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Strengthened Ore conditions for (s,t)-supereulerian graphs

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ABSTRACT

For integers $s \geq 0$ and $t \geq 0$, a graph G is (s,t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Pulleyblank (1979) showed that determining whether a graph is $(0,0)$-supereulerian, even when restricted to planar graphs, is NP-complete. Settling an open problem of Bauer, Catlin in (Catlin, 1988) showed that every simple graph G on n vertices with $\delta(G) \geq \frac{n}{5} - 1$, when n is sufficiently large, is $(0,0)$-supereulerian or is contractible to $K_{2,3}$. A function $j_0(s,t)$ has been found that every (s,t)-supereulerian graph must have edge connectivity at least $j_0(s,t)$.

For any nonnegative integers s and t, we obtain best possible Ore conditions to assure a simple graph on n vertices to be (s,t)-supereulerian as stated in the following.

(i) For any real numbers α and β with $0 < \alpha < 1$, there exists a family of finitely many graphs $\mathcal{F}(\alpha,\beta;s,t)$ such that if $\kappa'(G) \geq j_0(s,t)$ and if for any nonadjacent vertices $u,v \in V(G)$, $d_G(u)+d_G(v) \geq \alpha n+\beta$, then either G is (s,t)-supereulerian, or G is contractible to a member in $\mathcal{F}(\alpha,\beta;s,t)$.

(ii) If $\kappa'(G) \geq j_0(s,t)$ and if for any nonadjacent vertices $u,v \in V(G)$, $d_G(u)+d_G(v) \geq n-1$, then when n is sufficiently large, either G is (s,t)-supereulerian, or G is contractible to one of the six well specified graphs.

(iii) Suppose that $\delta(G) \geq 5$. If

$$\text{for any vertices } u,v,w \in V(G) \text{ with } E(G[\{u,v,w\}]) = \emptyset,\ d_G(u) + d_G(v) + d_G(w) > n-3. \tag{1}$$

then G is (s,t)-supereulerian if and only if $\kappa'(G) \geq j_0(s,t)$.

1. Introduction

We consider finite loopless graphs that may have parallel edges and follow [2] for undefined terms and notation. For a vertex subset or an edge subset X of a graph G, $G[X]$ denotes the subgraph induced by X. As in [2], we use $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\kappa'(G)$ to denote the minimum degree, the maximum degree, the connectivity and the edge-connectivity of a graph G, respectively. For a vertex $v \in V(G)$, define $N_G(v)$ to be the set of neighbors of v in G and $N_G[v] = N_G(v) \cup \{v\}$.

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A graph G is **supereulerian** if G has a spanning closed trail. Boesch, Suffel and Tindell in [1] initiated the study of the supereulerian problem, seeking characterizations of supereulerian graphs. Pulleyblank in [19] showed determining if a graph is supereulerian, even within planar graphs, is NP complete. In [15], the notion of (s, t)-supereulerian was formally brought in as a generalization of supereulerian graphs. For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Thus supereulerian graphs are precisely $(0, 0)$-supereulerian graphs. A number of research results on the (s, t)-supereulerian problem and similar topics have been obtained, as seen in [7,10,11,13–16,20,24], among others.

Let G be a graph, define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G) \text{ and } uv \notin E(G)\}$. Sufficient conditions in terms of σ_2 are commonly known as Ore conditions, due the well known result of Ore [18] on hamiltonian graphs. There have been many researches using Ore conditions to study supereulerian graphs, as can be found in Catlin's resourceful survey [4], as well as its later updates in [8,12].

As supereulerian graphs are $(0, 0)$-supereulerian graphs, it is natural to investigate (s, t)-supereulerian graphs for all possible values of s and t using Ore conditions. By definition, if a graph G is (s, t)-supereulerian, then $\kappa'(G) \geq t + 2$. This observation leads Xiong et al. in [23] to find a necessary condition for (s, t)-supereulerian graphs, as presented in the following proposition.

Proposition 1.1 (*Xiong et al. [23]*). *Let s, t be nonnegative integers. Define*

$$j_0(s, t) = \begin{cases} s + t + \frac{1-(-1)^s}{2} & \text{if } s \geq 1 \text{ and } s + t \geq 3, \\ t + 2 & \text{otherwise.} \end{cases} \tag{2}$$

If a graph G is (s, t)-supereulerian, then $\kappa'(G) \geq j_0(s, t)$.

For given nonnegative integers s and t, we shall use (2) as the definition of $j_0 := j_0(s, t)$ throughout the rest of this paper.

Motivated by the studies in hamiltonian graphs and supereulerian graphs, it is natural to seek best possible Ore condition to warrant a simple graph G to be (s, t)-supereulerian. To aim at a better result, we define a strengthened Ore type degree condition of a graph G as follows.

$$m_2(G) = \min\{\max\{d_G(u), d_G(v)\} : u, v \in V(G) \text{ and } uv \notin E(G)\}. \tag{3}$$

As $2m_2(G) \geq \sigma_2(G)$, the quantity $m_2(G)$ may be considered as a strengthened form of the Ore condition, and so both Theorem 1.1 and Theorem 1.2 can also be stated using Ore conditions in a straightforward way by replacing the condition $m_2(G) \geq f(n)$ with $\sigma_2(G) \geq 2f(n)$, for each of the lower bounds $f(n)$ stated in these theorems.

Theorem 1.1 involves a graph family $\mathcal{F}(a, b; s, t)$, for given integers s and t and real numbers a and b with $0 < a < 1$. Before we state Theorem 1.1, we shall describe this family $\mathcal{F}(a, b; s, t)$ and explain its finiteness. N is a finite number defined by a function of the given values a, b, s and t, and $\mathcal{F}(a, b; s, t)$ consists of all (s, t)-reduced graphs of order at most N. The (s, t)-reduced graphs will be introduced in Section 2.2, where it is shown that every (s, t)-reduced graph has it edge multiplicity bounded by $\max\{s + t, t + 1, 3\}$. Thus every graph $G \in \mathcal{F}(a, b; s, t)$ has order at most N and has its edge multiplicity bounded by a function of s and t, and so $\mathcal{F}(a, b; s, t)$ contains only finitely many graphs.

