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# Bounding $\ell$-edge-connectivity in edge-connectivity

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ABSTRACT

For a connected graph $G$, let $\kappa'(G)$ be the edge-connectivity of $G$. The $\ell$-**edge-connectivity** $\kappa'_\ell(G)$ of $G$ with order $n \geq \ell$ is the minimum number of edges that are required to be deleted from $G$ to produce a graph with at least $\ell$ components. It has been observed that while both $\kappa'(G)$ and $\kappa'_\ell(G)$ are related edge connectivity measures. In general, $\kappa'_\ell(G)$ cannot be upper bounded by a function of $\kappa'(G)$. Let $\overline{\kappa}'(G) = \max\{\kappa'(H) : H \subseteq G\}$ be the maximum subgraph edge-connectivity of $G$. We prove that for integers $k'$, $k$ and $\ell$ with $k' \geq k \geq 1$ and $\ell \geq 2$, each of the following holds.

(i) $\sup\{\kappa'_\ell(G) : \kappa'(G) = k, \overline{\kappa}'(G) = k'\} = k + (\ell - 2)k'$.

(ii) $\inf\{\kappa'_\ell(G) : \kappa'(G) = k, \overline{\kappa}'(G) = k'\} = \frac{k\ell}{2}$.



## 1. The problem

Graphs in this paper are finite and loopless. Undefined terms and notations can be found in [2]. For a graph $G$, $c(G)$ denotes the number of components of $G$. We write $H \subseteq G$ to mean $H$ is a subgraph of $G$. For vertex subsets $S$ and $S'$ of a graph $G$, define

$$[S, S']_G = \{uv \in E(G) : u \in S, \ \& \ v \in S'\}.$$

When $G$ is understood from the context, we often use $[S, S']$ for $[S, S']_G$. Define $\partial_G(S) = [S, V(G) - S]$, and when $S = \{v\}$, we often write $\partial_G(v)$ for $\partial_G(\{v\})$. If $H$ is a non-empty non-spanning subgraph of $G$, we often use $\partial_G(H)$ for $\partial_G(V(H))$. An **edge-cut** of a (not necessarily connected) graph $G$ is an edge subset of the form $\partial_G(S)$, for some nonempty proper subset $S$ of $V(G)$. A minimal edge-cut of $G$ is called a **bond**. The **edge-connectivity** $\kappa'(G)$ of a connected graph $G$ is the minimum cardinality of an edge-cut of $G$.

Matula [8] initiated the study of edge-connectivity of subgraphs. He defined

$$\overline{\kappa}'(G) = \max_{H \subset G} \kappa'(H),$$

and considered $\overline{\kappa}'(G)$ as a useful tool to investigate the cohesiveness of a network when modeled as a graph. Matula published a number of papers on the cohesiveness of networks, as seen in [8–10]. The extremal properties related to $\overline{\kappa}'(G)$ were investigated by Mader and others, which can be found in [5–7], among others. More generally, let $f(G)$ denote

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a density measure of $G$, one can define $\overline{f}(G) = \max\{f(H) : H \text{ is a subgraph of } G\}$. As indicated in [4], for certain network topology measures $f$, a network modeled as a graph $G$ with $f(G) = \overline{f}(G)$ is considered as uniformly dense in the measure $f$, and is of particular interest to be investigated. Thus we define a graph $G$ to be **edge-uniformly dense** if $\kappa'(G) = \overline{\kappa}'(G)$.

For an integer $\ell \geq 2$, Boesch and Chen [1] defined the $\ell$-**edge-connectivity** $\kappa'_\ell(G)$ of a connected graph $G$ of order $n \geq \ell$ to be the minimum number of edges that are required to be deleted from $G$ to produce a graph with at least $\ell$ components. As when $\ell = 2$, $\kappa'_2(G) = \kappa'(G)$, the edge-connectivity of $G$, the notion $\kappa'_\ell(G)$ is often considered as a generalization of edge-connectivity. Boesch and Chen proved the following.

**Theorem 1.1** (*Boesch and Chen [1]*). $\kappa'_\ell(K_n) = \frac{(\ell-1)(2n-\ell)}{2}$.

There have been quite a few studies on the behavior of $\kappa'_\ell(G)$, as seen in Oellermann's survey [11]. The following example indicates that, in general, $\kappa'_\ell(G)$ cannot be bounded above by a function of $\kappa'(G)$.

**Example 1.2.** Let $k, \ell, n$ be positive integers with $n \geq k+\ell \geq k+2$, and let $G := G(n, \ell, k)$ be a graph obtained from the complete graph $K_{n-1}$ by adding a new vertex $v_0 \notin V(K_{n-1})$ and by adding $k$ new edges joining $v_0$ to $k$ distinct vertices in $K_{n-1}$. As $n \geq k + \ell$, it is routine to verify that $\kappa'(G) = k$. Suppose that $X \subseteq E(G)$ is an edge subset such that $G - X$ has $\ell$ components $G_1, G_2, \ldots, G_\ell$ with $\kappa'_\ell(G) = |X|$. By symmetry, we assume that $v_0 \in V(G_1)$. If $V(G_1) = v_0$, then $\partial_G(v_0) \subseteq X$ and $X' := X - \partial_G(v_0)$ is an edge subset of $K_{n-1}$ with $|X'| \geq \kappa'_{\ell-1}(K_{n-1})$. In this case, by Theorem 1.1,

$$|X| = |\partial_G(v_0)| + |X'| \geq k + \frac{(\ell-2)(2n-\ell+1)}{2}.$$

If $|V(G_1)| \geq 2$, then we may assume that $\partial_G(v_0) \cap X = \emptyset$, and so $X \subseteq E(K_{n-1})$ with $|X| \geq \kappa'_\ell(K_{n-1})$. Thus by Theorem 1.1,

$$\kappa'_\ell(G) = |X| \geq \min\left\{k + \frac{(\ell-2)(2n-\ell+1)}{2}, \frac{(\ell-1)(2n-\ell-2)}{2}\right\}.$$

Since the number $n$ can be arbitrarily large, we conclude that $\kappa'_\ell(G)$ cannot be bounded by a function of $\kappa'(G)$.

