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On the Sizes of k-edge-maximal r-uniform Hypergraphs

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Abstract Let $H=(V,E)$ be a hypergraph, where V is a set of vertices and E is a set of non-empty subsets of V called edges. If all edges of H have the same cardinality r, then H is an r-uniform hypergraph; if E consists of all r-subsets of V, then H is a complete r-uniform hypergraph, denoted by K_n^r, where $n=|V|$. A hypergraph $H'=(V',E')$ is called a subhypergraph of $H=(V,E)$ if $V'\subseteq V$ and $E'\subseteq E$. The edge-connectivity of a hypergraph H is the cardinality of a minimum edge set $F\subseteq E$ such that $H-F$ is not connected, where $H-F=(V,E\setminus F)$. An r-uniform hypergraph $H=(V,E)$ is k-edge-maximal if every subhypergraph of H has edge-connectivity at most k, but for any edge $e\in E(K_n^r)\setminus E(H)$, $H+e$ contains at least one subhypergraph with edge-connectivity at least $k+1$.

Let k and r be integers with $k\ge 2$ and $r\ge 2$, and let $t=t(k,r)$ be the largest integer such that $\binom{t-1}{r-1}\le k$. That is, t is the integer satisfying $\binom{t-1}{r-1}\le k<\binom{t}{r-1}$. We prove that if H is an r-uniform k-edge-maximal hypergraph such that $n=|V(H)|\ge t$, then (i) $|E(H)|\le \binom{t}{r}+(n-t)k$, and this bound is best possible; (ii) $|E(H)|\ge (n-1)k-((t-1)k-\binom{t}{r})\lfloor \frac{n}{t}\rfloor$, and this bound is best possible.

Keywords Edge-connectivity; k-edge-maximal hypergraphs; r-uniform hypergraphs
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1 Introduction

For graph-theoretical terminologies and notation not defined here, we follow[3]. The edge-connectivity of a graph G, denoted by $\kappa'(G)$, is the the cardinality of a minimum edge set $F\subseteq E$ such that $G-F$ is not connected. The *complement* of a graph G is denoted by G^c. For $X\subseteq E(G^c)$, $G+X$ is the graph with vertex set $V(G)$ and edge set $E(G)\cup X$. We will use $G+e$ for $G+\{e\}$. The *floor* of a real number x, denoted by $\lfloor x\rfloor$, is the greatest integer not larger than x; the *ceiling* of a real number x, denoted by $\lceil x\rceil$, is the least integer greater than or equal to x. For two integers n and k, we define $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ when $k\le n$ and $\binom{n}{k}=0$ when $k>n$.

Given a graph G, Matula[9] defined the strength $\overline{\kappa}'(G)$ of G as $\max\{\kappa'(G'): G'\subseteq G\}$. For a positive integer k, the graph G is k-edge-maximal if $\overline{\kappa}'(G)\le k$ but for any edge $e\in E(G^c)$, $\overline{\kappa}'(G+e)>k$. Mader[8] and Lai[6] proved the following results.

Theorem 1.1. *Let $k\ge 1$ be an integer, and G be a k-edge-maximal graph on $n>k+1$ vertices. Each of the following holds.*

(i) (Mader[8]) $|E(G)|\le (n-k)k+\binom{k}{2}$. Furthermore, this bound is best possible.

(ii) (Lai[6]) $|E(G)|\ge (n-1)k-\lfloor\frac{n}{k+2}\rfloor\binom{k}{2}$. Furthermore, this bound is best possible.

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In [1] and [7], k-edge-maximal digraphs are investigated, and the upper bound and the lower bound of the sizes of the k-edge-maximal digraphs are determined, respectively. Motivated by these results, we will study k-edge-maximal hypergraphs in this paper.

Let $H = (V, E)$ be a hypergraph, where V is a finite set and E is a set of non-empty subsets of V, called edges. Throughout we will assume that every edge contains at least two vertices. An edge of cardinality 2 is just a graph edge. For a vertex $u \in V$ and an edge $e \in E$, we say u is incident with e or e is incident with u if $u \in e$ (we see the edge e as a subset of V). If all edges of H have the same cardinality r, then H is an r-uniform hypergraph; if E consists of all r-subsets of V, then H is a complete r-uniform hypergraph, denoted by K_n^r, where $n = |V|$. For $n < r$, the complete r-uniform hypergraph K_n^r is just the hypergraph with n vertices and no edges. The complement of a r-uniform hypergraph $H = (V, E)$, denoted by H^c, is the r-uniform hypergraph with vertex set V and edge set consisting of all r-subsets of V not in E. A hypergraph $H' = (V', E')$ is called a ubhypergraph of $H = (V, E)$, denoted by $H' \subseteq H$, if $V' \subseteq V$ and $E' \subseteq E$. Note that subhypergraph here is called a hypersubgraph in [2] and a strong subhypergraph in [4]. For $X \subseteq E(H^c)$, $H + X$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \cup X$; for $X' \subseteq E(H)$, $H - X'$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \setminus X'$. We use $H + e$ for $H + \{e\}$ and $H - e'$ for $H - \{e'\}$ when $e \in E(H^c)$ and $e' \in E(H)$. For $Y \subseteq V(H)$, we use $H[Y]$ to denote the hypergraph induced by Y, where $V(H[Y]) = Y$ and $E(H[Y]) = \{e \in E(H) : e \subseteq Y\}$. $H - Y$ is the hypergraph induced by $V(H) \setminus Y$.

