On Critical Unicyclic Graphs with Cutwidth Four

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Abstract: The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph $G$ on a line $P_n$ with $n = |V(G)|$ vertices in such a way that the maximum number of overlapping edges (i.e., the congestion) is minimized. A graph $G$ with a cutwidth of $k$ is $k$-cutwidth critical if every proper subgraph of $G$ has a cutwidth less than $k$ and $G$ is homeomorphically minimal. In this paper, we first verified some structural properties of $k$-cutwidth critical unicyclic graphs with $k > 1$. We then mainly investigated the critical unicyclic graph set $T$ with a cutwidth of four that contains fifty elements, and obtained a forbidden subgraph characterization of 3-cutwidth unicyclic graphs.

Keywords: graph labeling; cutwidth; critical graph; unicyclic graph

MSC: 05C75; 05C78; 90C27

1. Introduction

All graphs in this paper are finite, simple, and connected, with undefined notation following [1]. The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph $G$ on a path $P_n$ with $n = |V(G)|$ vertices in such a way that the maximum number of overlapping edges (i.e., the congestion) is minimized. As one of the most well-known optimization problems, it is also known as the minimum cut linear arrangement (or linear layout, optimal embedding, optimal labeling, etc.) problem [2]. Cutwidth has been extensively examined [2]. Computing cutwidth for general graphs is an $NP$-complete problem except for trees [3–6], and it remains $NP$-complete even if the input graph $G$ is restricted to planar graphs with a maximum degree of three [7]. Hence, a number of studies have focused on polynomial-time approximation algorithms for general graphs and polynomial-time algorithms for some special graphs to solve their cutwidth [2,8]. Relatively little work has been conducted on detecting special graph classes whose cutwidths can be computed polynomially [2] and critical graph classes with cutwidths of $k \geq 1$. Let $T_k(*k)$ be the set of critical graphs with the graph parameter $* = k$. From [9], $|T_1(c(G))| = 1$, $|T_2(c(G))| = 2$, $|T_3(c(G))| = 5$ (see Figure 1). For critical graphs with cutwidth $k \geq 4$, $|T_k(c(G))|$ has been unknown except that $|T_4(c(T))| = 18$, as reported by [10], where $T$ is a tree (see Figure 2). Similar studies have been conducted for the treewidth, pathwidth, and branchwidth of a graph $G$ (abbreviated by $tw(G)$, $pw(G)$, and $bw(G)$, respectively). A graph $G$ is said to be $k$-treewidth (pathwidth, branchwidth) critical if $tw(G)$ ($pw(G)$, $bw(G)$) = $k$ but $tw(G')$ ($pw(G')$, $bw(G')$) < $k$ for any minor $G'$ of $G$. From [11–13], $|T_3(tw(G))| = |T_3(bw(G))| = 1$, $|T_4(tw(G))| = |T_4(bw(G))| = 4$, $|T_3(pw(G))| = 110$. As shown in [14], the critical graphs for parameters with a similar nature are worthy of further study, and the number of these critical graphs for a given value of the parameter would be finite and have yet to be characterized. The cutwidth problem for graphs and a class of optimal embedding (or layout) problems have significant applications in VLSI layouts [15,16], network reliability [17], automatic graph drawing [18], information retrieval [19], urban drainage network design [20], and other domains.
particular, the cutwidth is related to a basic parameter, called the congestion, in designing microchip circuits [2,21,22]. Herein, a graph $G$ may be viewed as a model of the wiring diagram of an electronic circuit with the vertices representing components and the edges representing wires connecting them. When a circuit is laid out in a certain architecture (i.e., a path $P_n$), the maximum number of overlap wires is the congestion, which is one of major parameters in the determination of electronic performance. This motivates the enthusiasm for studying the cutwidth problem in graph theory practically. Theoretically, it appears to be closely related with other graph parameters such as pathwidth, treewidth, linear width, bandwidth, and modified bandwidth [8,23,24], among others. For instance, for any graph $G$ with vertices of degree bounded by an integer $d \geq 1$, $pw(G) \leq c(G) \leq d \cdot pw(G)$, where $c(G)$ and $pw(G)$ are cutwidth value and pathwidth value, respectively. In this paper, we mainly study the critical unicyclic graph set $T$ with a cutwidth of four that contains fifty elements.

For an integer $n > 0$, define $S_n = \{1, 2, ..., n\}$. A labeling of a graph $G = (V(G), E(G))$ with $|V(G)| = n$ is a bijection $f : V(G) \to S_n$, viewed as an embedding of $G$ into a path $P_n$ with vertices in $S_n$, where consecutive integers are the adjacent vertices. The cutwidth of $G$ with respect to $f$ is

$$c(G, f) = \max_{1 \leq j < n} |\{uv \in E(G) : f(u) \leq j < f(v)\}|,$$

where the minimum is taken over all labelings $f$. If $k = c(G, f)$, then $f$, and the embedding induced by $f$ is called a $k$-cutwidth embedding of $G$. A labeling $f$ attaining the minimum in (2) is an optimal labeling. For each $i$ with $i \in S_n$, let $u_i = f^{-1}(i)$ and $S_i = \{u_1, u_2, ..., u_i\}$. Define $\nabla f(S_i) = \{u_iu_j \in E : i \leq j < h\}$ is called the (edge) cut at $[j, j+1]$ with respect to $f$. From (2), we then have

$$c(G, f) = \max_{1 \leq j < n} |\nabla f(S_j)|.$$

An $f$-max cut of $G$ is a $\nabla f(S_j)$, achieving the maximum in (3).

For a graph $G$ and integer $i \geq 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, where $d_G(v)$ is the degree of vertex $v \in V(G)$. Any vertex in $D_1(G)$ is called a pendant vertex in $G$. For each $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For $V' \subset V(G)$ and $V' \neq \emptyset$, $G[V']$ is the subgraph of $G$ induced by $V'$. If $H, H'$ are subgraphs of $G$ and $X \subseteq E(G)$, then $G[X]$ is the subgraph of $G$ induced by $X$, $H \cup H' = G[E(H) \cup E(H')]$ and $H \cup X = G[E(H) \cup X]$. Specially, if $X = \{e\}$, then we write $G + e$ instead of $G \cup \{e\}$. If $G$ has a vertex $v \in D_2(G)$ with $N_G(v) = \{v_1, v_2\}$ and $v_1v_2 \notin E(G)$, then $G - v = G + v_1v_2$ and the graph obtained from $G - v$ by adding a new edge $v_1v_2$, is called a series reduction of $G$. A graph $G$ is homeomorphically minimal if $G$ does not have any series reductions. Two graphs $G'$ and $G''$ are homeomorphic if both of them can be obtained from the same graph $G$ by inserting new vertices of degree two into its edges. A graph $G$ is said to be $k$-cutwidth critical if $G$ is homeomorphically minimal with $c(G) = k$ such that every proper subgraph $G'$ of $G$ satisfies $c(G') < k$. The basic properties of cutwidth follow immediately from this definition.

![Figure 1](image-url)  
Figure 1. The 3-cutwidth critical graphs.
Figure 2. All elements of the 4-cutwidth critical tree set $T_4$.

**Lemma 1.** For graphs $G$ and $G'$, each of the following holds. (1) If $G'$ is a subgraph of $G$, then $c(G') \leq c(G)$. (2) If $G'$ is homeomorphic to $G$, then $c(G') = c(G)$.

**Lemma 2.** The unique 1-cutwidth critical graph is $K_2$. The only 2-cutwidth critical graphs are $K_3$ and $K_{1,3}$. All 3-cutwidth critical graphs are $H_1, H_2, H_3, H_4,$ and $H_5$ in Figure 1.

**Lemma 3** ([10]). A tree $T$ is 4-cutwidth critical if and only if $G \in T_4$, where $T_4 = \{ \tau_i : 1 \leq i \leq 18 \}$, as depicted in Figure 2.