As uniformly dense networks are of application importance (see [4]), it is of interest to study whether $\kappa'_\ell(G)$ can be bounded by a function of $\kappa'(G)$ among all edge uniformly dense graphs. To investigate this problem, for given integers $\ell$ and $k$, and a family $\mathcal{F}$ of connected graphs with order at least $\ell$, we define

$$\Phi(\ell, k; \mathcal{F}) = \sup\{\kappa'_\ell(G) : G \in \mathcal{F}, \kappa'(G) = k\} \text{ and } \phi(\ell, k; \mathcal{F}) = \inf\{\kappa'_\ell(G) : G \in \mathcal{F}, \kappa'(G) = k\}.$$

The main result of this paper is the following.

**Theorem 1.3.** *Let $\ell$ and $k$ be integers with $\ell > 1$ and $k \geq 1$, and $\mathcal{G}^u$ be the family of all edge-uniformly dense graphs of order at least $\ell$. Each of the following holds.*
*(i) $\Phi(\ell, k; \mathcal{G}^u) = (\ell-1)k$.*
*(ii) $\phi(\ell, k; \mathcal{G}^u) = \frac{k\ell}{2}$.*

In order to prove Theorem 1.3, we take a slightly more general approach to relax the edge-uniformly dense constraint by allowing bounded value of maximum subgraph edge-connectivity. Let $\ell, k, k'$ be integers with $\ell \geq 2$ and $k' > k \geq 1$ and $\mathcal{G}_{k'}$ denote the family of all connected graphs with order at least $\ell$ such that every graph $G \in \mathcal{G}_{k'}$ satisfies $\overline{\kappa}'(G) = k'$ and $\kappa'(G) = k$. As when $k = k'$, we have $\mathcal{G}_k = \mathcal{G}^u$, Theorem 1.3 will be the special case of Theorem 1.4 when $k = k'$.

**Theorem 1.4.** *Let $\ell$ and $k$ be integers with $\ell \geq 2$ and $k \geq 1$, and $\mathcal{G}_\ell$ be the family of all connected graphs of order at least $\ell$. Each of the following holds.*
*(i) $\Phi(\ell, k; \mathcal{G}_{k'}) = k + (\ell-2)k'$.*
*(ii) $\phi(\ell, k; \mathcal{G}_{k'}) = \frac{k\ell}{2}$.*

## 2. Proofs of the main results

A graph $G$ is a trivial graph if it has at least one vertex and is edgeless. Throughout the rest of this paper, we let $k', k, \ell$ be integers with $k' \geq k > 0$ and $\ell \geq 2$. While it is known that $\kappa'(G) = \kappa'_2(G)$, we will continue using $\kappa'(G)$ instead of $\kappa'_2(G)$ in our discussions.

**Lemma 2.1.** *Let $G$ be a connected graph. Then for any subgraph $H$ of $G$ with $|V(H)| \geq \ell$, we have $(\ell-2)\overline{\kappa}'(G) + \kappa'(H) \geq \kappa'_\ell(H)$.*

**Proof.** Let $H$ be a subgraph of $G$ with $|V(H)| \geq \ell$. Pick a minimum edge-cut $Z_1$ of $H$. Assuming that for a fixed $j$ with $1 \leq j < \ell-1$, $Z_j$ has been found. Then as $|V(H)| \geq \ell$, at least one component of $H - (\cup_{i=1}^{j} Z_i)$ is nontrivial. Fix one nontrivial component of $H - (\cup_{i=1}^{j} Z_i)$ and let $Z_{j+1}$ be a minimum edge-cut of this nontrivial component. Thus we have generated a sequence $(Z_1, Z_2, \ldots, Z_{\ell-1})$ of subsets of $H$, such that $|Z_1| = \kappa'(H)$ and for each $i$ with $2 \leq i \leq \ell-1$, $|Z_i| \leq \overline{\kappa}'(H) \leq \overline{\kappa}'(G)$. Furthermore, by our choices of the $Z_i$'s, $H - (\cup_{i=1}^{\ell-1} Z_i)$ has exactly $\ell$ components. Thus $(\ell-2)\overline{\kappa}'(G) + \kappa'(H) \geq \sum_{i=1}^{\ell-1} |Z_i| \geq \kappa'_\ell(H)$, which leads to Lemma 2.1. ■