For a hypergraph $H = (V, E)$ and two disjoint vertex subsets $X, Y \subseteq V$, let $E_H[X, Y]$ be the set of edges with non-empty intersecting with both X and Y and $d_H(X, Y) = |E_H[X, Y]|$. We use $E_H(X)$ and $d_H(X)$ for $E_H[X, V \setminus X]$ and $d_H(X, V \setminus X)$, respectively. If $X = \{u\}$, we use $E_H(u)$ and $d_H(u)$ for $E_H(\{u\})$ and $d_H(\{u\})$, respectively. The degree of u in H is the number of edges incident with u in H, which is $d_H(u)$ (Because we assume that every edge contains at least two vertices in this paper). The *minimum degree* $\delta(H)$ of H is defined as $\min\{d_H(u) : u \in V\}$; the maximum degree $\Delta(H)$ of H is defined as $\max\{d_H(u) : u \in V\}$. When $\delta(H) = \Delta(H) = k$, we call H k-regular.

For a nonempty proper vertex subset X of a hypergraph H, we call $E_H(X)$ an *edge-cut* of H. The *edge-connectivity* $\kappa'(H)$ of a hypergraph H is $\min\{d_H(X) : \emptyset \neq X \subsetneq V(H)\}$. By definition, $\kappa'(H) \leq \delta(H)$. We call a hypergraph H k-edge-connected if $\kappa'(H) \geq k$. A hypergraph is connected if it is 1-edge-connected. A maximal connected subhypergraph of H is called a *component* of H. It is easy to see that the edge-connectivity of a hypergraph H is the cardinality of a minimum edge set $F \subseteq E$ such that $H - F$ is not connected. Similarly, define the *strength* $\overline{\kappa}'(H)$ of H as $\max\{\kappa'(H') : H' \subseteq H\}$. An r-uniform hypergraph $H = (V, E)$ is k-edge-maximal if every subhypergraph of H has edge-connectivity at most k, but for any edge $e \in E(H^c)$, $H + e$ contains at least one subhypergraph with edge-connectivity at least $k + 1$. For any integer k with $k \geq \binom{n-1}{r-1}$, since $\kappa'(K_n^r) = \binom{n-1}{r-1} \leq k$ and there is no edge in $(K_n^r)^c$, we regard K_n^r as a k-edge maximal hypergraph. Thus H is a complete r-uniform hypergraph if H is a k-edge-maximal r-uniform hypergraph with $\binom{n-1}{r-1} \leq k$, where $n = |V(H)|$. For results on the connectivity of hypergraphs, see cf. [2, 4, 5] for references.

The main goal of this research is to determine, for given integers n, k and r, the extremal sizes of a k-edge-maximal r-uniform hypergraph on n vertices. Section 2 below is devoted to the study of some properties of k-edge-maximal r-uniform hypergraphs. In section 3, we give the upper bound of the sizes of k-edge-maximal r-uniform hypergraphs and characterize these k-edge-maximal r-uniform hypergraphs attained this bound. We obtain the lower bound of the sizes of k-edge-maximal r-uniform hypergraphs and show that this bound is best possible in section 4.

2 Properties of k-edge-maximal r-uniform Hypergraphs

For a 1-edge-maximal r-uniform hypergraph H with $n = |V(H)|$, we can verify that $\lceil \frac{n-1}{r-1} \rceil \leq |E(H)| \leq n - r + 1$. If H is the hypergraph with vertex set $V(H) = \{v_1, \cdots, v_n\}$ and edge set $E(H) = \{e_1, \cdots, e_{n-r+1}\}$, where $e_i = \{v_1, \cdots, v_{r-1}, v_{r-1+i}\}$ for $i = 1, \cdots, n - r + 1$, then H is a 1-edge-maximal r-uniform hypergraph H with $|E(H)| = n - r + 1$. The 1-edge-maximal r-uniform hypergraph K_r^r shows that the lower bound $\lceil \frac{n-1}{r-1} \rceil$ is also sharp. Thus, from now on, we always assume $k \geq 2$.

Definition 2.1. *For two integers k and r with $k, r \geq 2$, define $t = t(k, r)$ to be the largest integer such that $\binom{t-1}{r-1} \leq k$. That is, t is the integer satisfying $\binom{t-1}{r-1} \leq k < \binom{t}{r-1}$.*

Lemma 2.1. *Let $H = (V, E)$ be a k-edge-maximal r-uniform hypergraph on n vertices, where $k, r \geq 2$. Assume $n \geq t$ when $\binom{t-1}{r-1} = k$ and $n \geq t + 1$ when $\binom{t-1}{r-1} < k$, where $t = t(k, r)$. Then $\kappa'(H) = \overline{\kappa}'(H) = k$.*

Proof. Since H is k-edge-maximal, we have $\kappa'(H) \leq \overline{\kappa}'(H) \leq k$. In order to complete the proof, we only need to show that $\kappa'(H) \geq k$.

