven typical examples on G of Case 3 with the proof of Lemma 7.

Lemma 8. *Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_1) \ge 4$, $d_G(v_2) \ge 4$ in G, then G is 4-cutwidth critical if and only if $G \in \{\tau_1, \tau_2, \tau_3, \tau_4\}$.*

Proof. By Lemma 5, it suffices to show its necessity. Because G is 4-cutwidth critical, $T_1, T_3 \in \{K_{1,3}, F_2\}$ by Lemmas 6 and 7. For an optimal labeling f of G with $f(v_1) < f(v_2) < f(v_3)$, we can embed T_1 into the interval $[1, f(v_1)]$ with a congestion of three and T_3 into the interval $[f(v_3), n]$ with a congestion of three. So, T_2 must be $K_{1,3}$, because the congestion of the cycle edge of C_3 is two, which is embedded into the interval $(f(v_1), f(v_3))$. Thus, G is one of $\{\tau_1, \tau_2, \tau_3, \tau_4\}$. □

With an argument similar to that of Lemma 8, we can verify that the following two Lemmas 9 and 10 hold also.

Lemma 9. *Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_1) \ge 4$, $d_G(v_2) = 3$ and $d_G(v_3) = 3$ but $T_2 \ne K_2, T_3 \ne K_2$ in G; then, G is 4-cutwidth critical if and only if $G \in \{\tau_5, \tau_6, ..., \tau_{10}\}$.*

Lemma 10. *Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_i) = 3$ and $T_i \neq K_2$ for each $i \in \mathcal{S}_3$ in G; then, G is 4-cutwidth critical if and only if $G \in \{\tau_{11}, \tau_{12}, \tau_{13}\}$.*

Lemma 11. *Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_1) \geq 5$, $T_2 = K_2$, and $T_3 = K_2$ in G; then, G is 4-cutwidth critical if and only if $G \in \{\tau_{14}, \tau_{15}, \tau_{38}, \tau_{39}, \tau_{40}\}$.*

Proof. By Lemma 5, we only show its necessity. By $c(K_{1,7}) = 4$ (see τ_1' in Figure 2), $d_G(v) \leq 6$ for each $v \in V(G)$. So, $d_G(v_1) = 5$ or 6. We first consider the case of $d_G(v_1) = 5$, and let the three subtree components of $T_1 - v_1$ be $T_1^{(1)}, T_1^{(2)}$ and $T_1^{(3)}$, $T_1' = T_1^{(1)} + x_1v_1$, $T_1'' = T_1^{(2)} + x_2v_1$ and $T_1''' = T_1^{(3)} + x_3v_1$, respectively, where $x_1, x_2, x_3 \in N_G(v_1)$ and $x_1 \in V(T_1^{(1)}), x_2 \in V(T_1^{(2)}), x_3 \in V(T_1^{(3)})$.

Claim 2. *At most one subtree, say T_1''', among T_1', T_1'' and T_1''' is K_2.*

In fact, if $T_1' = K_2$ and $T_1'' = K_2$, then $c(G) = 3$ by direct computation, contrary to $c(G) = 4$. Now, let $T_1' = K_2$ but $T_1'' \neq K_2$. In this case, we have $c(T_1) = 4$. Otherwise, $c(T_1) = 3$, and by the minimality of G, $T_1 = H_2$ in Figure 1 (note that $T_1 \neq H_1$ because of $T_1'' \neq K_2$). For a labeling f of G with $f(v_1) < f(v_2) < f(v_3)$, we embed T_1 into the interval $[1, f(v_1)]$ with a congestion of three and $G - T_1$ into the interval $(f(v_1), n)$ with a congestion of three. In this way, $c(G, f) = 3$ resulting in $c(G) \leq 3$ by (2), a contradiction. Thus, by $T_1 \subset G$ and $c(T_1) = 4$, a contradiction to the minimality of G is obtained. Claim 2 holds.