Theorem 1.1. *For any nonnegative integers s and t and any real numbers a and b with $0 < a < 1$, there exists a family of finitely many graphs $\mathcal{F}(a, b; s, t)$ such that if G is a simple graph on n vertices with $\kappa'(G) \geq j_0(s, t)$ and*

$$m_2(G) \geq an + b, \tag{4}$$

then one of the following must hold.

(i) G is (s, t)-supereulerian.

(ii) G is contractible to a member in $\mathcal{F}(a, b; s, t)$.

Example 1.1. For some given values s and t, there exist graphs with edge connectivity at least $j_0(s, t)$ but not (s, t)-supereulerian.

(i) Let e_1, e_2 be a matching in the graph K_4. Then $K_4 - e_1$ does not have a spanning eulerian subgraph containing e_2, and so K_4 is not $(1, 1)$-supereulerian with $\kappa'(K_4) = 3 = j_0(1, 1)$.

(ii) Define $2K_3$ to be the graph with $V(2K_3) = \{v_1, v_2, v_3\}$ and $E(2K_3) = \{e_1, e_1', e_2, e_2', e_3, e_3'\}$ such that e_1 and e_1' are parallel edges joining v_1 and v_3, e_2 and e_2' are parallel edges joining v_2 and v_3, and e_3 and e_3' are parallel edges joining v_1 and v_2. Define $2K_3^{--} = 2K_3 - \{e_1', e_2'\}$ and $2K_3^{-} = 2K_3 - \{e_1'\}$. Then $2K_3^{--}$ does not have a spanning eulerian subgraph containing e_3 and e_3', $2K_3^{-} - \{e_2'\}$ does not have a spanning eulerian subgraph containing e_3 and e_3' and $2K_3 - \{e_1', e_2'\}$ does not have a spanning eulerian subgraph containing e_3 and e_3'. Hence $2K_3^{--}$ is not $(2, 0)$-supereulerian with $\kappa'(2K_3^{--}) = 2 = j_0(2, 0)$, $2K_3^{-}$ is not $(2, 1)$-supereulerian with $\kappa'(2K_3^{-}) = 3 = j_0(2, 1)$, and $2K_3$ is not $(2, 2)$-supereulerian with $\kappa'(2K_3) = 4 = j_0(2, 2)$.

(iii) For any integer $h \geq 1$, the graph $K_{2,2h+1}$ is not (0,0)-supereulerian, and $\kappa'(K_{2,2h+1}) = 2 = j_0(0, 0)$.

(iv) Let p, q be integers with $p \geq 2$ and $q \geq 1$. Let u_1, u_2 be two nonadjacent vertices of degree p in $K_{2,p}$. Form a graph $K_{2,p}^p$ from $K_{2,p}$ by adding q (parallel, if $q \geq 2$) edges $e_1, \ldots, e_q$ joining u_1 and u_2. Then $K_{2,2}^1$ does not have a spanning

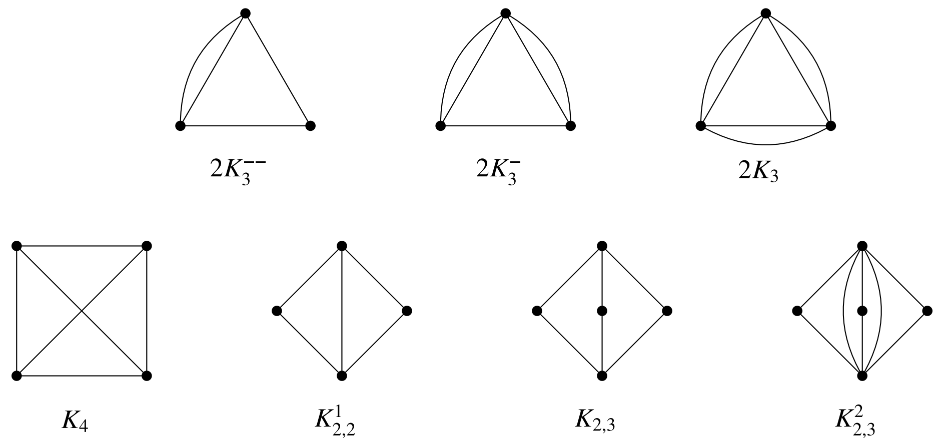

Fig. 1. Some special graphs.

eulerian subgraph containing e_1 and so $K_{2,2}^1$ is not (1,0)-supereulerian with $\kappa'(K_{2,2}^1) = 2 = j_0(1, 0)$; and $K_{2,3}^2$ does not have a spanning eulerian subgraph containing e_1 and e_2, and so $K_{2,3}^2$ is not (2,0)-supereulerian with $\kappa'(K_{2,3}^2) = 2 = j_0(2, 0)$.

Theorem 1.2. *Let s and t be two nonnegative integers. If G is a simple graph on $n \geq 51$ vertices with $\kappa'(G) \geq j_0(s, t)$ and*

$$m_2(G) \geq \frac{n}{2} - 2, \tag{5}$$

then one of the following must hold.
(i) G is (s, t)-supereulerian.
(ii) G is contractible to a member in $\{2K_3^{--}, 2K_3^{-}, 2K_3, K_4, K_{2,2}^1, K_{2,3}\}$.

In Fig. 1, we depict the graphs in Example 1.1 and Theorem 1.2 (ii).

An immediate consequence of Theorem 1.2 is the following corollary, using Ore condition to describe the property of being (s, t)-supereulerian.

Corollary 1.1. *Let s and t be two nonnegative integers. Suppose that G is a simple graph on $n \geq 51$ vertices and with $\delta(G) \geq 5$ such that*

$$\text{for any vertices } u, v \in V(G) \text{ with } uv \notin E(G),\ d_G(u) + d_G(v) \geq n - 4. \tag{6}$$

Then G is (s, t)-supereulerian if and only if $\kappa'(G) \geq j_0(s, t)$.