**Proposition 2.2** (*Zhang et al. Theorem 2.5 of [12]*). *Suppose that $G$ is a connected graph with $|V(G)| \geq \ell$. Then*

$$\kappa'_\ell(G) \geq \frac{\ell}{2}\kappa'(G).$$

**Proof.** Let $X$ be an edge subset of $E(G)$ such that $G - X$ has components $G_1, G_2, \ldots, G_\ell$ with $|X| = \kappa'_\ell(G)$. Thus for each $i$ with $1 \leq i \leq \ell$, $|\partial_G(V(G_i))| \geq \kappa'(G)$ and $2\kappa'_\ell(G) = 2|X| = \sum_{i=1}^{\ell} |\partial_G(V(G_i))| \geq \ell\kappa'(G)$. This proves the proposition. ■

By combining Lemma 2.1 (with $H = G$) and Proposition 2.2, we conclude that for any connected graph $G$ with $|V(G)| \geq \ell$,

$$(\ell - 2)\overline{\kappa}'(G) + \kappa'(G) \geq \kappa'_\ell(G) \geq \frac{\ell}{2}\kappa'(G). \tag{1}$$

### 2.1. Proof of Theorem 1.4(i)

To prove Theorem 1.4(i), we shall show, for given integers $k' \geq k \geq 1$ and $\ell \geq 2$, the existence of infinitely many graphs $G$ with $\kappa'(G) = k$, and $\overline{\kappa}'(G) = k'$ such that the upper bound in (1) will be reached. In this subsection, we shall construct an infinite family of graphs satisfying the expected edge-connectivity constraints and reaching the upper bound in (1), which implies Theorem 1.4(i). The following is the main result.

**Proposition 2.3.** *For any integers $k', k, \ell$ with $\ell \geq k'+1 > 2$ and $k' \geq k$, there exists an infinite graph family $\mathcal{F}_1 = \mathcal{F}_1(\ell, k', k)$ such that for any graph $H \in \mathcal{F}_1$, each of the following holds.*
*(i) (Example 1 and Theorem 1 of [5]) If $k' = k$, then $H$ is edge-uniformly dense with $\kappa'(H) = k$.*
*(ii) $\overline{\kappa}'(H) = k'$ and $\kappa'(H) = k$.*
*(iii) $\kappa'_\ell(H) = (\ell - 2)\overline{\kappa}'(H) + \kappa'(H)$.*

**Proof.** We are to construct this family of graphs to justify Proposition 2.3. For an integer $m \geq 1$, define $mG$ to be the disjoint union of $m$ copies of $G$. Hence $1G = G$. Following [2], for two vertex disjoint graphs $G, G'$, let $G \vee G'$ denote the **join** of $G$ and $G'$, which is a graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G') \cup \{uv : u \in V(G), v \in V(G')\}$. Extending a graph construction idea in [5], we construct the following graph family. For any integers $k', k$ and $n$ with $n > k + 1$, let $H_0 \cong K_{k'}$ be a complete graph with vertex set $V(H_0) = \{v_1, v_2, \ldots, v_{k'}\}$ and let the vertex set of $(n-k)K_1$ be $W := \{w_1, w_2, \ldots, w_{n-k}\}$. Define

$$H(k'; n - k) = (H_0 \vee (n-k)K_1) - \{w_1v_j : k + 1 \leq j \leq k'\}. \tag{2}$$

When $k' = k$, $H(k'; n - k)$ is precisely the same graph $H(k, n - k)$ constructed in [5]. We are to prove (ii) and (iii) of the proposition. Define

$$N_0 = \max\left\{2\ell + k, \ell + 2k' + k, 5\ell + k - 7\right\}. \tag{3}$$

and $\mathcal{F}_1 = \{H(k'; n - k) : k' \geq k, \ell \geq k + 1 > 2, n \geq N_0\}$. To prove (ii), we assume that $k' > k$ as otherwise we may turn to (i). Randomly pick a member $H \in \mathcal{F}_1$. By the definition of $\mathcal{F}_1$, $\partial_H(w_1)$ is the only edge cut in $H$ of size $k$, and so $\kappa'(H) = k$. Now let $H'$ be a subgraph of $H$ with $\overline{\kappa}'(H) = \kappa'(H')$. If $\kappa'(H') > k'$, then as every vertex in $W$ has degree at most $k'$ in $H$, we conclude that $V(H') \cap W = \emptyset$. Hence $H'$ is a subgraph of $H_0$, a complete graph of order $k'$. This implies that $k' < \kappa'(H') \leq k' - 1$, a contradiction. This implies that $\overline{\kappa}'(H) = \kappa'(H') \leq k'$. On the other hand, $H$ contains $K_{k'+1}$ as a subgraph, and so $\overline{\kappa}'(H) \geq \kappa'(K_{k'+1}) \geq k'$, implying that $\overline{\kappa}'(H) = k'$. This proves (ii).

It remains to prove (iii). Let $X \subseteq E(H)$ be an edge subset such that $H - X$ has at least $\ell$ components and that $|X| = \kappa'_\ell(H)$. By (1), it suffices to show that

$$|X| \geq (\ell - 2)k' + k.$$

As $|X| = \kappa'_\ell(H)$, $H - X$ must have exactly $\ell$ components $H_1, H_2, \ldots, H_\ell$. Since $\ell \geq k' + 1$ and $|V(H_0)| = k'$, without loss of generality, we may assume that $V(H_1) \cap V(H_0) = \emptyset$. This implies that $V(H_1) \subseteq W$ and $|V(H_1)| = 1$. Therefore, there must be at least one of $H_i$'s that consists of only one vertex in $W$. Without loss of generality, we assume that for some integer $s$ with $1 \leq s \leq \ell$ such that