From Claim 2, there are only two subcases considered: (1) $T_1''' \neq K_2$. From the minimality of G, $T_1 = H_2$ and $G = \tau_{14}$ or τ_{15}. (2) $T_1''' = K_2$. In this subcase, for an optimal labeling f of G with $f(x_1) = \max\{f(v) : v \in V(T_1^{(1)})\}, f(x_2) = \max\{f(v) : v \in V(T_1^{(2)})\}$ and $f(x_1) < f(x_2)$, because x_1v_1 and x_2v_1 are cut edges in G, under f, $T_1^{(1)}$ is embedded into the interval $[1, f(x_1)]$ with a congestion of three, $T_1^{(2)} + x_2v_1$ into the interval $(f(x_1), f(x_2)]$ with a congestion of four and $G - T_1^{(1)} - T_1^{(2)}$ $(= H_4$ in Figure 1) into the interval $[f(x_2) + 1, n]$ with a congestion of three, which leads to the conclusion that $c(G) = c(G, f) = 4$. Thus, by the minimality of G, $T_1' \cup T_1'' \in \mathcal{F}_2$ leading to the conclusion that $G \in \{\tau_{38}, \tau_{39}, \tau_{40}\}$. Likewise, for the case of $d_G(v_1) = 6$, using an argument similar to the case of $d_G(v_1) = 5$, we can verify that at least one of $\{\tau_{14}, \tau_{15}, \tau_{38}, \tau_{39}, \tau_{40}\}$ is a proper subgraph of G, contrary to the minimality of G. So, $d_G(v_1) \neq 6$. This completes the proof. □

Lemma 12. *Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle in G, and there are three disjoint graphs G_1, G_2, G_3 such that $G = K_{1,3} \circ (G_1', G_2', G_3')$. Then, G is 4-cutwidth critical if and only if $G \in \{\tau_{17}, \tau_{18}, ..., \tau_{28}\}$, where, for $j \in \mathcal{S}_3$,*

$$G_j' = \begin{cases} G_j & \text{if } v_j \notin V_p^{(j)} \text{ for } v_j \in V(G_j), \\ G_j - p_j & \text{otherwise} \end{cases}$$

in which $V_p^{(j)} \subset V(G_j)$ and there always are at least a pendent vertex $p_j \in D_1(G_j)$ such that $vp_j \in E(G_j)$ for each $v \in V_p^{(j)}$.

Proof. By Lemma 5, we only show its necessity and adopt the notation of Definition 1 (*iii*). As G is unicyclic and vertex p_i is pendent, C_3 is contained in one of $\{G_1, G_2, G_3\}$, say G_2. Thus, G_1, G_3 are subtrees in G. From the hypothesis that G is 4-cutwidth critical, for $j \in \mathcal{S}_3$ and $v_j \in V(G_j')$, G_j' and $G_j' + u_0v_j$ are 3-cutwidth critical in the cases of $v_j \notin V_p^{(j)}$ or $v_j \in V_p^{(j)}$ respectively after that the series reduction is implemented. Because, otherwise, it is not hard to obtain a noncritical graph with a cutwidth of four, a contradiction. Hence,

$G'_j \in \{H_1, H_2, H_3, H_4\}$ if $v_j \notin V_p^{(j)}$ and $G'_j + u_0 v_j \in \{H_1, H_2, H_3, H_4\}$ if $v_j \in V_p^{(j)}$. So, $G \in \{\tau_{17}, \tau_{18}, ..., \tau_{28}\}$ by the minimality of G. □

Similar to Lemma 12, a class of critical unicyclic graphs with a cutwidth of four has an interesting structure (see Definition 3 below). This structure together with that of Lemma 12 is called the decomposability of the critical unicyclic graphs with a cutwidth of four. From Corollary 3, $K_{1,3} \circ (K_{1,5}, K_{1,5}, K_{1,5})$ with $v_j \in D_1(K_{1,5})$ $(1 \leq j \leq 3)$ is 4-cutwidth critical after that the series reductions are carried out, so we may assume that $G - v$ has at most two $K_{1,4}$'s for any $v \in V(G)$ in the sequel.