Corollary 1.1 motivates another way to investigate different versions of a generalized form of the Ore condition. We also obtain the following result.

Theorem 1.3. *Let s and t be two nonnegative integers. Suppose that G is a simple graph on n vertices and with $\delta(G) \geq 5$. If*

$$\text{for any vertices } u, v, w \in V(G) \text{ with } E(G[\{u, v, w\}]) = \emptyset,\ d_G(u) + d_G(v) + d_G(w) > n - 3. \tag{7}$$

then G is (s, t)-supereulerian if and only if $\kappa'(G) \geq j_0(s, t)$.

In the next section, we summarize former results and needed tools in our arguments to prove the main results. The main results will be validated in the last three sections.

2. Preliminaries

Throughout this paper, we use $\mathbb{Z}_m$ to denote the set of all integers modulo m, for a given integer $m > 1$. We often use m instead of 0 to denote the additive identity in the group $\mathbb{Z}_m$ when $\mathbb{Z}_m$ is used as an index set. We use $H \subseteq G$ to mean that H is a subgraph of G. For vertex subsets $X, Y \subseteq V(G)$, define

$$E_G[X, Y] = \{xy \in E(G) : x \in X, y \in Y\} \text{ and } \partial_G(X) = E_G[X, V(G) - X],$$

and use $\partial_G(v)$ for $\partial_G(\{v\})$. If $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G and v_H is the vertex in G/H onto which H is contracted, then H is the preimage of v_H in G. A vertex v in the contraction G/X is nontrivial if its preimage in G has at least two vertices.

2.1. Reducing collapsible subgraphs

Collapsible graphs are introduced by Catlin in [3] as a useful tool to study eulerian subgraphs. Let $O(G)$ be the set of all odd degree vertices of G. A graph G is collapsible if for any subset R of $V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H with $O(H) = R$. By definition, the singleton graph K_1 is collapsible. As when $R = \emptyset$, a spanning connected subgraph H with $O(H) = R$ is a spanning closed trail of G, it follows that collapsible graphs are supereulerian graphs. Let $H_1, H_2, \ldots, H_c$ denote the list of all maximal collapsible subgraphs. The graph $G' = G/(\cup_{i=1}^{c} H_i)$ is the collapsible reduction of G, or simply the reduction of G in short. A graph equaling its own reduction is a collapsible reduced graph, or simply a reduced graph in short. Theorem 2.1 presents useful properties related to collapsible graphs.

Theorem 2.1. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Let $K_{3,3}^-$ be the graph obtained from $K_{3,3}$ by deleting an edge. Each of the following holds.*

(i) (Catlin, Theorem 3 of [3]) G is collapsible (or supereulerian, respectively) if and only if G/H is collapsible (or supereulerian, respectively). In particular, G is collapsible if and only if the reduction of G is K_1.

(ii) (Catlin, Theorem 5 of [3]) A graph is reduced if and only if it does not have a nontrivial collapsible subgraph.

(iii) (Lei et al. Theorem 2.1(v) of [16]) Let $X \subseteq E(G)$ be an edge subset of G. If $G - X$ is collapsible, then G has a spanning eulerian subgraph H with $X \subseteq E(H)$.

(iv) Collapsible graphs include all complete graphs with order at least 3, cycles of length at most 3, $K_{3,3}^-$ and $K_{3,3}$.

While part of Theorem 2.1(iv) are implicitly implied in [3], these statements can also be straightforwardly justified by applying Theorem 2.2(i). Following [5], let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning trees. Theorem 2.2(ii) can be obtained by applying Theorem 1.4 of [5] to maximal 2-connected subgraph of G.

Theorem 2.2. *Let G be a connected graph. Each of the following holds.*

(i) (Catlin et al. Theorem 1.3 of [5]) Suppose that $F(G) \le 2$. Then G is collapsible if and only if its reduction is not a member in $\{K_2, K_{2,t} : t \ge 1\}$; and G is supereulerian if and only if its reduction is not a member in $\{K_2, K_{2,2t+1} : t \ge 1\}$.

(ii) (Catlin et al. Theorem 1.4 of [5]) If $F(G) \le 2$ and $\kappa'(G) \ge 3$, then G is collapsible.

(iii) (Catlin et al. Lemma 2.3 of [5]) If G is a reduced graph with $|V(G)| \ge 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

(iv) (Theorem 2.4 of [9]) Let $C = v_1v_2v_3v_4v_5v_6v_1$ be a 6-cycle and let $v_0 \notin V(C)$ be a vertex. Define $K_{1,3}(1, 1, 1)$ to be the graph with vertex set $V(C) \cup \{v_0\}$ and edge set $E(C) \cup \{v_0v_1, v_0v_3, v_0v_5\}$. If G is a 2-edge-connected reduced graph with $F(G) \le 3$, $|V(G)| \le 7$ and at most 3 vertices of degree 2, then $G \in \{K_{2,3}, K_{1,3}(1, 1, 1)\}$.

Extending Theorems 1.1 and 1.3 of [6] by Catlin et al. the following lemma utilizes the spanning tree packing theorem of Nash-Williams [21] and Tutte [22], and indicates a relationship between edge connectivity and number of edge disjoint spanning trees.

Lemma 2.1 (*Catlin et al. [6] and Xiong et al. [23]*). *Let G be a connected graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0, 1\}$, $\ell \ge 2$ and $2 \le k \le \ell$. The following are equivalent.*

(i) $\kappa'(G) \geqslant 2\ell + \epsilon$.

(ii) For any $X \subseteq E(G)$ with $|X| \le 2\ell - k + \epsilon$, $G - X$ has k edge-disjoint spanning trees.