A connected graph $G$ with $|E(G)| = |V(G)|$ is called a unicyclic graph. The purpose of this paper is to characterize critical unicyclic graphs with a cutwidth of four and to present a forbidden subgraph characterization for unicyclic graphs with a cutwidth of three. Let $T = \{ \tau_i : 1 \leq i \leq 50 \}$ be the collection of the critical unicyclic graphs depicted in Figure 3. The main results of this paper are the following:

**Theorem 1.** A unicyclic graph $G$ is 4-cutwidth critical if and only if $G \in T$.

**Corollary 1.** A unicyclic graph $G$ has a cutwidth of three if and only if it does not contain any subgraph homeomorphic to any member in $T$.

The rest of this paper is as follows. Section 2 presents some preliminary results. The proof of Theorem 1 is given in Section 3 by a series of lemmas. We give a short remark in Section 4.
Figure 3. Cont.
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2. Preliminary Results

Throughout this section, for any integer \( n > 1 \), we always use \( P_n \) to denote the path with \( V(P_n) = S_n \) such that for all \( 1 \leq i < n, i \) and \( i + 1 \) are adjacent vertices in \( P_n \). In addition, because \( K_{1,2k-1} \) is \( k \)-cutwidth critical, as demonstrated by [10], we can let \( d_G(v) \leq 2k - 2 \) for each \( v \in V(G) \) in this paper.

The following observation is immediate from Lemma 1:

\[
\text{if } v \in V(G), \text{ then } c(G - v) \leq c(G). \tag{4}
\]

Definition 1.

(i) Let \( r \geq 0 \) be an integer, and \( v \) be a vertex of graph \( G \) with \( d_G(v) > r \). For \( v_1, v_2, \ldots, v_r \in N_G(v) \), define \( G(v; v_1, v_2, \ldots, v_r) \) as the component of \( G - \{v_1, v_2, \ldots, v_r\} \) that contains \( v \) (see an illustration in Figure 4a).

(ii) Let \( G, H \) be two disjoint graphs with \( u \in V(G) \) and \( v \in V(H) \). To identify \( u \) and \( v \), denoted as \( G \oplus_{u,v} H \), is to replace \( u, v \) with a single vertex \( z \) (i.e., \( u = v = z \)) incident to all the edges which are incident to \( u \) and \( v \), where \( z \) is called the identified vertex.

(iii) Let \( G_1, G_2 \) and \( G_3 \) be disjoint graphs \( D_3(K_{1,3}) = \{u_0\} \) and \( D_1(K_{1,3}) = \{u_1, u_2, u_3\} \). For each \( j \in S_3 \), pick \( v_j \in V(G_j) \). Define \( K_{1,3} \circ (G_1, G_2, G_3) \) as the graph obtained from the disjoint union \( G_1, G_2, G_3 \) and \( K_{1,3} \) by identifying \( u_j \) with \( v_j \) (again denoted as \( v_j \)) for each \( j \in S_3 \) (see Figure 4b).

Figure 3. All 4-cutwidth critical unicyclic graphs with optimal labelings.
(iv) If $|V(G)| \geq 2$, then define $\mathcal{M}(G) = \{G - uv : uv \in E(G) \text{ and } uv \text{ is not a cut edge}\} \cup \{G - v : v \in D_1(G)\}$ to be the family of all proper maximal subgraphs of $G$.

**Definition 2.** For a graph $G$ with $|V(G)| = n$, suppose that $v_0 \in V(G)$ is a vertex with $N_G(v_0) = \{v_1, v_2, \ldots, v_p\}$, $v_0v_1$ and $v_0v_2$ are two cut edges of $G$, $H'_1 = G(v_0, v_2, v_3, \ldots, v_p) - v_0$, $H'_2 = G(v_0, v_1, v_2)$ and $H'_3 = G(v_0, v_1, v_3, \ldots, v_p) - v_0$. For $1 \leq i \leq 3$, let $f_i : V(H'_i) \rightarrow S_{|V(H'_i)|}$ be an optimal labeling of $H'_i$, and let a labeling $f : V(G) \rightarrow S_n$ of $G$ be as follows: for $v \in V(G)$,

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(H'_1), \\ f_2(v) + |V(H'_2)| & \text{if } v \in V(H'_2), \\ f_3(v) + |V(G_1)| + |V(G_2)| + 1 & \text{if } v \in V(G_3). \end{cases}$$

Then the labeling $f$ is called a labeling by the order $(f_1, f_2, f_3)$ or $(V(H'_1), V(H'_2), V(H'_3))$ of $G$. For example, let $G = K_{1,3} \circ (G_1, G_2, G_3)$ with $v_0 = u_0$ in Figure 4b; then, $N_G(u_0) = \{v_1, v_2, v_3\}$, $H'_1 = G(u_0, v_2, v_3) - u_0 = G_1, H'_2 = G(u_0, v_1, v_3) = G_2 + u_0v_2$, and $H'_3 = G(u_0, v_1, v_2) - u_0 = G_3$. If $f_1$ is an optimal labeling of $H'_1$ for $1 \leq i \leq 3$, and for $v \in V(G)$, define

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1), \\ f_2(v) + |V(G_1)| & \text{if } v \in V(G_2) \cup \{v_0\}, \\ f_3(v) + |V(G_1)| + |V(G_2)| + 1 & \text{if } v \in V(G_3). \end{cases}$$

then, $f$ is a labeling of the order $(f_1, f_2, f_3)$ of $G$.

![Figure 4.](image)

**Theorem 2.** For any $v \in D_{\geq 3}(G)$, if there always are two vertices $v_1, v_2$ in $N_G(v)$ such that $ww_1, vv_2$ are cut edges in $G$, then $c(G) \leq k$ if and only if $c(G; v_1, v_2) \leq k - 1$.

**Proof.** We first provide a claim.

**Claim 1.** Let $v'_1, v'_2$ be a cut edge in $G$ and $V_1, V_2$ be the vertex sets of two components of $G - v'_1, v'_2$. Then, there exists an optimal labeling $f^*$ such that the vertices in each of $V_1$ and $V_2$ are labeled consecutively.

In fact, if $f$ is an optimal labeling of $G$ with $f(v'_1) < f(v'_2)$, then we can construct a labeling $f^*$ as follows. First, label the vertices of $V_1$ in the same order as $f$, and then label the vertices of $V_2$ in the same order as $f$. Because the edges in $G[V_1]$ and those in $G[V_2]$ are not overlapped, it follows that $c(G, f^*) \leq c(G, f)$. Thus, $f^*$ is also an optimal labeling of $G$.

Now, by using this observation, we proceed to prove Theorem 2. From the assumption that $wv_1, vv_2$ are cut edges of $G$, let $V_0, V_1, V_2$ be the vertex sets of three components of $G - \{v_0, v_1, v_2\}$, where $v_0 \in V_0, v_1 \in V_1, v_2 \in V_2$. Then, there exists an optimal labeling $f^*$ such that each of the vertices of $V_0, V_1, V_2$ are labeled consecutively. If $c(G, f^*) \leq k$, then, because the edges $vv_1$ and $vv_2$ give a congestion of one to $G[V_0]$, we have $c(G[V_0], f^*) \leq k - 1$. Thus, $c(G) \leq k$ implies $c(G; v_1, v_2) \leq k - 1$. Conversely, if $c(G[V_0], f^*) \leq k - 1$ and $V_1, V_2$ contain no vertices in $D_{\geq 3}(G)$, then $G[V_1]$ and $G[V_2]$ are two paths and so have a congestion of one. It follows that $c(G, f^*) \leq k$. If $V_1$ (or $V_2$) contains a vertex $v' \in D_{\geq 3}(G)$,
then there must be two cut edges \( v'v'_1, v'v'_2 \) in \( G \) by assumption. In this way, \( V(G) \) can be further decomposed into a sequence \( V_1, V_2, ..., V_r \) such that \( G[V_i] \) and \( G[V_{i+1}] \) are connected by a cut edge \( (1 \leq i < r) \). From \( c(G(v; v_i, v_j)) \leq k-1, c(G[V_i], f^i) \leq k-1 \) for \( 1 \leq i \leq r \). Hence, \( c(G, f^k) \leq k \), resulting in \( c(G) \leq k \). 