Let X be a minimum edge-cut of H, and let H_1 be a component of $H - X$ with minimum number of vertices and $H_2 = H - V(H_1)$. Denote $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Thus we have $X = E_H[V(H_1), V(H_2)]$, $n = n_1 + n_2$ and $n_1 \leq n_2$. To prove the lemma, we consider the following two cases.

Case 1. $E_{H^c}[V(H_1), V(H_2)] \neq \emptyset$.

Pick an edge $e \in E_{H^c}[V(H_1), V(H_2)]$. Since H is k-edge-maximal, we have $\overline{\kappa}'(H + e) > k$. Let $H' \subseteq H + e$ be a subhypergraph such that $\kappa'(H') \geq k+1$. By $\overline{\kappa}'(H) \leq k$, we have $e \in H'$. It follows that $(X \cup \{e\}) \cap E(H')$ is an edge-cut of H'. Thus $|X|+1 \geq |(X \cup \{e\})| \geq \kappa'(H') \geq k+1$, implying $|X| \geq k$. Thus $\kappa'(H) \geq k$.

Case 2. $E_{H^c}[V(H_1), V(H_2)] = \emptyset$.

Since $E_{H^c}[V(H_1), V(H_2)] = \emptyset$, we know that $E_H[V(H_1), V(H_2)]$ consists of all r-subsets of $V(H)$ intersecting both $V(H_1)$ and $V(H_2)$. Thus

$$|E_H[V(H_1), V(H_2)]| = \sum_{s=1}^{r-1} \binom{n_1}{s}\binom{n_2}{r-s} = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r}.$$

Let $g(x) = \binom{x}{r} + \binom{n-x}{r}$. It is routine to verify that $g(x)$ is a decreasing function when $1 \leq x \leq n/2$. If $n_1 \geq 2$, then as H is connected we have $r \leq n_1 \leq n/2$. Thus

$$\kappa'(H) = |E_H[V(H_1), V(H_2)]| = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} \geq \binom{n}{r} - \binom{2}{r} - \binom{n-2}{r} > \binom{n-1}{r-1} \geq \delta(H), \quad (2.1)$$

which contradicts to $\kappa'(H) \leq \delta(H)$. Thus, we assume $n_1 = 1$. Now we have

$$\kappa'(H) = |E_H[V(H_1), V(H_2)]| = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r} = \binom{n}{r} - \binom{1}{r} - \binom{n-1}{r} = \binom{n-1}{r-1} \geq \delta(H),$$

which implies $\kappa'(H) = \delta(H) = \binom{n-1}{r-1}$ and so H is a complete r-uniform hypergraph. Since $n \geq t$ when $\binom{t-1}{r-1} = k$ and $n \geq t + 1$ when $\binom{t-1}{r-1} < k$, we have $\kappa'(H) = \binom{n-1}{r-1} \geq k$. □

Lemma 2.2. *Suppose that $H = (V, E)$ is a k-edge-maximal r-uniform hypergraph, where $k, r \geq 2$. Let $X \subseteq E(H)$ be a minimum edge-cut of H and let H_1 be a union of some but not all components of $H - X$. Then H_1 is a k-edge-maximal r-uniform hypergraph.*

Proof. If H_1 is complete, then H_1 is k-edge-maximal by definition. Thus assume H_1 is not complete. For any edge $e \in E(H_1^c)$, $H + e$ has a subhypergraph H' with $\kappa'(H') \geq k + 1$ by

$E(H_1^c) \subseteq E(H^c)$. Since X is a minimum edge-cut of H, we have $|X| = \kappa'(H) \leq \overline{\kappa}'(H) \leq k$. Thus $X \cap E(H') = \emptyset$. As $e \in E(H') \cap E(H_1^c)$, we conclude that H' is a subhypergraph of H_1+e, and so $\overline{\kappa}'(H_1 + e) \geq k + 1$. Since $\overline{\kappa}'(H_1) \leq \overline{\kappa}'(H) \leq k$, it follows that H_1 is a k-edge-maximal r-uniform hypergraph. □

Lemma 2.3. *Let $H = (V, E)$ be a k-edge-maximal r-uniform hypergraph on n vertices, where $k, r \geq 2$. Assume $n \geq t$ when $\binom{t-1}{r-1} = k$ and $n \geq t + 1$ when $\binom{t-1}{r-1} < k$, where $t = t(k, r)$. Let $X \subseteq E(H)$ be a minimum edge-cut of H and let H_1 be a union of some but not all components of $H - X$. If $r \leq |V(H_1)| \leq n - 2$, then $|V(H_1)| \geq t$. Moreover, if H_1 is complete, then $|V(H_1)| = t$; if H_1 is not complete, then $|V(H_1)| \geq t + 1$.*

Proof. By Lemmas 2.1 and 2.2, we have $|X| = \kappa'(H) = k$ and H_1 is a k-edge-maximal r-uniform hypergraph, respectively. If H_1 is not complete, then there is a subhypergraph H_1' of H_1+e such that $\kappa'(H_1') \geq k+1$ for any $e \in E(H_1^c)$. Since $\binom{t-1}{r-1} \leq k$ and $\delta(H_1') \geq \kappa'(H_1') \geq k+1$, we have $|V(H_1)| \geq |V(H_1')| \geq t + 1$.