Definition 3. *Let C_3 be a unique cycle with a length of three in graph G, $v_0 \in V(G)$ with $|N_G(v_0)| \geq 4$, $G_i = (V_i, E_i)$ be a component of $G - v_0$ $(1 \leq i \leq j_0, 3 \leq j_0 \leq |N_G(v_0)|)$, v_0v_1, v_0v_2 be cut edges with $v_1 \in V_1$ and $v_2 \in V_2$, $\min\{c(G[V_j \cup \{v_0\}]) : j = 1,2,3\} \geq \max\{c(G[V_h \cup \{v_0\}]) : 4 \leq h \leq j_0\}$, and $\bar{G}_3 = \bigcup_{i=3}^{j_0} G[V_i \cup \{v_0\}]$. Then,*
(i) if $G_i \neq K_{1,4}$ for each $1 \leq i \leq j_0$, then define $\bar{G}_j = G[V_j \cup \{v_0\}] \cup G[E_0]$ for $j = 1,2$;
(ii) if $G_1 = K_{1,4}$, then define $\bar{G}_1 = G[V_1 \cup \{v_0\}] = K_{1,5}$, $\bar{G}_2 = G[V_2 \cup \{v_0\}] \cup G[E_0]$;
(iii) if $G_1 = K_{1,4}, G_2 = K_{1,4}$, then define $\bar{G}_j = G[V_j \cup \{v_0\}] = K_{1,5}$ for $j = 1,2$,
where $E_0 \neq \emptyset$ is an edge subset of $E(\bar{G}_3)$ but $E_0 \cap E(G_3) = \emptyset$.

In Definition 3, if $d_{\bar{G}_j}(v) = 2$ for some vertex $v \in V(\bar{G}_j)$, and $\bar{G}_j - v + x_1^j x_2^j$ is 3-cutwidth critical, then we also say that $\bar{G}_j$ is 3-cutwidth critical below, where $x_1^j, x_2^j \in N_{\bar{G}_j}(v)$. For examples, for Case (i), let $G = \tau_{31}$ with $C_3 = v_{15}v_{16}v_{17}v_{15}$ and $v_0 = v_{14}$ in Figure 3, $G_j = (V_j, E_j)$ be a component of $G - v_{14}$ and $G_j \neq K_{1,4}$ $(1 \leq j \leq 4)$, where $V_1 = \{v_i : 1 \leq i \leq 7\}, V_2 = \{v_i : 18 \leq i \leq 24\}, V_3 = \{v_{11}, v_{12}, v_{13}, v_{15}, v_{16}, v_{17}\}, V_4 = \{v_8, v_9, v_{10}\}$ and $\bar{G}_3 = G[V_3 \cup \{v_{14}\}] \cup G[V_4 \cup \{v_{14}\}] = G[\{v_i : 8 \leq i \leq 17\}]$ with an edge subset $E_0 = \{v_9v_8, v_9v_{10}\}$. Thus, $\bar{G}_1 = G[V_1 \cup \{v_{14}\}] \cup G[E_0] = G[V_1 \cup V_4 \cup \{v_{14}\}]$ and $\bar{G}_2 = G[V_2 \cup \{v_{14}\}] \cup G[E_0] = G[V_2 \cup V_4 \cup \{v_{14}\}]$. Likewise, for Cases (ii) and (iii), we can let $G = \tau_{30}$ and τ_{29}, respectively.

Lemma 13. *With the notation in Definition 3, if $\bar{G}_j$ is 3-cutwidth critical for each $j \in \mathcal{S}_3$, then G is 4-cutwidth critical.*

Proof. Without loss of generality, let G_1, G_2, G_3 satisfy (i) and $C_3 \subset \bar{G}_3$ by assumption. Then $\bar{G}_1, \bar{G}_2$ are subtrees in G. Due to the fact that $\bar{G}_j$ is 3-cutwidth critical for each $j \in \mathcal{S}_3$, $\bar{G}_3 \in \{H_3, H_4\}$ and $\bar{G}_1, \bar{G}_2 \in \{H_1, H_2\}$, it can be concluded that $G \in \{\tau_{16}, \tau_{31}, \tau_{34}, \tau_{36}, \tau_{40}\}$ via direct computation. So, G is 4-cutwidth critical by Lemma 5. Similarly, for Case (ii), $G \in \{\tau_{30}, \tau_{33}, \tau_{37}, \tau_{39}\}$; and for Case (iii), $G \in \{\tau_{29}, \tau_{32}, \tau_{35}, \tau_{38}\}$. So, the Lemma holds. □

Lemma 14. *With the notation in Definition 3, G is 4-cutwidth critical if and only if $G \in \{\tau_{16}, \tau_{29}, \tau_{30}, ..., \tau_{40}\}$.*

Proof. It suffices to show its necessity by Lemma 5. As the arguments are similar, we only consider the case that G_1, G_2, G_3 satisfy (i) of Definition 3. Furthermore, without loss of generality, let cycle $C_3 \subset \bar{G}_3$ by assumption, then $\bar{G}_1, \bar{G}_2$ are subtrees in G.