**Corollary 2.** With the notation in Theorem 2, for graph \( G \), if there exists a vertex \( v \in D_{\geq 3}(G) \) such that \( c(G(v; v_i, v_j)) \geq k-1 \) holds for any \( v_i, v_j \in N_G(v) \), then \( c(G) \geq k \), where \( v_{i0}, v_{i2} \) are cut edges in \( G \).

**Lemma 4.** Let graph \( G \) be \( k \)-cutwidth critical with \( D_1(G) \neq \emptyset \), and \( P_l = u_{01}u_{12}...u_l \) be a path with length \( 1 \). Then, \( c(G \oplus v_{00}, P_l) = c(G) = k \).

**Proof.** Let \( v_{02} \) be a pendant edge of \( G \), where \( z \) is a pendant vertex. We subdivide the edge \( v_{0z} \) into a path \( P \) with length \( l \) and denote the resulting graph with \( G' \). Then, by Lemma 1, \( c(G \oplus v_{00}, P_l) = c(G') = c(G) = k \).

**Theorem 3.** With the notation of Definition 1 (iii), let at least one of \( \{G_1, G_2, G_3\} \), say \( G_2 \), be \((k-1)\)-cutwidth critical with \( D_1(G_2) \neq \emptyset \). Then \( c(K_{1,3} \circ (G_1, G_2, G_3)) = k \).

**Proof.** Let \( G = K_{1,3} \circ (G_1, G_2, G_3) \). If \( d_G(v_i) = 2 \) for \( j \in S_3 \); then, the series reduction can be implemented without affecting \( c(G) = k \). Because \( u_0u_2 \) is a pendent edge of the subgraph \( G_2 \oplus u_2, u_3 \) and \( G_3 \) is \((k-1)\)-cutwidth critical with \( D_1(G_3) \neq \emptyset \). By Corollary 2 because \( c(G(u_0; v_i, v_j)) = k-1 \) for any \( v_i, v_j \in N_G(u_0) \). Thus, \( c(G) = k \), and the proof is complete.

**Corollary 3.** With the notation of Definition 1 (iii), for each \( j \in S_3 \), if \( G_j \) is \((k-1)\)-cutwidth critical with \( v_i \in D_1(G_j) \), then \( K_{1,3} \circ (G_1, G_2, G_3) \) is \( k \)-cutwidth critical.

**Proof.** Let \( G = K_{1,3} \circ (G_1, G_2, G_3) \). Because \( d_G(v_i) = 2 \) for each \( j \in S_3 \), three series reductions are carried out first. Furthermore, we still let \( N_G(u_0) = \{v_1, v_2, v_3\} \) for convenience.

First, by assumption and Theorem 3, \( c(G) = k \).

Second, we prove that \( G \) is \( k \)-cutwidth critical. It remains to be shown that, for any \( G' \in \mathcal{M}(G) \), \( c(G') \leq k-1 \). Because any \( G' \) is obtained by deleting a pendent edge \( xy \) or a nonpendent edge \( xy \in E(G) \) in \( G \), \( xy \notin \{u_0v_1, u_0v_2, u_0v_3\} \). Without loss of generality, let \( xy \in E(G_2) \). By assumption that \( G_j \) is \((k-1)\)-cutwidth critical for each \( j \in S_3 \), we have \( c(G_1 - u_0v_1) \leq k-2, c(G_2 - xy) \leq k-2 \) and \( c(G_3 - u_0v_3) \leq k-2 \). Thus, similar to (5), a labeling \( \omega : V(G') \rightarrow S_{|V(G')|} \) by the order \( V(G_1 - u_0v_1), V(G_2 - xy), V(G_3 - u_0v_3) \) is obtained and \( c(G', \omega') = k-1 \). So, \( c(G') \leq k-1 \) by (2), and \( G \) is \( k \)-cutwidth critical.

3. **Proof of Theorem 1**

We verify our main results by using a series of lemmas. Throughout this section, \( G \) denotes an uniciclic graph and \( C_t = v_1v_2...v_tv_1 \) denotes the unique cycle of \( G \) with \( t \geq 3 \). Furthermore, we have a convention that a graph \( H \) is designated to be homeomorphic to a subgraph of \( G \) if \( H \) can be obtained by deleting vertices or edges and some series reductions of \( G \) and \( c(H) = c(G) \). Because a cutwidth critical graph is homeomorphically minimal, if \( G \) is \( k \)-cutwidth critical then

\[
d_G(v_i) \geq 3 \text{ for } v_i \in V(C_t),
\]

unless \( v_i \) is a special vertex.
Let $x_0, w_0 \in D_3(H_2), y_0, z_0 \in D_1(H_2), u_0 \in D_1(K_{1,5})$ (see $H_2$ in Figure 1). Furthermore, let $F_1 = H_2 - y_0 - z_0, F_2 = H_2 - \{x_0, y_0, z_0\}, \mathcal{F}_1 = \{K_{1,5}, K_{1,3}, K_{1,5}, F_1, F_2\}$ (see $F_1, F_2$ in Figure 5a), and $\mathcal{F}_2 = \{K_{1,5} \oplus u_0 = 0, K_{1,5}' \oplus u_0 = 0, F_1, F_2 \}$ (see Figure 5b), where $K_{1,5}, F_1$ are copies of $K_{1,5}$ and $F_1, u'_0 \in D_1(K_{1,5})$, $x'_0 \in D_3(F'_1)$ are copies of $u_0$ and $x_0$, respectively. For an integer $p > 1$, we call a star $K_{1,p}$ centered at vertex $x$ if $d_{K_{1,p}}(x) = p$.

![Graphs F1 and F2](image1)

(a) Graphs $F_1$ and $F_2$

![All elements of F2](image2)

(b) All elements of $\mathcal{F}_2$

Figure 5. Two elements of $\mathcal{F}_1$ and set $\mathcal{F}_2$.

**Lemma 5.** Each member of set $\mathcal{T}$ is 4-cutwidth critical in Figure 3.

**Proof.** For a unicyclic graph $G$, let $C = v_1, v_2, \ldots, v_t$ be the unique cycle of $G$. Then, $G - E(C)$ is a forest of $t$ subtrees $T_1, T_2, \ldots, T_t$ where $T_t$ is called the $v_t$-branch leading from $v_t$. We first consider the case of $t = 3$ in which $G - E(C)$ has three subtrees $T_1, T_2, T_3$. For an optimal labeling $f$ of $G$, suppose that $f(v_1) < f(v_2) < f(v_3)$; then, the number set $S_n$ is divided into three intervals $I_1 = \{f(v_1), f(v_2)\}, I_2 = \{f(v_1), f(v_3)\}, I_3 = \{f(v_3), n\}$. Subtrees $T_1, T_2, T_3$ are then embedded into $I_1, I_2, I_3$ in different manners. We have the following classifications of 4-cutwidth critical unicyclic graphs.