Now we assume H_1 is a complete r-uniform hypergraph. Let $H_2 = H - V(H_1)$. If $n_1 = |V(H_1)| < t$, then, in order to ensure each vertex in H_1 has degree at least k in H (because $\delta(H) \geq \kappa'(H) = k$), we must have $n_1 = t - 1$ and $k = \binom{t-1}{r-1}$. Moreover, each vertex in H_1 is incident with exactly $\binom{t-2}{r-2}$ edges in $E_H[H_1, H_2]$, and thus $d_H(u) = k$ for each $u \in V(H_1)$. By (2.1), there is an e intersecting both $V(H_1)$ and $V(H_2)$ but $e \notin X$. Since $n_1 \geq r$, there is a vertex $w \in V(H_1)$ such that w is not incident with e. Then $d_{H+e}(w) = k$. This implies w is not contained in a $(k + 1)$-edge-connected subhypergraph of $H + e$. But then each vertex in $V(H_1) \setminus \{w\}$ has degree at most k in $(H + e) - w$, and thus each vertex in $V(H_1) \setminus \{w\}$ is not contained in a $(k + 1)$-edge-connected subhypergraph of $H + e$. This illustrates that there is no $(k + 1)$-edge-connected subhypergraph in $H + e$, a contradiction. Thus we have $n_1 \geq t$. If $n_1 > t$, then $\kappa'(H_1) = \binom{n_1-1}{r-1} \geq \binom{t}{r-1} > k$, contrary to H is k-edge-maximal. Therefore, $n_1 \leq t$, and thus $n_1 = t$ holds. □

3 The Upper Bound of the Sizes of k-edge-maximal r-uniform Hypergraphs

Definition 3.1. *Let n, k, r be integers such that $k, r \geq 2$ and $n \geq t$, where $t = t(k, r)$. A hypergraph $H \in \mathcal{M}(n; k, r)$ if and only if it is constructed as follows:*

(i) Start from the complete hypergraph $H_0 \cong K_t^r$;

(ii) If $n - t = s = 0$, then $H_s = H_0$. If $n - t = s \geq 1$, then we construct, recursively, H_i from H_{i-1} by adding a new vertex v_i and k new edges containing v_i and intersecting $V(H_{i-1})$ for $i = 1, \cdots, s$;

(iii) Set $H = H_s$.

It is known that $\kappa'(H) \leq \delta(H)$ holds for any hypergraph H. If $\kappa'(H) = \delta(H)$, then we say H is *maximal-edge-connected*. An edge-cut X of H is *peripheral* if there exists a vertex v such that $X = E_H(v)$. A hypergraph H is *super-edge-connected* if every minimum edge-cut of H is peripheral. By definition, every super-edge-connected hypergraph is maximal-edge-connected.

Lemma 3.1. *Let k and r be integers with $k, r \geq 2$. If $n \geq t$ when $\binom{t-1}{r-1} = k$ and $n \geq t + 1$ when $\binom{t-1}{r-1} < k$, where $t = t(k, r)$, then for any $H \in \mathcal{M}(n; k, r)$, we have*

(i) $\delta(H) = k$;

(ii) H is super-edge-connected; and

(iii) H is k-edge-maximal.

Proof. Let $H = H_s$, where H_s is recursively constructed from $H_0, \cdots, H_{s-1}$ as in Definition 3.1. Then $V(H_s) = V(H_0) \cup \{v_1, \cdots, v_s\}$. We will prove this lemma by induction on n.

(i) If $n = t$ and $\binom{t-1}{r-1} = k$, then $H \cong K_t^r$ and $\delta(H) = \binom{t-1}{r-1} = k$. If $n = t+1$ and $\binom{t-1}{r-1} < k$, then H is obtained from K_t^r by adding a new vertex v_1 and k edges with cardinality r such that each added edge is incident with v_1. Let $k = \binom{t-1}{r-1} + i$. As $\binom{t-1}{r-1} < k < \binom{t}{r-1}$, we have $1 \le i \le \binom{t-1}{r-2} - 1$. If there exists a vertex $u \in V(K_t^r)$ such that at most $i-1$ edges are incident with both u and v_1 in H, then by $k = \binom{t-1}{r-1} + i$, we have $|E_H[\{v_1\}, V(H) \setminus \{u, v_1\}]| > \binom{t-1}{r-1}$. But this can not happen because $|V(H) \setminus \{u, v_1\}| = t-1$. Thus for any vertex $u \in V(K_t^r)$, there are at least i edges incident with both u and v_1 in H. This implies $d_H(v) \ge \binom{t-1}{r-1} + i = k$ for any $u \in V(K_t^r)$. As $d_H(v_1) = k$, we have $\delta(H) = k$.

Now we assume $n \ge t+1$ when $\binom{t-1}{r-1} = k$ and $n \ge t+2$ when $\binom{t-1}{r-1} < k$. Since $H = H_s$ is obtained from H_{s-1} by adding a new vertex v_s and k edges with cardinality r such that each added edge is incident with v_s, then by the induction assumption that $\delta(H_{s-1}) = k$, we obtain $\delta(H) = \delta(H_s) = k$.