$$\text{every } H_j \text{ with } s \leq j \leq \ell \text{ consists of a single vertex in } W, \tag{4}$$

and the $H_j$'s ($s \leq j \leq \ell$) are so labeled that

$$|\partial_H(V(H_s))| \geq |\partial_H(V(H_{s+1}))| \geq \cdots \geq |\partial_H(V(H_\ell))|,$$

and that every $H_{j'}$ with $1 \leq j' \leq s - 1$ satisfies $V(H_{j'}) \cap V(H_0) \neq \emptyset$. Depending on whether an $H_i$ contains a vertex in $W$ or not, we further partitioned $H_1, \ldots, H_{s-1}$ into two parts and assume that there exists an integer $s' < s$ such that

$$\text{for any } H_t \in \{H_1, \ldots, H_{s'-1}\},\ V(H_t) \cap W = \emptyset, \tag{5}$$

and

$$\text{for any } H_t \in \{H_{s'}, \dots, H_{s-1}\},\ V(H_t) \cap W \neq \emptyset. \tag{6}$$

Thus for any $j$ with $1 \le j \le \ell$, $\partial_H(V(H_j)) \subseteq X$. By (2) and (4), we conclude that for any $H_j$ with $s \le j \le \ell$,

$$k \le |\partial_H(V(H_\ell))| \le k' = |\partial_H(V(H_j))|, \text{ where } s \le j \le \ell - 1. \tag{7}$$

By (5), for any $j$ with $1 \le j \le s' - 1$, $H_j$ is a subgraph of $H_0 \cong K_{k'}$. By (2), every vertex in this $H_j$ must be adjacent to every vertex in $W$, and so we conclude that for any $H_j$ with $1 \le j \le s' - 1$,

$$|\partial_H(V(H_j))| = |[V(H_j), W]_H| + |\partial_{H_0}(V(H_j))| = (n-k)|V(H_j)| + |\partial_{H_0}(V(H_j))|. \tag{8}$$

Fix an index $j$ with $s' \le j \le s - 1$, let $W_j = W \cap V(H_j)$, and $H'_j = H_j - W_j$. Thus $H'_j$ is a subgraph of $H_0 \cong K_{k'}$. By (2), we have, for any $H_j$ with $s' \le j \le s - 1$,

$$|\partial_H(V(H_j))| = |[W_j, V(H_0) - V(H'_j)]_H| + |\partial_{H_0}(V(H'_j))| + |[V(H'_j), W - W_j]_H|. \tag{9}$$

To estimate $X$, we set

$$\begin{aligned} X_1 &= \bigcup_{j=s}^{\ell} \partial_H(V(H_j)), \\ X'_2 &= \bigcup_{j=s'}^{s-1} [W_j, V(H_0) - V(H'_j)]_H \text{ and } X''_2 = \bigcup_{j=s'}^{s-1} \partial_{H_0}(V(H'_j)), \\ X'_3 &= \bigcup_{j=1}^{s'-1} [V(H_j), W]_H \text{ and } X''_3 = \bigcup_{j=1}^{s'-1} \partial_{H_0}(V(H_j)), \\ X'' &= X''_2 \cup X''_3. \end{aligned} \tag{10}$$

Thus $X = X_1 \cup X'_2 \cup X''_2 \cup X'_3 \cup X''_3$. Note that some of these sets defined in (10) could be empty. Recall that $\ell \ge s \ge 2$. If $s = 2$, then $H_1 = H_0 \cong K_k$ and so $X'_3 = [V(H_1), W]_H = \partial_H(H_1)$, $X'_2 = X''_2 = X''_3 = \emptyset$. Thus $X = X_1 = X'_3$, and so by (7), $\kappa'_\ell(H) = |X| \ge (\ell - 2)k' + k$. This, together with (1), implies (iii). Hence in the following we always assume that $s > 2$.

By their definitions in (10), $X_1$, $X'_2$, and $X''$ are mutually edge-disjoint, and each of $X_1$ and $X'_2$ is an edge-disjoint union, and $X'_3 \subseteq X_1 \cup X'_2$, whereas $X''$ is an edge subset of $H_0 \cong K_{k'}$ such that $H_0 - X''$ has $s - 1$ components. This gives us a way to apply (7), (8), (9) and Theorem 1.1 to estimate $X$, as follows.

$$\begin{aligned} |X| &= |X_1| + |X'_2| + |X''| \\ &\ge \sum_{j=s}^{\ell} |\partial_H(V(H_j))| + \sum_{j=s'}^{s-1} |[W_j, V(H_0) - V(H'_j)]_H| + \kappa'_{s-1}(K_{k'}) \\ &\ge (\ell - s)k' + k + \epsilon + \sum_{j=s'}^{s-1} |W_j| \cdot |V(H_0) - V(H'_j)| + \frac{(s-2)(2k' - s + 1)}{2}, \end{aligned} \tag{11}$$

where

$$\epsilon = \begin{cases} k' - k & \text{if } |\partial_H(V(H_\ell))| = k' \\ 0 & \text{if } |\partial_H(V(H_\ell))| = k. \end{cases}$$