2.2. Reducing (s, t)-contractible subgraphs

For a graph G and an integer i, we define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Let $\mathcal{L}_{s,t}$ denote the family of all (s, t)-supereulerian graphs. By definition, $K_1 \in \mathcal{L}_{s,t}$. A graph H is a contractible configuration of $\mathcal{L}_{s,t}$ (or (s, t)-contractible, in short), if for any graph G containing H as a subgraph, the following always holds:

$$G \in \mathcal{L}_{s,t} \text{ if and only if } G/H \in \mathcal{L}_{s,t}.$$

An elementary subdivision of an edge $e = uv \in E(G)$ is a new graph $G(e)$ from $G - e$ by adding a path uv_ev with v_e being a new vertex in $G(e)$. If $X \subseteq E(G)$ is an edge subset, then $G(X)$ denotes the resulting graph formed by elementarily subdividing each edge in X. With this definition, we have the following observations.

Observation 2.1. *For an edge subset $X \subseteq E(G)$, let $V_X = \{v_e : e \in X\}$, $E_X = \{uv_e, v_ev : e = uv \in X\}$ and $E'_X = \{v_ev : e = uv \in X\}$, and let $Y \subseteq E(G) - X$ be an edge subset disjoint from X. Each of the following holds.*

(i) $V_X = V(G(X)) - V(G)$ and $E_X = E(G(X)) - E(G)$.

(ii) There exists a bijection between X and $\{v_eu : e \in X\}$ and so $G(X)/E'_X \cong G$.

(iii) For any 2-edge-connected subgraph H' of $G(X)$ with $|E(H')| > 0$, and for any $e = uv \in X$, if $v_e \in V(H')$, then both $v_e u, v_e v \in E(H')$; and if $\{uv_e, vv_e\} \cap E(H') \neq \emptyset$, then $\{uv_e, vv_e\} \subset E(H')$. Thus in view of Observation 2.1(ii), $H = H'/(E'_X \cap E(H'))$ is a subgraph of G, called the restoration of H' in G.

(iv) G has a spanning eulerian subgraph H with $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$ if and only if $(G - Y)(X)$ is supereulerian.

(v) Let $J = (G - Y)(X)$ and J' be the reduction of J. If J is not supereulerian, then J' is also not supereulerian.

Observation 2.1(iii) defines the concept of restoration for a nontrivial subgraph H' of $G(X)$. To make this definition a complete one, we further define that, if H' contains a single vertex v, then the restoration of H' in G is the singleton graph $G[\{v\}]$ if $v \notin V_X$, or is an empty graph (a graph with no vertices nor edges) if $v \in V_X$. Observation 2.1(v) follows from Theorem 2.1(i). Using the notation in Observation 2.1, we define

$$h = |D_2(J')| \text{ and } h_1 = |D_2(J') \cap V_X|. \tag{8}$$

We define a relation "$\sim$" on the edge set $E(G)$ of a graph G such that $e_1 \sim e_2$ if $e_1 = e_2$, or if e_1 and e_2 form a cycle in G. It is routine to check that $\sim$ is an equivalence relation and edges in the same equivalence class are parallel edges with the same end vertices. We use $[uv]$ to denote the set of all edges between u and v in a graph, and shorten $|[uv]|$ to $|uv|$. If G is a graph, then $\mu(G) = \max\{|uv| : uv \in E(G)\}$ is the multiplicity of G. Thus for each edge $e \in E(G)$, the edges parallel to e in G induces a subgraph isomorphic to $|e|K_2$.

Chen, Chen and Luo prove an edge-connectivity sufficient condition for (s, t)-supereulerianicity, which is extended recently when the edge-connectivity is sufficiently large in Theorem 2.3(ii-2).

Theorem 2.3. *Let s, t be nonnegative integers and let G be a graph.*

(i) (Chen, Chen and Luo, Theorem 4.1 of [7]) If $\kappa'(G) \geq 4$, $t \leq \frac{\kappa'(G)}{2}$ and $s + t + 1 \leq \kappa'(G)$, then G is (s, t)-supereulerian.

(ii) (Proposition 2.6 of [16]) Each of the following holds.

(ii-1) If G is (s, t)-supereulerian, then any contraction of G is also (s, t)-supereulerian.

(ii-2) Suppose that H is a graph with $\kappa'(H) \geq \max\{s + t + 1, t + 2, 5\}$. Then H is (s, t)-supereulerian.

(ii-3) If $H = \ell K_2$ with $\ell \geq \max\{s + t + 1, t + 2, 4\}$, then G is (s, t)-supereulerian if and only if G/H is (s, t)-supereulerian.

Theorem 2.3(ii-1) indicates that $\mathcal{L}_{s,t}$ is closed under taking contraction, and, if $\ell \geq \max\{s + t + 1, t + 2, 4\}$, then ℓK_2 is a contractible configuration of $\mathcal{L}_{s,t}$. A graph Γ is (s, t)-reduced if Γ does not contain any nontrivial subgraph that is a contractible configuration of $\mathcal{L}_{s,t}$. For a graph G, the (s, t)-reduction of G, is the graph Γ formed from G by contracting all maximal (s, t)-contractible subgraphs of G. By definition, if Γ is the (s, t)-reduction of G, then

$$G \in \mathcal{L}_{s,t} \text{ if and only if } \Gamma \in \mathcal{L}_{s,t}. \tag{9}$$

Using the definitions for the graphs in Example 1.1 (ii), it is routine to verify that

$$K_4 \text{ is } (1, 1)\text{-reduced, } 2K_3^{--} \text{ is } (2, 0)\text{-reduced, } 2K_3^{-} \text{ is } (2, 1)\text{-reduced and } 2K_3 \text{ is } (2, 2)\text{-reduced.} \tag{10}$$

Definition 2.1. Let s and t be nonnegative integers, G be a graph, X and Y be disjoint edge subsets of G with $|X| \leq s$ and $|Y| \leq t$, and let $J = (G - Y)(X)$ and J' be the reduction of J. For any vertex $z \in V(J')$, let H'_z denote the preimages of z in J, and let H_z be the restorations of H'_z in $G - Y$. Define

$$M = G[\bigcup_{z \in V(J')} E(H_z)],$$

$$M' = J[\bigcup_{z \in V(J')} E(H'_z)],$$

$$X' = X \cap E(M') \text{ and } J'' = (G - Y)(X')/M'.$$

Define $Y' = \{uv \in Y :$ there exists a graph $L \in \{H_z : z \in V(J')\}$ such that $u, v \in V(L)\}$, and $Y'' = Y - Y'$.