(ii) If $n = t$ and $\binom{t-1}{r-1} = k$, then $H \cong K_t^r$ and $|E_H[X, V(H) \setminus X]| > \delta(H) = k$ for any $X \subseteq V(H)$ with $2 \le |X| \le n-2$ by (2.1). Thus H is super-edge-connected.

If $n = t+1$ and $\binom{t-1}{r-1} < k$, then H is obtained from K_t^r by adding a new vertex v_1 and k edges with cardinality r such that each added edge is incident with v_1. Let $k = \binom{t-1}{r-1} + i$. As $\binom{t-1}{r-1} < k < \binom{t}{r-1}$, we have $1 \le i \le \binom{t-1}{r-2} - 1$. In order to prove that H is super-edge-connected, we only need to verify that $d_H(X) > k$ for any $X \subseteq V(H) \setminus \{v_1\}$ with $2 \le |X| \le |V(H)| - 2$. If $|X| \le |V(H)| - 3$, then $|E_{K_t^r}[X, V(K_t^r) \setminus X]| > \binom{t-1}{r-1}$ by (2.1). Since for any vertex $u \in V(K_t^r)$, there are at least i edges incident with both u and v_1 in H (by the proof of (*i*)), we have $|E_H(X) \cap E_H(v_1)| \ge i$. Thus $d_H(X) = |E_{K_t^r}[X, V(K_t^r) \setminus X]| + |E_H(X) \cap E_H(v_1)| > \binom{t-1}{r-1} + i = k$. Assume $|X| = |V(H)| - 2$ and $V(H) \setminus X = \{v_1, w\}$. If $r \ge 3$, then $d_H(X) = |E_{K_t^r}[X, V(K_t^r) \setminus X]| + |E_H(X) \cap E_H(v_1)| = \binom{t-1}{r-1} + k > k$. If $r = 2$, then $d_H(X) = |E_{K_t^r}[X, V(K_t^r) \setminus X]| + |E_H(X) \cap E_H(v_1)| \ge \binom{t-1}{r-1} + k - 1 > k$.

Now we assume $n \ge t+1$ when $\binom{t-1}{r-1} = k$ and $n \ge t+2$ when $\binom{t-1}{r-1} < k$. On the contrary, assume H_s is not super-edge-connected. Then there is a minimum edge-cut $X = E_{H_s}[V(J_1), V(J_2)]$ of H_s with $|X| \le \delta(H_s) = k$, where J_1 is a component of $H_s - X$ and $J_2 = H_s - V(J_1)$ with $min\{|V(J_1)|, |V(J_2)|\} \ge 2$. Without loss of generality, assume $v_s \in V(J_1)$. If $E_{H_s}(v_s) \cap X \ne \emptyset$, then as $X \ne E_{H_s}(v_s)$, $X - E_{H_s}(v_s)$ is an edge-cut of H_{s-1}, and so $\kappa'(H_{s-1}) \le |X - E_{H_s}(v_s)| < k$, contradicts to the induction assumption that H_{s-1} is super-edge-connected. It follows that $E_{H_s}(v_s) \cap X = \emptyset$ and so $X = E_{H_{s-1}}[V(J_1 - v_s), V(J_2)]$ is an edge-cut of H_{s-1}. Since H_{s-1} is super-edge-connected, we conclude that either $|V(J_1 - v_s)| = 1$ or $|V(J_2)| = 1$. If $|V(J_2)| = 1$, then it contradicts to $min\{|V(J_1)|, |V(J_2)|\} \ge 2$. If $|V(J_1 - v_s)| = 1$, then $|V(J_1)| = 2$, $r = 2$ and $k = 1$, contrary to $k \ge 2$.

(iii) If $n = t$ and $\binom{t-1}{r-1} = k$, then $H \cong K_t^r$ is k-edge-maximal by the definition.

If $n = t+1$ and $\binom{t-1}{r-1} < k$, let $k = \binom{t-1}{r-1} + i$. As $\binom{t-1}{r-1} < k < \binom{t}{r-1}$, we have $1 \le i \le \binom{t-1}{r-2} - 1$. In order to prove that H is k-edge-maximal, it suffices to verify that $\overline{\kappa}'(H+e) \ge k+1$ for any $e \in E(H^c)$. By Definition 3.1, $H + e$ is obtained from K_t^r by adding a new vertex v_1 and $k+1$ edges with cardinality r such that each added edge is incident with v_1. If there exists a vertex $u \in V(K_t^r)$ such that at most i edges are incident with both u and v_1 in $H+e$, then by $k = \binom{t-1}{r-1} + i$, we have $|E_{H+e}[\{v_1\}, V(H) \setminus \{u, v_1\}]| > \binom{t-1}{r-1}$. But this can not happen because $|V(H+e) \setminus \{u, v_1\}| = t-1$. Thus for any vertex $u \in V(K_t^r)$, there are at least $i+1$ edges incident with both u and v_1 in $H+e$. This implies $d_{H+e}(u) \ge \binom{t-1}{r-1} + i + 1 = k+1$ for any $u \in V(K_t^r)$. By $d_{H+e}(v_1) = k+1$, we have $\delta(H+e) = k+1$. For any edge-cut W of $H+e$, if W is peripheral, then $|W| \ge \delta(H+e) = k+1$. Suppose W is not peripheral, and so $W - e$ is a non peripheral edge-cut of H. Since H is super-edge-connected, $|W| \ge |W-e| \ge \delta(H) + 1 = k+1$. Thus $\overline{\kappa}'(H+e) \ge \kappa(H+e) \ge k+1$.