Claim 3. *For each $j \in \mathcal{S}_3$, $\bar{G}_j$ is 3-cutwidth critical.*

In fact, if there is some $j_0 \in \mathcal{S}_3$, say $j_0 = 3$, such that $\bar{G}_3$ is not 3-cutwidth critical, then two cases need to be considered: (1) there are at least an edge $e \in E(\bar{G}_3)$ such that $c(\bar{G}_3 - e) \geq 3$; (2) $c(\bar{G}_3) \leq 2$. By assumption that G is 4-cutwidth critical, we can see that Case (1) is impossible by Lemma 13. Hence, it suffices to verify that Case (2) is also impossible. As $\bar{G}_1$ and $\bar{G}_2$ are 3-cutwidth critical, $c(G_1) \leq 2$ and $c(G_2) \leq 2$. Let f_1, f_2, f_3 be the optimal labelings of $G_1, \bar{G}_2$, and G_3, respectively. Then, similar to that of (5) in Definition 2, we can obtain a 3-cutwidth labeling $f : V(G) \mapsto \mathcal{S}_{|V(G)|}$ by the order

(f_1, f_2, f_3) with $c(G, f) = 3$, which implies $c(G) \le 3$, contrary to $c(G) = 4$. Similarily, for $j_0 \in \{1,2\}$, if $\bar{G}_{j_0}$ is not 3-cutwidth critical, then a similar contradiction can also be obtained. So, Case (2) is not possible and Claim 3 holds.

Thus, by Claim 3 and Lemma 13, $G \in \{\tau_{16}, \tau_{29}, \tau_{30}, ..., \tau_{40}\}$. The proof is completed. □

Lemma 15. *Let $t = 4$ in C_t. Then, G is 4-cutwidth critical if and only if $G \in \{\tau_{41}, \tau_{42}, ..., \tau_{45}\}$.*

Proof. By Lemma 5, it suffices to show its necessity. By Lemma 6, $T_i \in \mathcal{F}_1 \cup \mathcal{F}_2$ for each $v_i \in V(C_4)$. So, two cases need to be considered.

Case 1 $T_i \in \mathcal{F}_2$. In Lemma 6, we already showed that G is 4-cutwidth critical if and only if $G \in \{\tau_{41}, \tau_{42}, \tau_{43}\}$ in this case, omitted here.

Case 2 $T_i \in \mathcal{F}_1$. By (6), for each $v_i \in V(C_4)$, $d_G(v_i) \ge 3$ in G.

Claim 4. *There is a unique vertex, say v_1, such that $d_G(v_1) \ge 4$ in G.*

First, let $d_G(v_i) = 3$ for each $v_i \in V(C_4)$. In this case, there are at least three subtrees, say T_1, T_2, T_4, such that T_1, T_2, T_4 are all in $\{K_{1,5}, F_1\}$, which leads to the conclusion that one of $\{\tau_{11}, \tau_{12}, \tau_{13}\}$ is homeomorphic to a subgraph of G, which contradicts the minimality of G. Otherwise, by the fact that the cutwidth of each member of $\mathcal{F}_1$ is at most two, we can verify that $c(G) < 4$, contrary to $c(G) = 4$. In fact, for an optimal labeling f of G with $f(v_1) = \min\{f(v_i) : v_i \in V(C_t)\}$ and $f(v_4) = \max\{f(v_i) : v_i \in V(C_4)\}$, let $f(x) = \max\{f(v) : v \in V(T_1 - v_1)\}$, $f(y) = \min\{f(v) : v \in V(T_4 - v_4)\}$. Under f, we first embed T_1 into the interval $[1, f(v_1)]$ with a congestion of three and T_4 into the interval $[f(v_4), n]$ with a congestion of three, resulting in $T_1, T_4 \in \{K_{1,5}, F_1\}$. If $T_2, T_3 \in \{K_2, K_{1,3}, F_2\}$, then we can conclude that $c(G) = 3$ by embedding T_2 into the interval $(f(x), f(v_1))$ with a congestion of three and T_3 into the interval $(f(v_4), f(y))$ with a congestion of three. This is a contradiction, which leads to the conclusion that one of $\{T_2, T_3\}$ is in $\{K_{1,5}, F_1\}$.