Lemma 2.2 describes a relationship between the (collapsible) reduction of G and the (s, t)-reduction of G. Thus for a graph G, the (collapsible) reduction of G and the (s, t)-reduction of G may not be the same. The following lemmas are useful.

Lemma 2.2 (*Lemma 2.8 of [16]*). *We adopt the notation in Definition 2.1 and let $X'' = X - X'$. Each of the following holds.*

(i) $X'' \subseteq E(J'')$ and $J'' = (G - Y)(X')/M' = (G - Y'')(X')/M'$.

(ii) $J' = J''(X'') = ((G - Y'')/M)(X'')$.

(iii) If J is not supereulerian, then G can be contracted to an (s, t)-reduced and non (s, t)-supereulerian graph with order at most $|V(J')|$.

Lemma 2.3. *Let s and t be nonnegative integers.*

(i) Let K_{n_0} denote a complete graph of order $n_0 \geq \max\{s+t+3, 5\}$. Then for any disjoint edge subsets X and Y with $|X| \leq s$ and $|Y| \leq t$, both

$G - (X \cup Y)$ and $(G - Y)(X)$ are collapsible.

(ii) Every complete graph K_{n_0} with $n_0 \geq \max\{s + t + 3, 5\}$, is (s, t)-contractible.

Proof. (i) Let $K = K_{n_0}$. Define $K^1 = (K - Y)(X)$ and $K^2 = K - (X \cup Y)$. As $|X \cup Y| \le s+t$, and since $n_0 \ge \max\{s+t+3, 5\}$, either $n_0 = 5$ and $s+t \le 2$, or $n_0 \ge s+t+3$. If $n_0 = 5$ and $|X| + |Y| \le 2$, then as K^2 is obtained from K_5 by removing at most 2 edges, every edge in K^2 lies in a cycle of length 3. By the fact that K_3 is collapsible, we conclude that K^2 is also collapsible. Now assume that $n_0 - 1 \ge s+t+2$. Let $\ell = \lfloor \frac{n_0-1}{2} \rfloor$. Then $|X \cup Y| \le s+t \le n_0 - 3 \le 2\ell - 2$. By Lemma 2.1 with $k = 2$ we conclude that K^2 has 2-edge-disjoint spanning trees, and so by Theorem 2.3(i), K^2 is collapsible. Since K^2 is a collapsible subgraph of K^1, and K^1/K^2 is a graph with $|X|$ 2-cycles, it follows Theorem 2.1(iv) that K^1/K^2 is collapsible. By Theorem 2.1(i), K^1 is collapsible. This proves Lemma 2.3(i).

To prove (ii), we let G be a graph and $K \cong K_{n_0}$ be a subgraph of G such that $n_0 \ge \max\{s+t+3, 5\}$. In the rest of the proof, we view that K is an induced subgraph of G, and so $K = G[V(K_{n_0})]$. By Theorem 2.3(ii-1), it suffices to assume that $G/K \in \mathcal{L}(s,t)$ to prove that $G \in \mathcal{L}(s,t)$.

If $G \notin \mathcal{L}(s,t)$, then there exist disjoint subsets X, Y of $E(G)$ with $|X| \le s$ and $|Y| \le t$ such that $G - Y$ does not have a spanning eulerian subgraph containing edges in X. By Observation 2.1(iv), $(G-Y)(X)$ is not supereulerian. Define $X_1 = X \cap E(K)$, $Y_1 = Y \cap E(K)$, $X_2 = X - X_1$ and $Y_2 = Y - Y_1$. Let v_K be the vertex in G/K onto which K is contracted. Thus by definition, X_2, Y_2 are disjoint edge subsets of G/K. Since $G/K \in \mathcal{L}(s,t)$, by Observation 2.1(iv), $G/K - Y_2$ has a spanning eulerian subgraph containing all edges in X_2. Thus $(G/K - Y_2)(X_2)$ is supereulerian. Let $G^1 = (G-Y)(X)$ and $K^1 = (K - Y_1)(X_1)$. Then K^1 is a subgraph of G^1. By (i), K^1 is a collapsible subgraph of G^1, and as $G^1/K^1 = (G/K - Y_2)(X_2)$ is supereulerian, it follows by Theorem 2.1(i) that $G^1 = (G-Y)(X)$ is supereulerian. This contradiction justifies the lemma. ■

Lemma 2.4 (*Liu et al. Lemma 3.1 of [17]*)**.** *Let G be a simple graph and $W \subseteq V(G)$ be a subset with $\delta_W = \min\{d_G(w) : w \in W\}$. If $|\partial_G(W)| < \delta_W$, then $|W| \ge \delta_W + 1$.*

Lemma 2.5. *Let $c_0 > 0$ be a constant, G be a simple graph with $m_2 = m_2(G)$, and W_1 and W_2 be two vertex subsets of G with $W_1 \cap W_2 = \emptyset$ satisfying that for $i \in \{1,2\}$, $|\partial_G(W_i)| \le c_0$. Each of the following holds.*

(i) If for some $W \in \{W_1, W_2\}$, $G[W]$ is not spanned by a complete graph, then $|W| \ge m_2 + 1 - c_0$.

(ii) If for some $w_1 \in W_1$ and $w_2 \in W_2$, $w_1w_2 \notin E(G)$, then $\max\{|W_1|, |W_2|\} \ge m_2 + 1 - c_0$.