Now we assume $n \geq t+1$ when $\binom{t-1}{r-1} = k$ and $n \geq t+2$ when $\binom{t-1}{r-1} < k$. On the contrary, assume H_s is not k-edge-maximal. Then there is an edge $e \in E(H_s^c)$ such that $\overline{\kappa}'(H_s + e) \leq k$. If $e \in E(H_{s-1}^c)$, then by induction assumption, $\overline{\kappa}'(H_{s-1} + e) \geq k+1$, a contradiction. Hence $e \notin E(H_{s-1}^c)$. Since H_s is obtained from H_{s-1} by adding a new vertex v_s and k edges incident with v_s, we have $e \in E_{H_s+e}(v_s)$.

Let $Y = E_{H_s+e}[V(F_1), V(F_2)]$ be a minimum edge-cut of $H_s + e$ with $|Y| \leq k$, where F_1 is a component of $(H_s + e) - Y$ and $F_2 = (H_s + e) - V(F_1)$. Since H_s is super-edge-connected, we have $\kappa'(H_s) = \delta(H_s) = k$, and so $e \notin Y$ and $Y \neq E_{H_s}(v_s)$. This implies $Y \subseteq E(H_s)$. Without loss of generality, assume that $v_s \in V(F_1)$. By H_{s-1} is super-edge-connected, we have $\kappa'(H_{s-1}) = \delta(H_{s-1}) = k$. If $Y \cap E_{H_s}(v_s) \neq \emptyset$, then as $Y \neq E_{H_s}(v_s)$, $Y - E_{H_s}(v_s)$ is an edge-cut of H_{s-1}. It follows that $\kappa'(H_{s-1}) \leq |Y - E_{H_s}(v_s)| < k = \kappa'(H_{s-1})$, a contradiction. Hence we must have $Y \cap E_{H_s}(v_s) = \emptyset$, and so $Y \subseteq E(H_s) - E_{H_s}(v_s) = E(H_{s-1})$. By H_{s-1} is super-edge-connected, there exists a vertex $w \in V(H_{s-1})$ such that $Y = E_{H_{s-1}}(w)$. As $N_{H_s}(v_s) \cup \{v_s\} \subseteq V(F_1)$, we have $V(F_2) = \{w\}$.

Let $H' = H_s - w$. Then $e \in E((H')^c)$. If $w \in V(H_s) \setminus V(H_0)$, then $H' \in \mathcal{M}(n-1; k, r)$. If $w \in V(H_0)$, then by $d_{H_s}(w) = |Y| = k$, we have $d_{H_1}(w) = k$. By Definition 3.1, there are exactly $k - \binom{t-1}{r-1}$ edges containing $\{w, v_1\}$ in H_1 and $|E_{H_1}[v_1, V(H_0) \setminus w]| = \binom{t-1}{r-1}$. Thus the hypergraph induced by $(V(H_0) \setminus \{w\}) \cup \{v_1\}$ in H_s is complete, and so $H' \in \mathcal{M}(n-1; k, r)$. By induction assumption, $\overline{\kappa}'(H' + e) \geq k+1$, and so $\overline{\kappa}'(H_s + e) \geq \overline{\kappa}'(H' + e) \geq k+1$, contrary to $\overline{\kappa}'(H_s + e) \leq k$. □

Theorem 3.2. *Let H be a k-edge-maximal r-uniform hypergraph on n vertices, where $k, r \geq 2$. If $n \geq t$, where $t = t(k, r)$, then each of the following holds.*

(i) $|E(H)| \leq \binom{t}{r} + (n-t)k$.

(ii) $|E(H)| = \binom{t}{r} + (n-t)k$ if and only if $H \in \mathcal{M}(n; k, r)$.

Proof. By Definition 3.1, we have $|E(H)| = \binom{t}{r} + (n-t)k$ if $H \in \mathcal{M}(n; k, r)$.

We will prove the theorem by induction on n. If $n = t$, then by H is k-edge-maximal and $\binom{t-1}{r-1} \leq k$, we have $H \cong K_t^r$. Thus $|E(H)| = \binom{t}{r} + (n-t)k$ and $H \in \mathcal{M}(n; k, r)$.

Now suppose $n > t$. We assume that if $t \leq n' < n$ and if H' is a k-edge-maximal r-uniform hypergraph with n' vertices, then $|E(H')| \leq \binom{t}{r} + (n'-t)k$ and $H' \in \mathcal{M}(n'; k, r)$ if $|E(H')| = \binom{t}{r} + (n'-t)k$.

Let X be a minimum edge-cut H. By Lemma 2.1, we have $|X| = k$. We consider two cases in the following.

Case 1. There is a component, say H_1, of $H - X$ such that $|V(H_1)| = 1$.