Let $n' = \sum_{j=s'}^{s-1} |W_j|$. Then by (2), $n' = (n - k) - (\ell - s + 1)$. Without loss of generality, we may assume that

$$|V(H'_{s'})| \ge |V(H'_{s'+1})| \ge \cdots \ge |V(H'_{s-1})|.$$

Suppose first that $|V(H'_{s'})| \le \frac{k'}{2}$. Then for any $j$ with $s' \le j \le s - 1$, we have $|V(H_0) - V(H'_j)| = k' - |V(H'_j)| \ge \frac{k'}{2}$. By (3), we have $n \ge 2\ell + k$. This, together with $\ell \ge s > 2$, implies that $n' = n - k - \ell + s - 1 \ge 2s - 2$. Hence by (11), we have

$$\begin{aligned} |X| &\ge (\ell - s)k' + k + \epsilon + \sum_{j=s'}^{s-1} |W_j| \cdot |V(H_0) - V(H'_j)| + \frac{(s-2)(2k' - s + 1)}{2} \\ &\ge (\ell - s)k' + k + \epsilon + \frac{k'}{2} \sum_{j=s'}^{s-1} |W_j| + \frac{(s-2)(2k' - s + 1)}{2} \\ &= (\ell - s)k' + k + \epsilon + \frac{k'n'}{2} + \frac{(s-2)(2k' - s + 1)}{2} \end{aligned}$$

$$\geq (\ell - s)k' + k + \epsilon + \frac{k'(2s-2)}{2} + \frac{(s-2)(2k'-s+1)}{2}$$
$$> (\ell - s)k' + k + \epsilon + k'(s-1) + \frac{(s-2)(2k'-s+1)}{2} > (\ell-2)k' + k.$$

Hence in the following, we may assume that $|V(H'_{s'})| > \frac{k'}{2}$.

**Case 2.4.** $|W_{s'}| \geq \frac{n'}{2}$.

By (2), every vertex in $W_{s'}$ is adjacent to every vertex in $V(H_0) - V(H_{s'})$. As $H_1, H_2, \dots, H_{s'-1}, H_{s'+1}, \dots, H_{s-1}$ are vertex disjoint subgraphs of $H_0$, it follows that

$$|V(H_0) - V(H_{s'})| = \sum_{j=1}^{s'-1} |V(H'_j)| + \sum_{j=s'+1}^{s-1} |V(H'_j)| \geq s - 2.$$

By (11) and by (3), we have $n \geq 2k' + k + \ell$, and so $n' \geq 2k'$.

$$|X| \geq (\ell - s)k' + k + \epsilon + \sum_{j=s'}^{s-1} |W_j| \cdot |V(H_0) - V(H'_j)| + \frac{(s-2)(2k'-s+1)}{2}$$
$$\geq (\ell - s)k' + k + \epsilon + |W_{s'}| \cdot |V(H_0) - V(H_{s'})| + \frac{(s-2)(2k'-s+1)}{2}$$
$$\geq (\ell - s)k' + k + \epsilon + \frac{n'}{2}(s-2) + \frac{(s-2)(2k'-s+1)}{2}$$
$$= (\ell - s)k' + k + \epsilon + \frac{2k'}{2}(s-2) + \frac{(s-2)(2k'-s+1)}{2}$$
$$> (\ell - s)k' + k + \epsilon + (s-2)k' \geq (\ell-2)k' + k.$$

**Case 2.5.** $|W_{s'}| < \frac{n'}{2}$.

Since $|V(H'_{s'})| > \frac{k'}{2}$, it follows that for any $j$ with $s'+1 \leq j \leq s-1$, $|V(H'_j)| \leq \sum_{i=s'+1}^{s-1} |V(H'_i)| = |V(H_0)| - |V(H'_{s'})| < \frac{k'}{2}$. As $|W_{s'}| < \frac{n'}{2}$, we have $\sum_{j=s'+1}^{s-1} |W_j| = n' - |W_{s'}| > \frac{n'}{2}$. By (3), $n \geq 5\ell + k - 7$ and so $n' = n - k - \ell + s - 1 \geq 4\ell - 8 \geq 4(s-2)$. Thus by (11), we have

$$|X| \geq (\ell - s)k' + k + \epsilon + \sum_{j=s'}^{s-1} |W_j| \cdot |V(H_0) - V(H'_j)| + \frac{(s-2)(2k'-s+1)}{2}$$
$$> (\ell - s)k' + k + \epsilon + \frac{k'}{2} \sum_{j=s'+1}^{s-1} |W_j| = (\ell - s)k' + k + \epsilon + \frac{k'(n' - |W_{s'}|)}{2}$$
$$> (\ell - s)k' + k + \epsilon + \frac{kn'}{4} \geq (\ell - s)k' + k + k'(s-2) = (\ell-2)k' + k.$$

As we always have $|X| \geq (\ell-2)k' + k$, by (1), we prove Proposition 2.3. ■

### 2.2. Proof of Theorem 1.4(ii)

To prove Theorem 1.4(ii), we shall show, for given integers $k' \geq k \geq 1$ and $\ell \geq 2$, the existence of infinitely many graphs $G$ with $\kappa'(G) = k$, and $\overline{\kappa}'(G) = k'$ such that the lower bound in (1) will be reached. Following [3], we introduce circulant graphs and some definitions for constructing graphs to be used in our arguments.