Proof. Suppose that $G[W]$ is not spanned by a complete graph. Then there exist vertices $w_1', w_2' \in W$ such that $w_1'w_2' \notin E(G)$. Suppose that $d_G(w_1') \ge d_G(w_2')$ and let $V_1 = N_G[w_1'] \cap W$ and $V_2 = N_G[w_1'] - W$. Then $|W| \ge |V_1| = |N_G[w_1']| - |V_2| \ge m_2 + 1 - |\partial_G(W)| \ge m_2 + 1 - c_0$. This proves (i).

Now assume that for some $w_1 \in W_1$ and $w_2 \in W_2$, $w_1w_2 \notin E(G)$. By symmetry, we may assume that $d_G(w_1) \ge d_G(w_2)$. Then $|W_1| \ge |N_G[w_1]| - |\partial_G(W_1)| \ge m_2 + 1 - c_0$. This proves (ii). ■

3. Proof of Theorem 1.1

Throughout this section, a and b denote real numbers with $0 < a < 1$, and s and t denote two given nonnegative integers, and $j_0 = j_0(s,t)$. Unless otherwise stated, let G be a simple graph on n vertices with $\kappa'(G) \ge j_0(s,t)$ such that (4) holds. When disjoint edge subsets X and Y in a graph G are given, we let $J = (G-Y)(X)$, and let J' be the reduction of J, and adopt the notation in Observation 2.1 for the definition of V_X and E_X, and define h and h_1 as in (8). Let

$$V_X' = \{v \in V_X : v \in V(J')\} \tag{11}$$

denote the set of trivial vertices in J' each of whose preimage is a single vertex in V_X. For each vertex $z \in V(J') - V_X'$, let H_z' be the preimage of z in J and H_z be the restoration of H_z' in G.

Lemma 3.1. *Suppose that s, t are non-negative integers, G is a simple graph with $m_2 = m_2(G)$ and $\kappa'(G) \ge j_0$. Let $m_0 = \max\{s+t+3, 5\}$, $X, Y \subseteq E(G)$ be disjoint edge subsets with $|X| \le s$ and $|Y| \le t$, and let J' be the reduction of $(G-Y)(X)$ such that J' is not supereulerian. Then each of the following holds.*

(i) Suppose that m_2 satisfies (4). Define

$$Z_{a,b} = Z_{a,b}(G) = \{v \in V(G) : d_G(v) < an + b\}. \tag{12}$$

Then $G[Z_{a,b}]$ is a complete subgraph of G.

(ii) Suppose that m_2 satisfies (4) and Z_{ab} is defined in (12). If Z' is the vertex subset in J' such that the preimage of each vertex in Z' contains a vertex in $Z_{a,b}$, then $|Z'| < m_0$.

(iii) If $J' = K_{2,h}$ for some odd integer $h \ge 3$, with $h - h_1 \le 1$, then G can be contracted to a member in $\{2K_3^{--}, 2K_3^{-}, 2K_3\}$.

(iv) If $J' = K_{2,h}$ for some integer $h \ge 3$ with $h - h_1 \ge 2$, then either $t \in \{1,2\}$, $h - h_1 = 2$ and edges in Y are joining vertices in the preimages of the two vertices in $D_2(J') - V_X'$, or $t = 0$ and $0 \le s \le 2$.

Proof. If there exist distinct vertices $z_1, z_2 \in Z_{a,b}$ with $z_1z_2 \notin E(G)$, then by the definition of $Z_{a,b}$ and by (4), we have $an + b > \max\{d_G(z_1), d_G(z_1)\} \ge m_2 \ge an + b$, a contradiction. This proves (i).

Suppose that $|Z'| \ge m_0$. Then by Lemma 3.1(i), $G[Z_{a,b}]$ is a complete graph. As $|X \cup Y| \le s+t$, it follows from Lemma 2.3 that $G[Z_{a,b}] - (X \cup Y)$ is a collapsible subgraph of J. Hence each collapsible subgraph of J that contains a vertex in $Z_{a,b}$

must be contracted into one vertex in J', and so $|Z'| = 1$. This leads to a contradiction that $1 = |Z'| \geq m_0$, and so (ii) must hold.

Assume that $J' = K_{2,h}$ for some odd integer $h \geq 3$ with $D_h(J') = \{u_1, u_2\}$. Let $k = |Y \cap E_G[V(H_{u_1}), V(H_{u_2})]|$. Then $k \leq |Y| \leq t$. Suppose first that $h = h_1$. Then by (8), $3 \leq h = h_1 \leq |V_X| = |X| \leq s$, and so we have $j_0 \leq \kappa'(G) \leq |E_G[V(H_{u_1}), V(H_{u_2})]| = h + k \leq s + t$. Since either $h < s$ or $s = h$ is odd, it follows by (2) that $j_0 \leq \kappa'(G) \leq h+k \leq j_0 - 1$, a contradiction. Therefore we assume that $h - h_1 = 1$. Let $z \in D_2(J') - V'_X$. Then by (8), $3 \leq h = h_1 + (h - h_1) = |V'_X| + 1 \leq |X| + 1 \leq s + 1$, and so $s \geq 2$. If $t = 0$, then by (2), we have $s \leq j_0 \leq \kappa'(G) \leq |\partial_G(V(H_z))| = 2$. Hence we must have $s = 2$ and $h = 3$. It follows that $G/(H_{u_1} \cup H_{u_2} \cup H_z) \cong 2K_3^{--}$, and so Lemma 3.1(iii) holds in this case. Thus we may assume that $t > 0$. Then $t + 2 \leq \kappa'(G) \leq |\partial_G(V(H_z)) - Y| + |\partial_G(V(H_z)) \cap Y| \leq d_{J'}(z) + t = 2 + t$, and so both $\partial_G(V(H_z)) \cap Y = Y$ and $\kappa'(G) = j_0(s, t) = t + 2$. This implies, by (2) and $s \geq 2$, that $s = 2$ and $t \in \{1, 2\}$. If $t = 1$, then by symmetry, we assume that the only edge in Y is incident with a vertex in $V(H_{u_1})$ in G. Thus $G/(H_{u_1} \cup H_{u_2} \cup H_z) \cong 2K_3^-$. Assume that $t = 2$. Let $t_1 = |\partial_G(V(H_z)) \cap \partial_G(V(H_{u_1})) \cap Y|$ and $t_2 = |\partial_G(V(H_z)) \cap \partial_G(V(H_{u_2})) \cap Y|$. Then we have $t = t_1 + t_2$. Moreover, $\partial_G(V(H_{u_1})) = \partial_G(V(H_z)) \cap \partial_G(V(H_{u_1})) \cup X \cup \{e'\}$, and so $t_1 + 2 + 1 = |\partial_G(V(H_{u_1}))| \geq \kappa'(G) \geq 2 + t$. Hence $t_1 \geq t - 1$. Similarly, $t_2 \geq t - 1$ and so $t = t_1 + t_2 \geq 2t - 2$. Thus we must have $t = 2$ with $t_1 = t_2 = 1$. It follows that $G/(H_{u_1} \cup H_{u_2} \cup H_z) \cong 2K_3$, once again, Lemma 3.1(iii) holds in this case as well.