Let $H_2 = H - V(H_1)$. By Lemma 2.2, H_2 is k-edge-maximal. Since $|V(H_2)| = n-1 \geq t$, by induction assumption, we have $|E(H_2)| \leq \binom{t}{r} + (n-1-t)k$ and $H_2 \in \mathcal{M}(n-1; k, r)$ if $|E(H_2)| = \binom{t}{r} + (n-1-t)k$. Thus $|E(H)| = |E(H_2)| + k \leq \binom{t}{r} + (n-t)k$. If $|E(H)| = \binom{t}{r} + (n-t)k$, then $|E(H_2)| = \binom{t}{r} + (n-1-t)k$ and $H_2 \in \mathcal{M}(n-1; k, r)$. Thus, by $|V(H_1)| = 1$ and $|X| = k$, we have $H \in \mathcal{M}(n; k, r)$ if $|E(H)| = \binom{t}{r} + (n-t)k$.

Case 2. Each component of $H - X$ has at least two vertices.

Let H_1 be a component of $H - X$ and $H_2 = H - V(H_1)$. By Lemma 2.2, both H_1 and H_2 are k-edge-maximal. Assume $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n_1 + n_2 = n$. Since each edge contains r vertices, we have $n_1, n_2 \geq r$. By Lemma 2.3, we have $n_1, n_2 \geq t$. By induction assumption, we have $|E(H_i)| \leq \binom{t}{r} + (n_i - t)k$ and $H_i \in \mathcal{M}(n_i; k, r)$ if $|E(H_i)| = \binom{t}{r} + (n_i - t)k$ for $i \in \{1, 2\}$. Thus

$$\begin{aligned}
|E(H)| =& |E(H_1)| + |E(H_2)| + k \\
\leq& \tbinom{t}{r} + (n_1 - t)k + \tbinom{t}{r} + (n_2 - t)k + k \\
=& \tbinom{t}{r} + (n_1 + n_2 - t)k + \tbinom{t}{r} - (t-1)k
\end{aligned}$$

$$\begin{aligned}
&\le \tbinom{t}{r} + (n_1+n_2-t)k + \tbinom{t}{r} - (t-1)\tbinom{t-1}{r-1}\\
&= \tbinom{t}{r} + (n_1+n_2-t)k + \Big(\frac{t}{r} - (t-1)\Big)\tbinom{t-1}{r-1}\\
&\le \tbinom{t}{r} + (n-t)k.
\end{aligned}$$

If $|E(H)| = \binom{t}{r} + (n-t)k$, then $\frac{t}{r} - (t-1) = 0$ and $k = \binom{t-1}{r-1}$, which imply $t = r = 2$ and $k = 1$, contrary to $k \ge 2$. Thus $|E(H)| < \binom{t}{r} + (n-t)k$ holds. □

If $r = 2$, then H is a graph and $t = k+1$. Mader's [8] result for the upper bound of the sizes of k-edge-maximal graphs is a corollary of Theorem 3.2.

Corollary 3.3[8]**.** *Let G be a k-edge-maximal graph with n vertices, where $k \ge 2$. If $n \ge k+1$, then we have $|E(G)| \le \binom{k+1}{2} + (n-k-1)k = \binom{k}{2} + (n-k)k$. Furthermore, $|E(G)| = \binom{k}{2} + (n-k)k$ if and only if $G \in \mathcal{M}(n;k,2)$.*

4 The Lower Bound of the Sizes of k-edge-maximal r-uniform Hypergraphs

Theorem 4.1. *Let H be a k-edge-maximal r-uniform hypergraph with n vertices, where $k, r \ge 2$. If $n \ge t$, where $t = t(k,r)$, then we have $|E(H)| \ge (n-1)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n}{t} \rfloor$.*

Proof. We will prove the theorem by induction on n. If $n = t$, then by H is k-edge-maximal and $\binom{t-1}{r-1} \le k$, we have $H \cong K_t^r$. Thus $|E(H)| = \binom{t}{r} = (n-1)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n}{t} \rfloor$.

Now suppose $n > t$. We assume that if $t \le n' < n$ and if H' is a k-edge-maximal r-uniform hypergraph with n' vertices, then $|E(H')| \ge (n'-1)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n'}{t} \rfloor$.

Let X be a minimum edge-cut H. By Lemma 2.1, we have $|X| = k$. We consider two cases in the following.

Case 1. There is a component, say H_1, of $H - X$ such that $|V(H_1)| = 1$.

Let $H_2 = H - V(H_1)$. By Lemma 2.2, H_2 is k-edge-maximal. Since $|V(H_2)| = n-1 \ge t$, by induction assumption, we have $|E(H_2)| \ge (n-2)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n-1}{t} \rfloor$. Thus

$$\begin{aligned}
|E(H)| =& |E(H_2)| + k\\
\ge& (n-1)k - ((t-1)k - \tbinom{t}{r})\Big\lfloor \frac{n-1}{t} \Big\rfloor\\
\ge& (n-1)k - ((t-1)k - \tbinom{t}{r})\Big\lfloor \frac{n}{t} \Big\rfloor,
\end{aligned}$$

the last inequality holds because $(t-1)k - \binom{t}{r} \ge (t-1)\binom{t-1}{r-1} - \frac{t}{r}\binom{t-1}{r-1} \ge 0$.

Case 2. Each component of $H - X$ has at least two vertices.