1. Type 3A (including $v_1$ to $v_t$): $T_1$ is embedded in $I_1$ with a congestion of three, $T_2$ is embedded in $I_2$ with a congestion of four, and $T_3$ is embedded in $I_3$ with a congestion of three. Herein, $T_1$ and $T_3$ are the star $K_{1,3}$ with center $v_1$ or the two stars $K_{1,3}$ with an identifying leaf at $v_i (i=1,3)$ (see $F_2$ in Figure 5a). Let $T_1$ and $T_3$ combine with the two edges in $C_3$ incident with $v_1$. Then, $T_1$ and $T_3$ are the 3-cutwidth critical tree $H_1 = K_{1,5}$ or the 3-cutwidth critical tree $H_2$ with a central edge (i.e., similar to $u_0, x_0$ in $H_2$) contracted. As to $T_2$ embedded in $I_2$ with a congestion of four, the cycle $C_3$ yields a congestion of two in this interval, and we have to choose $T_2$ as a 2-cutwidth critical tree, namely, a $K_{1,3}$ such that either $d_C(v_2) = 3$ or $d_C(v_2) = 5$. For this type of construction, the maximum congestion is four, that is, $c(G) = 4$. Furthermore, for any edge $e \in E(G)$, if $e \notin E(C)$, then the deletion of $e$ reduces the congestion two of cycle-edge in $I_2$ by one. Hence, $T_2$ embedded in $I_2$ has a congestion of three, and so $c(G - e) < 4$. If $e \notin E(C)$, for $v_t$ with $T_t = K_{1,3}$ and $d_C(v_t) = 5$ for each $1 \leq j \leq 3$, we can let $e \in E(T_2)$. Because $T_2 - e = K_{1,3} - e$ has a congestion of one, we can embed $T_1$ in $I_1$, $T_2 - e$ in $f(v_2), f(v_3)$, and $T_3$ in $f(v_3) - 1, n - 1$. Thus, $c(T_1 - e) = 3$. For $T_2, T_3$, and $T_4$, such that $d_C(v_2) = 3$, we can let $e \in E(T_1)$; then, $T_1 - e$ in $I_1$ has a congestion of two. Hence, we embed $T_2 - e$ in $I_1$ such that $I_1$ has a congestion of three (for example, $T_1$ is an $F_2$ at $v_1$ for $T_4$), and the same is true for the case of $e \in E(T_3)$. Hence, $c(G - e) < 4$. Thus, $G$ is 4-cutwidth critical.

2. Type 3B (including $v_5$ to $v_3$): $f(v_1), f(v_2), f(v_3)$ are consecutive and $I_2 = \{f(v_2)\}$, $T_1$ is embedded in $I_1$ with a congestion of three, $T_2$ is embedded in $I_2$ with a congestion of four, $T_3$ is embedded in $I_3$ with a congestion of three. Herein, we denote the subtree $H_2$ obtained by deleting two leaves in the same branch (say $y_0, z_0$ in Figure 1) with $F_1$, and denote the subtree of $H_2$ obtained by deleting three vertices in the same branch (say $y_0, y_0, z_0$ in Figure 1) with $F_2$ (see Figure 5a). Then, $T_1$ is a star $K_{1,3}, K_{1,3}, F_1, F_2$, or $T_3$ and $T_3$ is a star $K_{1,5}$. Note that if $T_1 = K_{1,3}$ then $T_1 = H_1$, if $T_2 = F_1$ then $T_3 = H_2$, where $H_1$ and $H_2$ are 3-cutwidth critical. Because $T_2$ and $T_3$ are embedded in $I_3$ consecutively and an edge of $T_2$ incident with $v_2$ strides over all edges of $T_3$, we see that the overlapped edges of $T_2$ and $T_3$ give rise to a congestion of four in the embedding. Hence, $c(G) = 4$. Furthermore, for any edge $e \in E(G)$, if $e \notin E(C)$, then $G - e$ is a tree made up with $H_1$ and $H_2$, which has a cutwidth of three. Thus, $c(G - e) < 4$. Otherwise, we may assume $e \in E(T_2)$ (we
may change the order of $T_1$, $T_2$, $T_3$ if necessary). Then, $c(T_2 - e) = 2$, and so the embedding of $T_2$ and $T_3$ in $I_3$ gives a congestion of three by making $\min\{f(v) : v \in V(T_2 - e - v_2)\} = f(v_2) + 1$ and $\max\{f(v) : v \in V(T_2 - e - v_2)\} = \min\{f(v) : v \in V(T_3 - v_3)\} = 1$. Thus, we have $c(G - e) < 4$. Hence, $G$ is 4-cutwidth critical.

(3) Type 3C (including $\tau_{34}, \tau_{35}$ and $\tau_{38}$ to $\tau_{40}$): $T_2$ and $T_3$ are decomposed and embedded into different intervals. For $\tau_{34}$ and $\tau_{35}$, $T_1$ is an $H_2$, and it is decomposed into two stars $K_{1,3}$ embedded in $I_1$ and one star $K_{1,3}$ embedded in $I_2$. The star $K_{1,3}$ in $I_2$ and the two cycle edges give rise to the congestion of four in $I_2$. For $\tau_{38}$, $T_1$ is decomposed into two stars $K_{1,3}$ and $K_2$, where a star $K_{1,3}$ and a $K_2$ are embedded in $I_1$ and $I_3$, respectively. $\tau_{39}$ and $\tau_{40}$ are similar. Similar to the previous cases, it can be shown that $G$ is 4-cutwidth critical.

(4) Type 3D (including $\tau_{36}$ to $\tau_{37}$): This type of unicyclic graphs are obtained from 4-cutwidth critical trees by making the following local transformations: the star $K_{1,3}$ is transformed into a triangle $K_3$ (for example, $H_2$ is transformed into $H_3$, see Figure 1) and the star $H_1 = K_{1,5}$ is transformed into a ‘sun’ $H_4$. Because these local transformations do not change the congestion of two of $K_{1,3}$ or the congestion of three of $K_{1,5}$, this part of the proof is based on Lemma 3. Let $T_i'$ be the 4-cutwidth critical trees in Lemma 3 (see Figure 2). Then, for $\tau_{36}$ to $\tau_{37}$ in Figure 3, we have the following correspondences: $\tau_{36}$ is from $\tau_{37}'$ to $\tau_{37}$, $\tau_{38}$ is from $\tau_{39}'$ to $\tau_{39}$, $\tau_{39}$ is from $\tau_{40}'$ to $\tau_{40}$, $\tau_{40}$ is from $\tau_{41}'$ to $\tau_{41}$, and $\tau_{41}$ is from $\tau_{42}'$ to $\tau_{42}$. Thus, each of $\{\tau_{36}, \tau_{37}, ..., \tau_{37}\}$ is 4-cutwidth critical.

For $i \geq 4$, we have the similar arguments as follows.

(5) Type 4A (including $\tau_{41}$ to $\tau_{43}$): Similar to above, for each $G \in \{\tau_{41}, \tau_{42}, \tau_{43}\}$, $c(G) = 4$. To show that $G$ is 4-cutwidth critical, we take $e \in E(G)$. If $e \in E(C_4)$, then $G - e$ is a tree made up with $T_1$ in $F_2$ and $T_2 = K_2$ for $2 \leq i \leq 4$, which has a cutwidth of three. Thus, $c(G - e) < 4$. If $e \in E(T_1)$, then $T_1 - e$ has a congestion of three, and so $c(G - e) < 4$. If $e \in E(T_2)$ with $T_2 = K_2$ for $2 \leq i \leq 4$, then $G - e$ is a proper subgraph of one of $\tau_{38}$ to $\tau_{40}$, and so, $c(G - e) < 4$.

(6) Type 4B (including $\tau_{44}$ to $\tau_{48}$): Similar to the previous cases, the cutwidth of $G \in \{\tau_{44}, \tau_{45}\}$ is four. Now let $e \in E(G)$. If $e \in E(C_4)$, then $G - e$ is a tree made up with $T_i \in \{K_2, K_{1,3}, F_2\}$ for $1 \leq i \leq 4$, which has a cutwidth of three. Thus, $c(G - e) < 4$. For $e \in E(T_i)$ with any $1 \leq i \leq 4$, we can always find a labeling $f'$ of $G - e$ such that $c(G - e, f') \leq 3$. So, $c(G - e) \leq 3$ leading to that $G$ is 4-cutwidth critical.

(7) Type 5A (including $\tau_{46}$ to $\tau_{48}$): Similar to that of Type 4A, omitted here.