We now prove (iv) and assume that $D_2(J') - V'_X = \{z_1, z_2, \ldots, z_{h-h_1}\}$. By definition of J', we have $\partial_G(V(H_{z_i})) \subseteq \partial_{J'}(z_i) \cup Y$, and so $t + 2 \leq \kappa'(G) \leq |\partial_G(V(H_{z_i}))| \leq d_{J'}(z_i) + t = 2 + t$, for each i with $1 \leq i \leq h - h_1$. It follows that $\partial_G(V(H_{z_i})) = \partial_{J'}(z_i) \cup Y$. As $z_i, z_j \in D_2(J')$ for $1 \leq i \leq j \leq h - h_1$, we have $\partial_G(V(H_{z_i})) \cap \partial_G(V(H_{z_j})) = \emptyset$, and so either $t = 0$ or both $t > 0$ and $h - h_1 = 2$. Assume first that $t > 0$. Then $t + 2 \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}) \cup V(H_{z_2}))| = d_{J'}(z_1) + d_{J'}(z_2) = 4$, leading to $t \in \{1, 2\}$. This, together with $|\partial_G(V(H_{z_i}))| = d_{J'}(z_i) + |Y|$ for $i \in \{1, 2\}$, implies that $Y = E_G[V(H_{z_1}), V(H_{z_2})]$, and so Lemma 3.1(iv) holds in this case. Hence we may assume that $t = 0$ and so $2 \leq j_0 \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = d_{J'}(z_1) = 2$. This implies that, by (2), $s \in \{0, 1, 2\}$. This completes the proof of the lemma. ■

Proof of Theorem 1.1. Choose an integer $c = c(a, s, t)$ such that

$$c \geq \max\left\{\frac{10a}{1 + a(2s + t + 6)} + 1, 4\right\}, \tag{13}$$

and define $N = N(a, b, s, t)$ to be an integer satisfying

$$N \geq \max\left\{\frac{(1 + a)(1 + |b|)}{a^2}, \frac{c + t + |b| + 1}{a}, \frac{1 + a(2s + t + 6)}{a}, \frac{(c + 1)[a(2s + t) + 1] + 2a(3c - 2)}{a(c - 3)}\right\}. \tag{14}$$

We use $\mathcal{F} = \mathcal{F}(a, b; s, t)$ to denote the family of all (s, t)-reduced and non (s, t)-supereulerian graphs of order at most N. By Theorem 2.3(ii-3), every graph G in $\mathcal{F}$ has multiplicity at most $\max\{s+t, t+1, 3\}$. As there are only finitely many graphs with order N and with edge multiplicity bounded by $\max\{s + t, t + 1, 3\}$, $\mathcal{F}$ is a family of finitely many graphs. To prove Theorem 1.1, we argue by contradiction, and assume that there exists a counterexample graph G to Theorem 1.1 with $n = |V(G)|$ minimized among all counterexample to the theorem. As G is a minimum counterexample, and by Lemma 2.2, we have the following observations.

Observation 3.1. *Each of the following holds for this graph G.*

(i) $\kappa'(G) \geq j_0(s, t)$.

(ii) G is not (s, t)-supereulerian and cannot be contracted to a member in $\mathcal{F}$, and so $n \geq N + 1$.

(iii) There exist disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$ such that $G - Y$ does not have a spanning closed trail that contains all edges in X.

As G is a counterexample, G is not (s, t)-supereulerian and cannot be contracted to a member in $\mathcal{F}$. In order to justify Observation 3.1(ii), assume that $n \leq N$. Then let G' be an (s, t)-reduction of G. As G is not (s, t)-supereulerian, it follows by Theorem 2.3(ii) that G' is also not (s, t)-supereulerian with $|V(G')| \leq |V(G)| \leq N$. As an (s, t)-reduction of G, G' is (s, t)-reduced, and so $G' \in \mathcal{F}$. This is a contradiction to the fact that G cannot be contracted to a member in $\mathcal{F}$. Hence we must have Observation 3.1(ii). Observation 3.1(iii) follows from the definition as G is not (s, t)-supereulerian.

In the following, we let X and Y be the edge subsets defined in Observation 3.1(iii), $J = (G - Y)(X)$, J' be the reduction of J, and adopt the notation in Observation 2.1 for the definition of V_X and E_X; define h and h_1 as in (8), and V'_X as in (11). As $\kappa'(G) \geq j_0(s, t)$, it follows by Observation 2.1(iv) and (v) that,

$$\kappa'(J') \geq \kappa'(J) \geq 2, J' \text{ is not supereulerian, and } F(J') \geq 2. \tag{15}$$

Claim 3.1. $F(J') \geq 3$.