**Definition 2.6.** Let $\ell, n$ be integers with $\ell \geq 2$ and $n > 1$ and denote the additive cyclic group as $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$.
(i) Let $S \subseteq \mathbb{Z}_n - \{0\}$ be a subset such that for an element $a \in \mathbb{Z}_n$, $a \in S$ if and only if $-a \in S$, where $-a$ is the additive inverse of $a$. Define the **circulant graph** $C(\mathbb{Z}_n, S)$ to be the graph with vertex set $\mathbb{Z}_n$, where $ij \in C(\mathbb{Z}_n, S)$ if and only if $i - j \in S$. The set $S$ is called its **connection set**. Using the definition of Cayley graphs in [3], circulant graphs are Cayley graphs.
(ii) Let $G$ and $J$ be vertex disjoint graphs and let $v \in V(G)$ be a vertex which is adjacent to (not necessarily distinct) vertices $v_1, v_2, \dots, v_d$ with edges $e_i = vv_i$, $1 \leq i \leq d$. Let $u_1, u_2, \dots, u_d$ be (not necessarily distinct) vertices in $J$. Define a graph $G(v; J)$ from the disjoint union of $(G - v) \cup J$ by adding edges $\{u_i v_i \ : \ 1 \leq i \leq d\}$. As the choices of $u_i$'s and $v_j$'s are not unique, $G(v; J)$ represents a family of graphs. For simplicity, we shall use $G(v; J)$ to denote any graph in the family, and say that we blow up the vertex $v$ to the graph $J$.

(iii) Let $G$ be a graph with distinct vertices $z_1, z_2, \dots, z_\ell$, and let $J_1, J_2, \dots, J_\ell$ be mutually vertex disjoint graphs, each of which is also disjoint from $G$. Let $G(z_1, z_2, \dots, z_\ell; J_1, J_2, \dots, J_\ell)$ denote the family of graphs obtained by, for each $i$ with $1 \le i \le \ell$, blowing up the vertex $z_i$ to the graph $J_i$. When there is no need to emphasize the vertices $z_1, z_2, \dots, z_\ell$, we often use $G(J_1, J_2, \dots, J_\ell)$ for $G(z_1, z_2, \dots, z_\ell; J_1, J_2, \dots, J_\ell)$. For notational simplicity, we shall use $G(J_1, J_2, \dots, J_\ell)$ to denote any member in the family.

To prove Theorem 1.4(ii), we need a few more tools for the construction of the needed graph families. A graph $G$ is **vertex transitive** if the automorphism group of $G$ acts transitively on $V(G)$.

**Lemma 2.7.** *Let $G$ be a connected graph. Each of the following holds.*
*(i) (Theorem 3.1.2, [3]) Cayley graphs are vertex transitive. As circulant graphs are Cayley graphs, all circulant graphs are vertex transitive.*
*(ii) (Lemma 3.3.3, [3]) If $G$ is vertex transitive, then $G$ is a regular graph with $\kappa'(G) = \delta(G)$.*
*(iii) Every vertex transitive graph is edge-uniformly dense.*
*(iv) Suppose that $G$ and $J$ are two vertex disjoint edge-uniformly dense graphs with $\kappa'(G) = \kappa'(J) = k$. Then $G(v; J)$ is also edge-uniformly dense with $\kappa'(G(v; J)) = k$.*
*(v) Suppose that $G$, $J_1$, $J_2$, …, $J_\ell$ are mutually vertex disjoint edge-uniformly dense graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ such that $n \ge \ell$, and let $j$ be an integer with $1 \le j \le \ell$. If $\kappa'(G) = \kappa'(J_1) = \cdots = \kappa'(J_j) = k$, then $G(u_1, u_2, \dots, u_j; J_1, J_2, \dots, J_j)$ is also edge-uniformly dense with $\kappa'(G(u_1, u_2, \dots, u_j; J_1, J_2, \dots, J_j)) = k$.*

**Proof.** Let $G$ be a vertex transitive graph with $d = \kappa'(G)$. Then $G$ is $d$-regular. Let $H \subseteq G$ be a subgraph with $\overline{\kappa}'(G) = \kappa'(H)$. If $H = G$, then $G$ is edge-uniformly dense. Assume that $H$ is a proper subgraph of $G$. Since $G$ is connected and $d$-regular, $\kappa'(H) \le \delta(H) < d$, and so $\kappa'(G) \le \overline{\kappa}'(G) = \kappa'(H) < d = \kappa'(G)$, a contradiction. Thus $G$ must be edge-uniformly dense. This justifies (iii).

Now suppose that $G$ and $J$ are edge-uniformly dense with $\kappa'(G) = \kappa'(J) = k$, and let $v \in V(G)$ be a vertex. First we explain that $\kappa'(G(v; J)) = \kappa'(G)$. Observe that $\kappa'(G(v; J)) \le \kappa'(G(v; J)/J) = \kappa'(G) = k$. Thus to show $\kappa'(G(v; J)) = \kappa'(G)$, it suffices to show that every edge cut of $G(v; J)$ has size at least $k$. Let $X$ be a minimal edge cut of $G(v; J)$. If $X \cap E(J) = \emptyset$, then $X \subseteq E(G)$ is an edge cut of $G$, whence $|X| \ge k$. Now assume that $X \cap E(J) \ne \emptyset$. Since $X$ is minimal, $X \cap E(J)$ must be an edge cut of $J$, and so $|X| \ge |X \cap E(J)| \ge \kappa'(J) = k$. Thus we must have $\kappa'(G(v; J)) = \kappa'(G) = k$.