(8) Type 5B (including $\tau_{49}$ only): The labeling $f$ of $\tau_{49}$ in Figure 3 implies that $c(\tau_{49}) \leq 4$. $\tau_{49} - E(C_5)$ has five subtrees $T_1, T_2, T_3, T_4, T_5$ each of which is a star $K_{1,3}$, and each $v_i \in V(C_5)$ ($1 \leq i \leq 5$) is a pendent vertex of $T_i$ correspondingly. Without loss of generality, for an optimal labeling $f$ of $\tau_{49}$, let $f(v_i) = j_i$ and $j_1 < j_2 < j_3 < j_4 < j_5$. Clearly, $|\nabla f(S_{j_i})| \geq 2$ for $1 \leq i \leq 4$. Because $T_i$ has a congestion of two for each $1 \leq i \leq 5$, $t = 5$, there is at least a vertex $x \in V(T_2) \cup V(T_3)$ (or $x \in V(T_3) \cup V(T_4)$) with $f(x) = j$ such that $|\nabla f(S_{j})| = 4$. From (3), $c(\tau_{49}, f) = 4$, resulting in $c(\tau_{49}) = 4$ by (2). On the other hand, $\tau_{49}$ has two maximal proper subgraphs where one is obtained by deleting a pendent edge $e \notin V(C_5)$, the other is obtained by deleting any cycle edge $e \in V(C_5)$. For each maximal proper subgraph $G - e$, we can always find a labeling $f'$ such that $c(G - e, f') \leq 3$ easily. Thus, $c(G - e) \leq 3$, leading to the finding that $G$ is 4-cutwidth critical.

(9) Type 6A (including $\tau_{50}$ only): Similar to that of Type 5B, omitted here. This completes the proof.

**Lemma 6.** Let $G$ be a 4-cutwidth critical graph with unique cycle $C_i$ and $t \geq 4$. Then each of the following holds.

(i) If, for each $v_i \in V(C_i)$ ($1 \leq i \leq t$), each member in $F_2$ is not an induced subgraph of $T_i$, then $T_i \in F_1$. 


(ii) If there is at least a vertex $v_i \in V(C_i)$ such that one of $F_2$ is an induced subgraph of $T_i$, then $T_i \in \mathcal{F}_2$.

**Proof.** (i) From (6), $d_G(v_i) \geq 3$. First, from the assumption that $G$ is 4-cutwidth critical, it follows that $c(G - xy) = 3$ for any $xy \in E(G)$ and $c(T_i) \leq 3$ for each $v_i \in V(C_i)$. For the edge $xy$, there are three cases to consider.

Case 1 $x \in V(C_i)$, $y \notin V(C_i)$. In this case, $C_i \subset G - xy$. So, by the minimality of $H_3$ and $H_4$ (see $H_3, H_4$ in Figure 1), either $H_3$ or $H_4$ is a subgraph of $G - xy$ resulting in that either $F_2$ or $K_2$ is contained in some $T_i$, say $T_1$.

Case 2 $x \notin V(C_i)$, $y \notin V(C_i)$. Similar to that of Case 1, we can conclude that $F_2$ or $K_2$ is also contained in some $T_i$, say $T_1$.

Case 3 $x \in V(C_i)$, $y \in V(C_i)$. Clearly, $G - xy$ is a 3-cutwidth tree. So, by the minimality of $H_1$ and $H_2$ in Figure 1, either $H_1$ or $H_2$ is a subgraph of $G - xy$ leading to the conclusion that either $K_1,5$ or $F_1$ is contained in some $T_i$ with $v_i \neq v_1$.

In addition, because $K_{1,3}$ is a proper subgraph of any of $\{K_{1,5}, F_1, F_2\}$ and $t \geq 4$, we can conclude that there is at least a vertex $v_i \in V(C_i)$ such that $T_i = K_{1,3}$ with $d_G(v_i) = 3$.

Otherwise, $T_i \in \mathcal{F}_1 \setminus \{K_{1,3}\}$ for every $i \in S_i$. In this case, we can verify that either $c(G) = 3$ (contradicting $c(G) = 4$) or one of $\{\tau_{44}, ..., \tau_{45}\}$ is homeomorphic to a subgraph of $G$ contradicting the minimality of $G$. Thus, $T_i \in \mathcal{F}_1$ for $i \in S_i$.

(ii) Assume that one member of $F_2$ is a subgraph of some $T_i$ and $v_{i_0} = u_0$ (or $x_0$) by homeomorphism. Because $t \geq 4$ and $d_G(v_i) \geq 3$ for each $i \neq i_0$ by (6), one of $\{\tau_{44}, \tau_{45}, \tau_{43}\}$ must be either a subgraph of $G$ or homeomorphic to a subgraph of $G$, contrary to the minimality of $G$. Hence, $T_{i_0} \in \mathcal{F}_2$. Thus, by Lemma 5, $G$ is 4-cutwidth critical with $T_{i_0} \in \mathcal{F}_2$ for $v_{i_0} \in V(C_i)$ if and only if $G \in \{\tau_{44}, \tau_{45}, \tau_{43}\}$. This completes the proof. □

**Lemma 7.** Let $G$ be a 4-cutwidth critical graph with cycle $C_i$. Then, $t \leq 6$.

**Proof.** This is a proof by contradiction. Assume that $t \geq 7$; then, $T_i \notin \mathcal{F}_2$ for each $i \in S_i$. This is because, otherwise, one of $\{\tau_{41}, \tau_{42}, \tau_{43}\}$ is homeomorphic to a subgraph $G'$ of $G$ in which the cycle $C_4$ is subdivided into $C_i$. So, $c(G') = 4$, contradicting the conclusion that $G$ is 4-cutwidth critical. Thus, $T_i \in \mathcal{F}_1$ by Lemma 6.

For each $i \in S_i$, if $T_i = K_{2,3}$, then direct computation yields that $c(G) = 3$. This implies that at least a $T_i \in \mathcal{F}_1 \setminus \{K_{2,3}\}$. In addition, because $c(T_i) = 4$, there are at most two vertices $v_i, v_{i_1} \in V(C_i)$ such that $T_{i_1} = K_{1,3}$ centered at $v_i$ and $T_{i_2} = K_{3,1}$ centered at $v_{i_1}$. In the sequel, let $f : V(G) \rightarrow S_n$ be an optimal 4-cutwidth labeling with $f(v_1) = \min\{f(v_2) : v_1 \in V(C_i)\}$ and $f(v_{i_0}) = \max\{f(v_2) : v_2 \in V(C_i)\}$ for some $2 \leq h \leq t$, and embed $T_i$ into the interval $[1, f(v_1)]$, $T_i (i \neq 1, h)$ into the interval $(f(v_1), f(v_{i_0}))$ and $T_h$ into the interval $(f(v_{i_0}), n]$.

Case 1 $\min\{d_G(v_1), d_G(v_{i_0})\} \geq 4$, i.e., $T_1 = K_{1,3}$ is centered at $v_1$ or $F_2$ with $d_G(v_1) = 4$, and $T_h = K_{1,3}$ is centered at $v_{i_0}$ or $F_2$ with $d_G(v_{i_0}) = 4$ because $T_i \in \mathcal{F}_1$ for each $i \in S_i$. Thus, the congestions of $T_1$ and $T_h$ are at most three under $f$. Because $G$ is 4-cutwidth critical and each cycle edge $v_i \in C_i$ has a congestion of two, the subtree $T_i$ for $i \neq 1, h$ must be 1- or 2-cutwidth critical, namely, $T_i = K_{2,3}$ or $K_{1,3}$. If each $T_i = K_{2,3}$, then $c(G) = 3$ by direct computation. So, there are at least a vertex $v_{i_0}$ ($i_0 \neq 1, h$) such that $T_{i_0} = K_{1,3}$ in $G$, which results in the conclusion that one of $\{\tau_{2}, \tau_{3}, \tau_{4}\}$ is homeomorphic to a subgraph of $G$; this is a contradiction. Hence, this case is not possible.

Case 2 $\max\{d_G(v_1), d_G(v_{i_0})\} \geq 4$ and $\min\{d_G(v_1), d_G(v_{i_0})\} = 3$, say $d_G(v_1) \geq 4$, $d_G(v_{i_0}) = 3$.

From the minimality of $G$, in this case, $T_1$ is either $K_{1,3}$ centered at $v_1$ or $F_2$, $T_h$ is $K_{1,3}$ not centered at $v_{i_0}$. For each $i \neq 1, h$, if $T_i = K_{2,3}$ then $c(G) = 3$ by the direct computation, contrary to $c(G) = 4$. So, there is at least a $T_i$ except $T_1$ and $T_h$ such that $T_i \in \{K_{1,3}, K_{1,5}, F_1\}$.