Second, let $d_G(v_2) \ge 4$, i.e., $T_2 = F_2$ or $K_{1,3}$ centered at v_2. If $T_3 = K_2$ and $T_4 = K_2$, then $c(G) = 3$, contradicting $c(G) = 4$. So, at least one of $\{T_3, T_4\}$ is a $K_{1,3}$. However, in this case, one of $\{\tau_2, \tau_3, \tau_4\}$ is homeomorphic to a subgraph of G, contrary to the minimality of G. So, Claim 4 holds.

Because $\tau_{41}, \tau_{42}, \tau_{43}$ are 4-cutwidth critical, there is at least a subtree T_i $(i = 2,3,4)$ such that $T_i \ne K_2$. In addition, $K_{1,3} \subset K_{1,5}$ and $F_2 \subset F_1$. So, by Claim 4 and the minimality of G, G must be among the six graphs in Figure 8. From direct computation, only graphs (c) and (f) are 4-cutwidth critical, which are τ_{44} and τ_{45} in Figure 8, respectively. Thus, $G \in \{\tau_{41}, \tau_{42}, \tau_{43}, \tau_{44}, \tau_{45}\}$. □

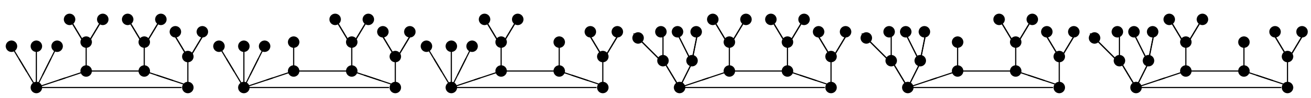

Figure 8. (**a–f**) Six possible graphs on G with the proof of Lemma 15.

Lemma 16. *Let $t = 5$ in C_t. Then, G is 4-cutwidth critical if and only if $G \in \{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}$.*

Proof. By Lemma 5, it suffices to prove its necessity. Because G is 4-cutwidth critical and $t = 5$, by Lemmas 6 and 15, $T_i \in \mathcal{F}_1$ for each $i \in \mathcal{S}_5$.

Claim 5. $d_G(v_i) = 3$ *for each $v_i \in V(C_5)$, and at most two subtrees T_{i_1} and T_{i_2} are in $\{K_{1,5}, F_1\}$.*

In fact, if there exists at least one vertex, say v_1, such that $d_G(v_1) \ge 4$, then, with an argument similar to that of Lemma 15, we can verify that one of $\{\tau_1, \tau_2, ..., \tau_{10}, \tau_{44}, \tau_{45}\}$ is homeomorphic to a subgraph of G, which is a contradiction. If there is another $T_{i_3} \in$

$\{K_{1,5}, F_1\}$, then one of $\{\tau_{11}, \tau_{12}, \tau_{13}\}$ is homeomorphic to a subgraph of G, which is another contradiction. Claim 5 holds.