Next, we shall show that $\overline{\kappa}'(G(v; J)) = \kappa'(G) = k$. Suppose that $H$ is a subgraph of $G(v; J)$ with $\overline{\kappa}'(G(v; J)) = \kappa'(H)$. If $E(H) \cap E(J) = \emptyset$, then $H$ is a subgraph of $G(v; J)/J = G$. As $G$ is edge-uniformly dense, we have $\overline{\kappa}'(G(v; J)) = \kappa'(H) \le \kappa'(G(v; J)/J) = \kappa'(G)$. Thus we may assume that $E(H) \cap E(J) \ne \emptyset$. Let $J_1, J_2, \dots, J_\ell$ be the connected components of the edge induced subgraph $J[E(H) \cap E(J)]$.

If $\ell \ge 2$, then add a set $W$ of new edges that connects the connected components $J_1, J_2, \dots, J_\ell$ so that, in the graph $H + W$ obtained by adding the edges in $W$ to $H$, $(H + W)[W \cup (\cup_{i=1}^{\ell} E(J_i))]$ is a connected graph. (If $\ell = 1$, then let $W = \emptyset$.) As we are adding edges to $H$, we have $\kappa'(H + W) \ge \kappa'(H)$. By definition, we have $(H + W)/[W \cup (\cup_{i=1}^{\ell} E(J_i))] = (H \cup J)/J$, which is a subgraph of $G(v; J)/J = G$. It follows that

$$k = \overline{\kappa}'(G) \ge \kappa'((H \cup J)/J) \ge \kappa'((H + W)/[W \cup (\cup_{i=1}^{\ell} E(J_i))]) \ge \kappa'(H + W) \ge \kappa'(H).$$

Since $\kappa'(H) = \overline{\kappa}'(G(v; J)) \ge \kappa'(G(v; J)) = \kappa'(G) = k$, we conclude that we always have $\overline{\kappa}'(G(v; J)) = \kappa'(G) = k$. This proves (iv).

The conclusion (v) follows from Definition 2.6 and Lemma 2.7(iv), arguing by induction on $j$. ■

**Lemma 2.8.** *Suppose that $h$ and $k$ are two integers with $h > k > 0$, and $G$ and $J$ are two vertex disjoint graphs with $k = \kappa'(G) \le \overline{\kappa}'(G) \le h$ and $\kappa'(J) = h$. Then each of the following holds.*
*(i) $G(v; J)$ satisfies with $\overline{\kappa}'(G(v; J)) \ge h$.*
*(ii) If $J$ is uniformly dense, then $\overline{\kappa}'(G(v; J)) = h$.*

**Proof.** By Definition 2.6, $J$ is a subgraph of $G(v; J)$ and so $\overline{\kappa}'(G(v; J)) \ge \kappa'(J) = h$. Hence (i) holds. It suffices to show (ii). Let $H$ be a subgraph of $G(v; J)$ with $\kappa'(H) = \overline{\kappa}'(G(v; J))$. As $\kappa'(J) = h$, we may assume that $H \ne J$. Assume first that $E(H) \cap E(J) = \emptyset$. Then by Definition 2.6, $H$ is a subgraph of $G(v; J)/J = G$. Hence

$$h = \kappa'(J) \le \overline{\kappa}'(G(v; J)) = \kappa'(H) \le \overline{\kappa}'(G) \le h,$$

forcing $\overline{\kappa}'(G(v; J)) = h$. Hence we assume that $E(H) \cap E(J) \ne \emptyset$. Let $J_1, J_2, \dots, J_\ell$ denote the connected components of $H[E(H) \cap E(J)]$, the edge induced subgraph in $H$. Let $W$ be a set of new edges such that in the graph $H + W$ obtained by adding the edges in $W$ to $H$, $(H + W)[W \cup (\cup_{i=1}^{\ell} E(J_i))]$ is a connected graph. Again we have

$$\kappa'(H) \le \kappa'(H + W) \le \kappa'((H + W)/[W \cup (\cup_{i=1}^{\ell} E(J_i))]) = \kappa'((H \cup J)/J).$$

As $(H \cup J)/J$ is a subgraph of $G(v; J)/J = G$, it follows that

$$h = \kappa'(J) \le \overline{\kappa}'(G(v; J)) = \kappa'(H) \le \kappa'((H \cup J)/J) \le \overline{\kappa}'(G) \le h,$$

and so we also have $\overline{\kappa}'(G(v; J)) = h$. This proves (ii). ■

Let $\ell, k$ and $k'$ be integers with $\ell \geq 3$ and $k' \geq k \geq 2$, we are to construct a graph family $\mathcal{G}(\ell, k', k)$ with some of the desirable properties to facilitate our justification for Theorem 1.4(ii).

**Example 2.9.** Suppose that $\ell$ and $k$ are given integers such that for some integer $s > 1$, $\ell = (k+1)s$. Let $S \subseteq \mathbb{Z}_\ell$ be the subset $S = \{s, 2s, \dots, (k-1)s, ks\}$. Then as $\ell = (k+1)s$, for any $a \in S$, we also have $-a \in S$. Thus $G = C(\mathbb{Z}_\ell, S)$ satisfies the following properties.
(i) $G$ is $k$-regular with $\kappa'(G) = k$.
(ii) $\kappa'_\ell(G) = \frac{k\ell}{2}$.