This results in the conclusion that one of $\{\tau_{44}, \tau_{45}\}$ must be homeomorphic to a subgraph of $G$, contrary to the minimality of $G$. For example, if $t = 7$ and $h = 4$, $\tau_{44}$ must be
homeomorphic to Figure 6a (or Figure 6b), while $\tau_45$ is homeomorphic to Figure 6c. So, this case is impossible.

Case 3 $\max\{d_G(v_1),d_G(v_h)\} = \min\{d_G(v_1),d_G(v_h)\} = 3$, i.e., $d_G(v_i) = 3$ for each $1 \leq i \leq t$. By Lemma 6 and the minimality of $G$, $T_i \in F_1 \setminus \{F_2\}$ for each $1 \leq i \leq t$. Because $\tau_{11}, \tau_{12},$ and $\tau_{13}$ are 4-cutwidth critical, at most two subtrees of $G - E(G)$, say $T_1$ and $T_h$, are in $\{K_{1,5}, F_1\}$. So, similar to Cases 1 and 2, either one of $\tau_{40}, \ldots, \tau_{50}$ is homeomorphic to a subgraph of $G$ or $c(G) < 4$ (see seven typical cases in Figure 7 each of whose cutwidth is three by homomorphism), contradicting the conclusion that $G$ is 4-cutwidth critical. So, this case is also impossible.

To sum up, we have $t \leq 6$. This completes the proof. □

![Figure 6. (a–c) Three examples of Case 2 with the proof of Lemma 7.](image)

![Figure 7. (a–h) Seven typical examples on $G$ of Case 3 with the proof of Lemma 7.](image)

**Lemma 8.** Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_1) \geq 4, d_G(v_2) \geq 4$ in $G$, then $G$ is 4-cutwidth critical if and only if $G \in \{\tau_1, \tau_2, \tau_3, \tau_4\}$.

**Proof.** By Lemma 5, it suffices to show its necessity. Because $G$ is 4-cutwidth critical, $T_1, T_3 \in \{K_{1,3}, F_2\}$ by Lemmas 6 and 7. For an optimal labeling $f$ of $G$ with $f(v_1) < f(v_2) < f(v_3)$, we can embed $T_1$ into the interval $[1, f(v_1)]$ with a congestion of three and $T_3$ into the interval $[f(v_2), n]$ with a congestion of three. So, $T_2$ must be $K_{1,3}$, because the congestion of the cycle edge of $C_3$ is two, which is embedded into the interval $(f(v_1), f(v_3))$. Thus, $G$ is one of $\{\tau_1, \tau_2, \tau_3, \tau_4\}$. □

With an argument similar to that of Lemma 8, we can verify that the following two Lemmas 9 and 10 hold also.

**Lemma 9.** Suppose that $C_3 = v_1v_2v_3v_1$ is a unique cycle with $d_G(v_1) \geq 4, d_G(v_2) = 3$ and $d_G(v_3) = 3$ but $T_2 \neq K_2, T_3 \neq K_2$ in $G$; then, $G$ is 4-cutwidth critical if and only if $G \in \{\tau_5, \tau_6, \ldots, \tau_{10}\}$.


**Lemma 10.** Suppose that \( C_3 = v_1v_2v_3v_1 \) is a unique cycle with \( d_G(v_i) = 3 \) and \( T_i \neq K_2 \) for each \( i \in S_3 \) in \( G \); then, \( G \) is 4-cutwidth critical if and only if \( \tau \in \{ \tau_{11}, \tau_{12}, \tau_{13} \} \).

**Lemma 11.** Suppose that \( C_3 = v_1v_2v_3v_1 \) is a unique cycle with \( d_G(v_1) \geq 5 \), \( T_2 = K_2 \), and \( T_3 = K_2 \) in \( G \); then, \( G \) is 4-cutwidth critical if and only if \( \tau \in \{ \tau_{14}, \tau_{15}, \tau_{38}, \tau_{39}, \tau_{40} \} \).

**Proof.** By Lemma 5, we only show its necessity. By \((K_1, T_1) = 4 \) (see \( T_1 \) in Figure 2), \( d_G(v) \leq 6 \) for each \( v \in V(G) \). So, \( d_G(v_1) = 5 \) or 6. We first consider the case of \( d_G(v_1) = 5 \), and let the three subtree components of \( T_1 - v_1 \) be \( T_1^{(1)}, T_1^{(2)} \) and \( T_1^{(3)} \). Thus, \( T_1^{(1)} + x_1v_1, T_1'' = T_1^{(2)} + x_2v_1 \), and \( T_1'' = T_1^{(1)} + x_3v_1 \), respectively, where \( x_1, x_2, x_3 \in N_G(v_1) \) and \( x_1 \in V(T_1^{(1)}), x_2 \in V(T_1^{(2)}), x_3 \in V(T_1^{(3)}) \).

**Claim 2.** At most one subtree, say \( T_1'' \), among \( T_1', T_1'' \) and \( T_1''' \) is \( K_2 \).

In fact, if \( T_1' = K_2 \) and \( T_1'' = K_2 \), then \( c(G) = 3 \) by direct computation, contrary to \( c(G) = 4 \). Now, let \( T_1' = K_2 \) but \( T_1'' \neq K_2 \). In this case, we have \( c(T_1) = 4 \). Otherwise, \( c(T_1) = 3 \), and by the minimality of \( G, T_1 = H_2 \) in Figure 1 (note that \( T_1 \neq H_1 \) because of \( T_1' \neq K_2 \)). For a labeling \( f \) of \( G \) with \( f(v_1) < f(v_2) < f(v_3) \), we embed \( T_1 \) into the interval \([1, f(v_1)] \) with a congestion of three and \( G - T_1 \) into the interval \((f(v_1), n] \) with a congestion of three. In this way, \( c(G, f) = 3 \) resulting in \( c(G) \leq 3 \) by (2), a contradiction. Thus, by \( T_1 \subset G \) and \( c(T_1) = 4 \), a contradiction to the minimality of \( G \) is obtained. Claim 2 holds.

From Claim 2, there are only two subcases considered: (1) \( T_1'' \neq K_2 \). From the minimality of \( G, T_1 = H_2 \) and \( c(G) = \tau_{14} \) or \( \tau_{15} \). In this subcase, for an optimal labeling \( f \) of \( G \) with \( f(x_1) = \max\{f(v) : v \in V(T_1^{(1)}) \} \), \( f(x_2) = \max\{f(v) : v \in V(T_1^{(2)}) \} \) and \( f(x_1) < f(x_2) \), because \( x_1v_1 \) and \( x_2v_1 \) cut edges in \( G \), under \( f \), \( T_1' \) is embedded into the interval \([1, f(x_1)] \) with a congestion of three, \( T_1'' + x_2v_1 \) into the interval \((f(x_1), f(x_2)] \) with a congestion of four and \( G - T_1'' \) \( = H_4 \) in Figure 1 into the interval \([f(x_2) + 1, n] \) with a congestion of three, which leads to the conclusion that \( c(G) = c(G, f) = 4 \). Thus, by the minimality of \( G, T_1' \cup T_1'' \in F_2 \) leading to the conclusion that \( G \in \{ \tau_{38}, \tau_{39}, \tau_{40} \} \). Likewise, for the case of \( d_G(v_1) = 6 \), using an argument similar to the case of \( d_G(v_1) = 5 \), we can verify that at least one of \( \{ \tau_{14}, \tau_{15}, \tau_{38}, \tau_{39}, \tau_{40} \} \) is a proper subgraph of \( G \), contrary to the minimality of \( G \). So, \( d_G(v_1) \neq 6 \). This completes the proof. \( \square \)

**Lemma 12.** Suppose that \( C_3 = v_1v_2v_3v_1 \) is a unique cycle in \( G \), and there are three disjoint graphs \( G_1, G_2, G_3 \) such that \( G = K_1, G_2, G_3 \). Then, \( G \) is 4-cutwidth critical if and only if \( G \in \{ \tau_{17}, \tau_{18}, ..., \tau_{28} \} \), where, for \( j \in S_3 \),

\[
G_j' = \begin{cases} 
G_j & \text{if } v_j \notin \operatorname{V}_P^{(j)} \\
G_j - p_j & \text{otherwise}
\end{cases}
\]

in which \( \operatorname{V}_P^{(j)} \subset V(G_j) \) and there always are at least a pendent vertex \( p_j \in D_1(G_j) \) such that \( \forall p_j \in E(G_j) \) for each \( v \in \operatorname{V}_P^{(j)} \).