By the minimality of G, if each of $\{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}$ is homeomorphic to a subgraph of G, then G does not need to be considered. Similarly, if G is not homeomorphic to any of $\{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}$, then any homeomorphic subgraph of G is not also considered by (4). Thus, except for $\tau_{46}, \tau_{47}, \tau_{48}$, and τ_{49}, it is possible that G is among the five graphs in Figure 9 by Claim 5. However, by direct computations, $c(G) = 3$ for each graph G in Figure 9, contrary to $c(G) = 4$. So, $G \in \{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}$. $\square$

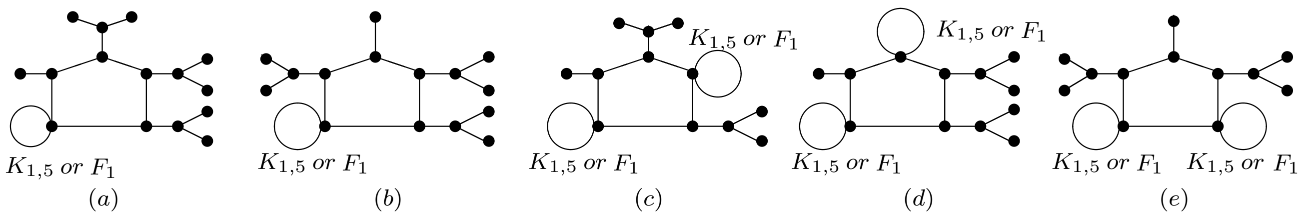

Figure 9. (**a**–**e**) Possible graphs on G with the proof of Lemma 16.

Lemma 17. *Let $t = 6$ in C_t. Then, G is 4-cutwidth critical if and only if $G = \tau_{50}$.*

Proof. By Lemma 5, it suffices to prove its necessity. Similar to those of Lemmas 15 and 16, we can verify that $d_G(v_i) = 3$ for each $i \in \mathcal{S}_6$ and $T_{i_1}, T_{i_2}, T_{i_3} \in \{K_{1,3}\}$, $T_{i_4}, T_{i_5}, T_{i_6} \in \{K_2\}$. Thus, G is among the following three graphs in Figure 10. By direct computations, we can see that only graph (c) is 4-cutwidth critical, and $(c) = \tau_{50}$. So, $G = \tau_{50}$ and the Lemma holds. $\square$

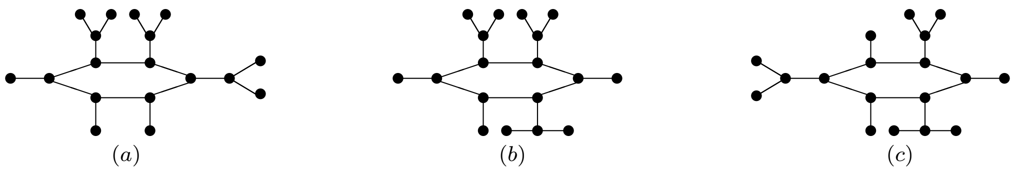

Figure 10. (**a**–**c**) Three possible graphs with the proof of Lemma 17.

Proof of Theorem 1. By Lemmas 5, 8–12, and 14–17, the desired result holds. $\square$

4. Remarks

In this paper, fifty critical unicyclic graphs with a cutwidth of four were obtained, during which a decomposable property of some 4-cutwidth critical unicyclic graphs was also obtained (see Lemma 12 and Definition 3). For an integer $k \geq 4$, although it seems to be difficult to find all k-cutwidth critical graphs, some structural properties of some of them can be found definitively. In fact, as the decomposability of k-cutwidth critical trees [25] and some special non-tree graphs with uncomplicated structure [26], a similar decomposable property of 4-cutwidth critical unicyclic graphs, which is contained in Lemma 12 and Definition 3, can be generalized to k-cutwidth critical graphs even if these graphs are multicyclic graphs. For instance, in Lemma 12, if any element of $\{G_1, G_2, G_3\}$ is a critical unicyclic graph with a cutwidth of $k-1$, then we can verify that $K_{1,3} \circ (G'_1, G'_2, G'_3)$ is a critical unicyclic graph with a cutwidth of k. Clearly, if $v_j \notin V_p^{(j)}$ with $1 \leq j \leq 3$, then $\{G'_j + u_0 v_j : 1 \leq j \leq 3\}$ is a decomposition of $K_{1,3} \circ (G'_1, G'_2, G'_3)$. Regarding the critical multicyclic graphs with a cutwidth of at least four, their general structural properties have yet to be known. Additionally, the application of critical unicyclic graphs with a cutwidth of four to some realistic fields, such as social and biological networks, multivariate cryptography, and other fields, is worth studying. These will be the objects for further study in future works.

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