**Proof.** By Definition 2.6, the degree of vertex in $G$ is equal to $|S| = k$. By Lemma 2.7(ii), $G$ is a $k$-regular graph with $\kappa'(G) = k$. It remains to show (ii). Since $|V(G)| = \ell$, it follows that $\kappa'_\ell(G) = |E(G)| = \frac{1}{2}\sum_{v \in V(G)} d_G(v) = \frac{k\ell}{2}$. ∎

**Lemma 2.10** (*Theorem 1 and Corollary 3 of [5]*). *Let $k \geq 2$ be an integer. For any integer $n \geq k+1$, there exist edge-uniformly dense graphs $H$ with $|V(H)| = n$ and $\kappa'(H) = \overline{\kappa}'(H) = k$.*

**Proposition 2.11.** *Suppose that $\ell, k'$ and $k$ are given integers with $k' \geq k \geq 1$ and $\ell \geq 2$. There exists an infinite family $\mathcal{G}(\ell, k', k)$ of graphs such that for any $H \in \mathcal{G}(\ell, k', k)$, we have the following properties.*
*(i) $\kappa'(G) = k$ and $\overline{\kappa}'(G) = k'$.*
*(ii) $\kappa'_\ell(G) = \frac{k\ell}{2}$.*

**Proof.** Suppose that $\ell, k'$ and $k$ are given with the indicated relations. By Example 2.9, there exists a graph $C(\mathbb{Z}_\ell, S)$ such that it satisfies Example 2.9(i) and (ii) with

$$V(C(\mathbb{Z}_\ell, S)) = \{u_1, u_2, \dots, u_\ell\}.$$

Pick an integer $j$ with $1 \leq j \leq \ell$. By Lemma 2.10, there exist edge-uniformly dense graphs $J_1, J_2, \dots, J_j$ such that $\kappa'(J_i) = \overline{\kappa}'(J_i) = k$, for each $i$ with $1 \leq i \leq j$; and edge-uniformly dense graphs $J_{j+1}, J_{j+2}, \dots, J_\ell$ such that $\kappa'(J_{i'}) = \overline{\kappa}'(J_{i'}) = k'$, for each $i'$ with $j+1 \leq i' \leq \ell$. Define

$$G_1 = C(\mathbb{Z}_\ell, S)(u_1, u_2, \dots, u_j; J_1, J_2, \dots, J_j) \tag{12}$$

as in Definition 2.6(iii). By Lemma 2.7(v), $G_1$ is edge-uniformly dense with $\kappa'(G_1) = k$. We can view $u_{j+1}, \dots, u_\ell$ as vertices in $G_1$. Using the notation in Definition 2.6, let

$$G = G_1(u_{j+1}, \dots, u_\ell; J_{j+1}, J_{j+2}, \dots, J_\ell). \tag{13}$$

By Lemma 2.8 and arguing by induction on $\ell - j$, we conclude that $\kappa'(G) = k$ and $\overline{\kappa}'(G) = k'$, and so $G$ satisfies Proposition 2.11(i).

We shall show that $G$ satisfies Proposition 2.11(ii). By Definition 2.6(iii), (12) and (13), we have

$$G/(J_1 \cup J_2 \cup \dots \cup J_\ell) = C(\mathbb{Z}_\ell, S).$$

Thus by Example 2.9(ii), $\kappa'_\ell(G) \leq \kappa'_\ell(C(\mathbb{Z}_\ell, S)) = \frac{k\ell}{2}$. By (1), we much have $\kappa'_\ell(G) = \frac{k\ell}{2}$, which implies Proposition 2.11(ii).

Let $\mathcal{G}(\ell, k', k)$ denote the graph family of graphs $G$ constructed in the steps above. Then every graph $G \in \mathcal{G}(\ell, k', k)$ satisfies Proposition 2.11(i) and (ii). Hence Proposition 2.11 follows. ∎

By (1) and Proposition 2.11, we conclude that Theorem 1.4(ii) must hold. This completes the proof of the theorem.

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## References

[1] F.T. Boesch, S. Chen, A generalization of line conectivity and optimally invulnerable graphs, SIAM J. Appl. Math. 34 (1978) 657–665.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[3] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.
[4] A.M. Hobbs, Survivability of networks under attack, in: John G. Michaels, Kenneth H. Rosen (Eds.), Applications of Discrete Mathematics, 1991, pp. 332–353.
[5] H.-J. Lai, The size of strength-maximal graphs, J. Graph Theory 14 (1990) 187–197.
[6] H.-J. Lai, C.-Q. Zhang, Edge-maximal $(k, l)$-graphs, J. Graph Theory 18 (1994) 227–240.
[7] D. Mader, Minimale $n$-fach kantenzusammenhängende graphen, Math. Ann. 191 (1971) 21–28.
[8] D.W. Matula, The cohesive strength of graphs, in: The Many Facets of Graph Theory, Springer-Verlag, Berlin, 1969, pp. 215–221.
[9] D.W. Matula, K-components, clusters, and slicings in graphs, SIAM J. Appl. Math. 22 (1972) 21–28.
[10] D.W. Matula, Subgraph connectivity numbers of a graph, in: A. Dold, B. Eckmann (Eds.), in: Lecture Notes in Mathematics, vol. 642, Springer, New York, 1976, pp. 371–383.
[11] O.R. Oellermann, Explorations into graph connectivity, Not. S. Afr. Math. Soc. 20 (1988) 117–151.
[12] L. Zhang, K. Hennayake, H.-J. Lai, Y. Shao, A lower bound of the $\ell$-edge-connectivity and optimal graphs, J. Combin. Math. Combin. Comput. 66 (2008) 79–95.