**Proof.** By Lemma 5, we only show its necessity and adopt the notation of Definition 1 (iii). As \( G \) is unicyclic and vertex \( p_j \) is pendent, \( C_3 \) is contained in one of \( \{ G_1, G_2, G_3 \} \), say \( G_2 \). Thus, \( G_1, G_3 \) are subtrees in \( G \). From the hypothesis that \( G \) is 4-cutwidth critical, for \( j \in S_3 \) and \( v_j \in V(G_j'), G_j' \) and \( G_j + u_0v_j \) are 3-cutwidth critical in the cases of \( v_j \notin \operatorname{V}_P^{(j)} \) or \( v_j \in \operatorname{V}_P^{(j)} \) respectively after that the series reduction is implemented. Because, otherwise, it is not hard to obtain a noncritical graph with a cutwidth of four, a contradiction. Hence,
\[ G'_i \in \{H_1, H_2, H_3, H_4\} \text{ if } v_j \notin V_p^{(j)} \text{ and } G'_i + u_0[v_j] \in \{H_1, H_2, H_3, H_4\} \text{ if } v_j \in V_p^{(j)}. \] So, \( G \in \{\tau_17, \tau_18, \ldots, \tau_28\} \) by the minimality of \( G \). \( \square \)

Similar to Lemma 12, a class of critical unicyclic graphs with a cutwidth of four has an interesting structure (see Definition 3 below). This structure together with that of Lemma 12 is called the decomposability of the critical unicyclic graphs with a cutwidth of four. From Corollary 3, \( K_{3,3} \cup (K_{1,5}, K_{1,5}) \) with \( v_j \in D_1(K_{1,5}) \) (1 \( \leq j \leq 3 \)) is 4-cutwidth critical after that the series reductions are carried out, so we may assume that \( G - v \) has at most two \( K_{1,4} \)'s for any \( v \in V(G) \) in the sequel.

**Definition 3.** Let \( C_j \) be a unique cycle with a length of three in graph \( G \), \( v_0 \in V(G) \) with \( |N_C(v_0)| \geq 4 \), \( G_i = (V_i, E_i) \) be a component of \( G - v_0 \) \( (1 \leq i \leq j_0, 3 \leq j_0 \leq |N_C(v_0)|) \), \( v_0v_1, v_0v_2 \) be cut edges with \( v_1 \in V_1 \) and \( v_2 \in V_2 \), \( \min\{c(G[V_i \cup \{v_0\}]): j = 1, 2, 3\} \geq \max\{c(G[V_i \cup \{v_0\}]): 4 \leq h \leq j_0\}, \) and \( G_3 = \bigcup_{h=3}^{j_0} G[V_i \cup \{v_0\}] \). Then,

(i) if \( G_i \neq K_{1,4} \) for each \( 1 \leq i \leq j_0 \), then define \( G'_1 = G[V_i \cup \{v_0\}] \cup G[E_0] \) for \( j = 1, 2; \)

(ii) if \( G_1 = K_{1,4} \), then define \( G'_1 = G[V_i \cup \{v_0\}] = K_{1,5} \), \( G_2 = G[V_2 \cup \{v_0\}] \cup G[E_0]; \)

(iii) if \( G_1 = K_{1,4}, G_2 = K_{1,4} \), then define \( G'_1 = G[V_i \cup \{v_0\}] = K_{1,5} \) for \( j = 1, 2; \)

where \( E_0 \neq \emptyset \) is an edge subset of \( E(G_3) \) but \( E_0 \cap E(G_3) = \emptyset \).

In Definition 3, if \( d_{G_j}(v) = 2 \) for some vertex \( v \in V(G_j) \), and \( G_j - v + x_i^j x_i^j \) is 3-cutwidth critical, then we also say that \( G_j \) is 3-cutwidth critical below, where \( x_i^j, y_i^j \in N_{G_j}(v) \). For examples, for Case (i), let \( G = \tau_{31} \) with \( C_3 = v_15v_16v_17v_15 \) and \( v_0 = v_{14} \).

**Lemma 13.** With the notation in Definition 3, if \( G_j \) is 3-cutwidth critical for each \( j \in S_3 \), then \( G \) is 4-cutwidth critical.

**Proof.** Without loss of generality, let \( G_1, G_2, G_3 \) satisfy (i) and \( C_3 \subset G_3 \) by assumption. Then \( G_1, G_2 \) are subtrees in \( G \). Due to the fact that \( G_i \) is 3-cutwidth critical for each \( j \in S_3 \), \( G_3 \subset \{H_3, H_4\} \) and \( G_1 \subset \{H_1, H_2\} \), it can be concluded that \( G \in \{\tau_{16}, \tau_{31}, \tau_{34}, \tau_{36}, \tau_{40}\} \) via direct computation. So, \( G \) is 4-cutwidth critical by Lemma 5. Similarly, for Case (ii), \( G \in \{\tau_{30}, \tau_{33}, \tau_{37}, \tau_{39}\} \); and for Case (iii), \( G \in \{\tau_{29}, \tau_{32}, \tau_{35}, \tau_{38}\} \). So, the Lemma holds. \( \square \)

**Lemma 14.** With the notation in Definition 3, \( G \) is 4-cutwidth critical if and only if \( G \in \{\tau_{16}, \tau_{29}, \tau_{30}, \ldots, \tau_{40}\} \).

**Proof.** It suffices to show its necessity by Lemma 5. As the arguments are similar, we only consider the case that \( G_1, G_2, G_3 \) satisfy (i) of Definition 3. Furthermore, without loss of generality, let cycle \( C_3 \subset G_3 \) by assumption, then \( G_1, G_2 \) are subtrees in \( G \).

**Claim 3.** For each \( j \in S_3 \), \( G_j \) is 3-cutwidth critical.

In fact, if there is some \( j_0 \in S_3 \), say \( j_0 = 3 \), such that \( G_3 \) is not 3-cutwidth critical, then two cases need to be considered: (1) there are at least an edge \( e \in E(G_3) \) such that \( c(G_3 - e) \geq 3 \); (2) \( c(G_3) \leq 2 \). By assumption that \( G \) is 4-cutwidth critical, we can see that Case (1) is impossible by Lemma 13. Hence, it suffices to verify that Case (2) is also impossible. As \( G_1 \) and \( G_2 \) are 3-cutwidth critical, \( c(G_1) \leq 2 \) and \( c(G_2) \leq 2 \). Let \( f_1, f_2, f_3 \) be the optimal labelings of \( G_1, G_2, \) and \( G_3 \), respectively. Then, similar to that of (5) in Definition 2, we can obtain a 3-cutwidth labeling \( f: V(G) \mapsto S_{|V(G)|} \) by the order
Lemma 15. Let \( t = 4 \) in \( C_t \). Then, \( G \) is 4-cutwidth critical if and only if \( G \in \{ \tau_{41}, \tau_{42}, \ldots, \tau_{45} \} \).

Proof. By Lemma 5, it suffices to show its necessity. By Lemma 6, \( T_i \in \mathcal{F}_1 \cup \mathcal{F}_2 \) for each \( v_i \in V(C_4) \). So, two cases need to be considered.

Case 1: \( T_i \in \mathcal{F}_2 \). In Lemma 6, we already showed that \( G \) is 4-cutwidth critical if and only if \( G \in \{ \tau_{41}, \tau_{42}, \tau_{43} \} \) in this case, omitted here.

Case 2: \( T_i \in \mathcal{F}_1 \). By (6), for each \( v_i \in V(C_4) \), \( d_G(v_i) \geq 3 \) in \( G \).