Let H_1 be a component of $H - X$ and $H_2 = H - V(H_1)$. By Lemma 2.2, both H_1 and H_2 are k-edge-maximal. Assume $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n_1 + n_2 = n$. Since each edge contains r vertices, we have $n_1, n_2 \ge r$. By Lemma 2.3, we have $n_1, n_2 \ge t$. By induction assumption, we have $|E(H_i)| \ge (n_i - 1)k - ((t-1)k - \binom{t}{r})\lfloor \frac{n_i}{t} \rfloor$ for $i \in \{1, 2\}$. Thus

$$\begin{aligned}
|E(H)| =& |E(H_1)| + |E(H_2)| + k\\
\ge& (n_1-1)k - ((t-1)k - \tbinom{t}{r})\Big\lfloor \frac{n_1}{t} \Big\rfloor + (n_2-1)k - ((t-1)k - \tbinom{t}{r})\Big\lfloor \frac{n_2}{t} \Big\rfloor + k\\
=& (n-1)k - ((t-1)k - \tbinom{t}{r})\Big(\Big\lfloor \frac{n_1}{t} \Big\rfloor + \Big\lfloor \frac{n_2}{t} \Big\rfloor\Big)\\
\ge& (n-1)k - ((t-1)k - \tbinom{t}{r})\Big\lfloor \frac{n_1+n_2}{t} \Big\rfloor
\end{aligned}$$

$$=(n-1)k-((t-1)k-\tbinom{t}{r})\left\lfloor\frac{n}{t}\right\rfloor.$$

The theorem thus holds. □

Definition 4.1. *Let k,t,r be integers such that $t>r>2$, $k=\binom{t-1}{r-1}$ and $kr\geq 2t$. Assume $n=st$, where $s\geq 2$. For any tree T with $V(T)=\{v_1,\cdots,v_s\}$, we define a family of r-uniform hypergraphs $\mathcal{N}(T)$ as follows. Firstly, we replace each v_i by a complete r-uniform hypergraph $K_t^r(i)$ with t vertices. Then whenever there is an edge $v_iv_j\in E(T)$, we add a set E_{ij} of k edges with cardinality r such that (i) $e\subseteq V(K_t^r(i))\cup V(K_t^r(j))$, $e\cap V(K_t^r(i))\neq\emptyset$ and $e\cap V(K_t^r(j))\neq\emptyset$ for any $e\in E_{ij}$, and (ii) each vertex in $V(K_t^r(i))\cup V(K_t^r(j))$ is incident with some edge in E_{ij} (we can do this because $kr\geq 2t$).*

Theorem 4.2. *If $H\in\mathcal{N}(T)$, then H is a k-edge-maximal r-uniform hypergraph.*

Proof. By definition, $\overline{\kappa}'(H)\leq k$. We will prove the theorem by induction on s. If $s=2$, then $|V(H)|=2t$ and $\delta(H)\geq\binom{t-1}{r-1}+1=k+1$. Since $K_t^r(1)$ and $K_t^r(2)$ are super-edge-connected and $\delta(H)\geq\binom{t-1}{r-1}+1=k+1$, each edge-cut of H except for $E_H[V(K_t^r(1),V(K_t^r(2)]$ has cardinality at least $k+1$. For any $e\in E(H^c)$, we have $e\in E_{H^c}[V(K_t^r(1),V(K_t^r(2)]$. Thus every edge-cut of $H+e$ has cardinality at least $k+1$, that is, $\kappa'(H+e)\geq k+1$. This shows $\overline{\kappa}'(H+e)\geq\kappa'(H+e)\geq k+1$, and thus H is k-edge-maximal.

Now suppose $s\geq 3$. We assume that each hypergraph constructed in Definition 4.1 with less than st vertices is k-edge-maximal. In the following, we will show that each H in $\mathcal{N}(T)$ with st vertices is also k-edge-maximal.

By contradiction, assume that there is an edge $e\in E(H^c)$ such that $\overline{\kappa}'(H+e)\leq k$. Let $E_{H+e}[X,V(H)\setminus X]$ be an edge-cut in $H+e$ with cardinality at most k. Since $K_t^r(i)$ is super-edge-connected for $1\leq i\leq s$ and $\delta(H)\geq k+1$, edge-cuts in H with cardinality at most k are these E_{ij}, where $v_iv_j\in E(T)$. Thus $E_{H+e}[X,V(H)\setminus X]=E_{ij}$ for some $1\leq i,j\leq s$ with $v_iv_j\in E(T)$. Then $e\in E_{H+e}(H_i+e)$, where H_i is a component of $H-E_{ij}$. Since $H_i\in\mathcal{N}(T_i)$, where T_i is a components of $T-v_iv_j$, by induction assumption, H_i+e contains a subhypergraph H' with $\kappa'(H')\geq k+1$. But H' is also a subhypergraph of $H+e$, contrary to $\overline{\kappa}'(H+e)\leq k$. □

For any $H\in\mathcal{N}(T)$, we have $|E(H)|=(n-1)k-((t-1)k-\binom{t}{r})\lfloor\frac{n}{t}\rfloor$. By Theorem 4.2, H is k-edge-maximal. Thus, the lower bound given in Theorem 4.1 is best possible.

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