Claim 4. There is a unique vertex, say \( v_1 \), such that \( d_G(v_1) \geq 4 \) in \( G \).

First, let \( d_G(v_i) = 3 \) for each \( v_i \in V(C_4) \). In this case, there are at least three subtrees, say \( T_1, T_2, T_4 \), such that \( T_1, T_2, T_4 \) are all in \( \{ K_{1,5}, F_1 \} \), which leads to the conclusion that one of \( \{ \tau_{11}, \tau_{12}, \tau_{13} \} \) is homeomorphic to a subgraph of \( G \), which contradicts the minimality of \( G \). Otherwise, by the fact that the cutwidth of each member of \( \mathcal{F}_1 \) is at most two, we can verify that \( c(G) < 4 \), contrary to \( c(G) = 4 \). In fact, for an optimal labeling \( f \) of \( G \) with \( f(v_i) = \min \{ f(v_j) : v_j \in V(C_4) \} \) and \( f(v_i) = \max \{ f(v_j) : v_j \in V(C_4) \} \), let \( f(x) = \max \{ f(v) : v \in V(T_1 - v_1) \} \), \( f(y) = \min \{ f(v) : v \in V(T_4 - v_4) \} \). Under \( f \), we first embed \( T_1 \) into the interval \([1, f(v_1)]\) with a congestion of three and \( T_4 \) into the interval \([f(v_4), n]\) with a congestion of three, resulting in \( T_1, T_4 \in \{ K_{1,5}, F_1 \} \). If \( T_2, T_3 \in \{ K_{1,3}, F_2 \} \), then we can conclude that \( c(G) = 3 \) by embedding \( T_2 \) into the interval \((f(x), f(v_1)]\) with a congestion of three and \( T_3 \) into the interval \((f(v_4), f(y)]\) with a congestion of three. This is a contradiction, which leads to the conclusion that one of \( \{ T_2, T_3 \} \) is in \( \{ K_{1,5}, F_1 \} \).

Second, let \( d_G(v_2) \geq 4 \), i.e., \( T_2 = F_2 \) or \( K_{1,3} \) centered at \( v_2 \). If \( T_3 = K_2 \) and \( T_4 = K_2 \), then \( c(G) = 3 \), contradicting \( c(G) = 4 \). So, at least one of \( \{ T_3, T_4 \} \) is a \( K_{1,3} \). However, in this case, one of \( \{ \tau_2, \tau_3, \tau_4 \} \) is homeomorphic to a subgraph of \( G \), contrary to the minimality of \( G \). So, Claim 4 holds.

Because \( \tau_{41}, \tau_{42}, \tau_{43} \) are 4-cutwidth critical, there is at least a subtree \( T_i \) (\( i = 2, 3, 4 \)) such that \( T_i \neq K_2 \). In addition, \( K_{1,3} \subset K_{1,5} \) and \( F_2 \subset F_1 \). So, by Claim 4 and the minimality of \( G \), \( G \) must be among the six graphs in Figure 8. From direct computation, only graphs \( (c) \) and \( (f) \) are 4-cutwidth critical, which are \( \tau_{44} \) and \( \tau_{45} \) in Figure 8, respectively. Thus, \( G \in \{ \tau_{41}, \tau_{42}, \tau_{43}, \tau_{44}, \tau_{45} \} \).

Figure 8. (a–f) Six possible graphs on \( G \) with the proof of Lemma 15.

Lemma 16. Let \( t = 5 \) in \( C_t \). Then, \( G \) is 4-cutwidth critical if and only if \( G \in \{ \tau_{46}, \tau_{47}, \tau_{48}, \tau_{49} \} \).

Proof. By Lemma 5, it suffices to prove its necessity. Because \( G \) is 4-cutwidth critical and \( t = 5 \), by Lemmas 6 and 15, \( T_i \in \mathcal{F}_1 \) for each \( i \in S_5 \).

Claim 5. \( d_G(v_i) = 3 \) for each \( v_i \in V(C_5) \), and at most two subtrees \( T_{i_1} \) and \( T_{i_2} \) are in \( \{ K_{1,5}, F_1 \} \).

In fact, if there exists at least one vertex, say \( v_1 \), such that \( d_G(v_1) \geq 4 \), then, with an argument similar to that of Lemma 15, we can verify that one of \( \{ \tau_{11}, \tau_{12}, \ldots, \tau_{10}, \tau_{44}, \tau_{45} \} \) is homeomorphic to a subgraph of \( G \), which is a contradiction. If there is another \( T_{i_3} \in \)
\{K_{1,5}, F_1\}, then one of \{\tau_{11}, \tau_{12}, \tau_{13}\} is homeomorphic to a subgraph of \(G\), which is another contradiction. Claim 5 holds.

By the minimality of \(G\), if each of \{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\} is homeomorphic to a subgraph of \(G\), then \(G\) does not need to be considered. Similarly, if \(G\) is not homeomorphic to any of \{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}, then any homeomorphic subgraph of \(G\) is not also considered by (4). Thus, except for \(\tau_{46}, \tau_{47}, \tau_{48}\), and \(\tau_{49}\), it is possible that \(G\) is among the five graphs in Figure 9 by Claim 5. However, by direct computations, \(c(G) = 3\) for each graph \(G\) in Figure 9, contrary to \(c(G) = 4\). So, \(G \in \{\tau_{46}, \tau_{47}, \tau_{48}, \tau_{49}\}\). □

![Figure 9. (a–e) Possible graphs on \(G\) with the proof of Lemma 16.](image)

**Lemma 17.** Let \(t = 6\) in \(C_t\). Then, \(G\) is 4-cutwidth critical if and only if \(G = \tau_{50}\).

**Proof.** By Lemma 5, it suffices to prove its necessity. Similar to those of Lemmas 15 and 16, we can verify that \(d_G(v_i) = 3\) for each \(i \in S_6\) and \(T_{i_1}, T_{i_2}, T_{i_3} \in \{K_{1,3}\}, T_{i_4}, T_{i_5}, T_{i_6} \in \{K_2\}\). Thus, \(G\) is among the following three graphs in Figure 10. By direct computations, we can see that only graph (c) is 4-cutwidth critical, and (c) = \(\tau_{50}\). So, \(G = \tau_{50}\) and the Lemma holds. □

![Figure 10. (a–c) Three possible graphs with the proof of Lemma 17.](image)

**Proof of Theorem 1.** By Lemmas 5, 8–12, and 14–17, the desired result holds. □

**4. Remarks**

In this paper, fifty critical unicyclic graphs with a cutwidth of four were obtained, during which a decomposable property of some 4-cutwidth critical unicyclic graphs was also obtained (see Lemma 12 and Definition 3). For an integer \(k \geq 4\), although it seems to be difficult to find all \(k\)-cutwidth critical graphs, some structural properties of some of them can be found definitively. In fact, as the decomposability of \(k\)-cutwidth critical trees [25] and some special non-tree graphs with uncomplicated structure [26], a similar decomposable property of 4-cutwidth critical unicyclic graphs, which is contained in Lemma 12 and Definition 3, can be generalized to \(k\)-cutwidth critical graphs even if these graphs are multicyclic graphs. For instance, in Lemma 12, if any element of \{\(G_1, G_2, G_3\)\} is a critical unicyclic graph with a cutwidth of \(k - 1\), then we can verify that \(K_{1,3} \circ (G_1', G_2', G_3')\) is a critical unicyclic graph with a cutwidth of \(k\). Clearly, if \(v_i \notin V_p^{(j)}\) with \(1 \leq j \leq 3\), then \{\(G_1' + u_0v_j : 1 \leq j \leq 3\)\} is a decomposition of \(K_{1,3} \circ (G_1', G_2', G_3')\). Regarding the critical multicyclic graphs with a cutwidth of at least four, their general structural properties have yet to be known. Additionally, the application of critical unicyclic graphs with a cutwidth of four to some realistic fields, such as social and biological networks, multivariate
cryptography, and other fields, is worth studying. These will be the objects for further study in future works.

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