Suppose by contradiction that $F(J') = 2$. By Theorem 2.2(i), we have $J' = K_{2,h}$. By Observation 3.1(iii) and Observation 2.1(iv), J is not supereulerian. By Theorem 2.1(i), J' is not supereulerian. Thus h must be an odd integer and so $h \geq 3$. By (10), Lemma 3.1(iii), (iv) and Observation 3.1(ii), we may assume that either $t \in \{1, 2\}$ and $h - h_1 = 2$, or $t = 0$ and $0 \leq s \leq 2$. Let $D_h(J') = \{u_1, u_2\}$, and let H'_{u_1}, H'_{u_2} be the preimages of u_1 and u_2 in J, respectively; and let H_{u_1} and H_{u_2} be the restorations of H'_{u_1} and H'_{u_2} in $G - Y$, respectively. Define $k = |Y \cap E_G[V(H_{u_1}), V(H_{u_2})]|$. Then $k \leq |Y| \leq t$.

Suppose first that both $t \in \{1, 2\}$ and $h - h_1 = 2$. Let z_1, z_2 be the vertices in $D_2(J') - V'_X$. By symmetry, we assume that $|V(H_{z_1})| \geq |V(H_{z_2})|$. By Lemma 3.1(iv), $Y = E_G[V(H_{z_1}), V(H_{z_2})]$. If $t = 1$, then $3 \leq j_0(s, 2) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 3$, and so by (2), $s \in \{1, 2\}$. If $s = 1$, then as h is odd, we must have $h_1 = s = 1$, and so G can be contracted to K_4 with vertices z_1, z_2, u_1, u_2, contrary to Observation 3.1(ii) with $s = t = 1$. Therefore we must have $t = 2$, and so $4 \leq j_0(s, 1) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 4$, forcing $\kappa'(G) = 4$ and $s \in \{1, 2\}$. As $h \geq 3$ is odd, we must have $s = 1$, and so $\kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 3$, contrary to the fact that $\kappa'(G) = 4$.

Hence we may assume that $t = 0$ and $0 \leq s \leq 2$. As $t = 0$, $Y = \emptyset$, and for any i with $1 \leq i \leq h - h_1$ and any $v_i \in V(H_{z_i})$, the set $\{v_1, v_2, \ldots, v_{h-h_1}\}$ is a stable set in G. We may assume that the labeling of these vertices satisfies

$$|V(H_{z_1})| \leq |V(H_{z_2})| \leq \ldots \leq |V(H_{z_{h-h_1}})|.$$

As $|\partial_G(H_{z_i})| = 2$, it follows from Lemma 2.5 with $c_0 = 2$ that

$$\sum_{i=2}^{h-h_1} |V(H_{z_i})| \geq (h - h_1 - 1)(an + b - 1). \tag{16}$$

This leads to $n - 3 \geq n - (|V(H_{u_1})| + |V(H_{u_2})| + |V(H_{z_1})|) \geq \sum_{i=2}^{h-h_1} |V(H_{z_i})| \geq (h - h_1 - 1)(an + b - 1)$. Thus

$$h - h_1 \leq \frac{n-3}{an+b-1} + 1 \leq \frac{n}{an+b-1} + 1 = \frac{1}{a} + \frac{1-b}{a(an+b-1)} + 1. \tag{17}$$

By (14), we have $n > N \geq \frac{(1+a)(1+|b|)}{a^2} \geq \frac{1-b}{a}$ and so $an + b - 1 > 0$, and $n > N \geq \frac{(1+a)(1+|b|)}{a^2} \geq \frac{(1+a)(1-b)}{a^2}$, and so $\frac{1-b}{a(an+b-1)} < 1$. Hence by (17),

$$h - h_1 \leq \frac{1}{a} + \frac{1-b}{a(an+b-1)} + 1 < \frac{1}{a} + 2.$$

It follows by $h_1 \leq |X| \leq s$ and (14) that $|V(J')| = 2 + h = 2 + h_1 + (h - h_1) < 2 + s + \frac{1}{a} + 2 = \frac{1}{a} + s + 4 \leq \frac{1+a(2s+t+6)}{a} \leq N$. By Lemma 2.2, G can be contracted to an (s, t)-reduced graph with at most N vertices, which is in $\mathcal{F}$, contrary to Observation 3.1(ii). This contraction proves the claim.

Let $n' = |V(J')|$. By Claim 3.1, $F(J') \geq 3$ and so by Theorem 2.2(iii), we have

$$4n' - 2|E(J')| \geq 10. \tag{18}$$

In the arguments below, we are to use the following notation and terms. For each vertex $v \in V(J') - V'_X$, let H'_v be the maximal collapsible subgraph in J which is the contraction preimage of v, and let H_v be the restoration of H'_v. Thus H_v is a subgraph of G. Define $n' = |V(J')|$, $Z_{a,b}$ as in (12) and

$$Z_c = \{v \in V(J') : d_{J'}(v) \leq c\}, Z' = \{v \in Z_c : V(H_z) \cap Z_{a,b} \neq \emptyset\}, \text{ and } Z'' = Z_c - (Z' \cup V'_X). \tag{19}$$

Claim 3.2. *Each of the following holds.*

(i) $|Z'| \leq s + t + 5$ and $|Z_c \cap V'_X| \leq s$.

(ii) Suppose that $n'' = |Z''| > 0$ and denote $Z'' = \{z_1, z_2, \ldots, z_{n''}\}$. Then there exist vertices $v_1, v_2, \ldots, v_{n''}$ such that for each i with $1 \leq i \leq n''$, $v_i \in V(H_{z_i})$ such that $N_G[v_i] \subseteq V(H_{z_i})$.

(iii) $|Z''| \leq \frac{1}{a} + 1$, and $|Z_c| \leq \frac{1+a(2s+t+6)}{a}$.

(iv) $n' \leq N$.

By Lemma 3.1(ii), we have $|Z'| \leq m_0 \leq s + t + 5$. By definition, $|Z_c \cap V'_X| \leq |V_X| = |X| \leq s$. Hence Claim 3.2(i) follows.

Denote $Z'' = \{z_1, z_2, \ldots, z_{n''}\}$ with $n'' = |Z''|$. For each $z_i \in Z''$, by (19), $V(H_{z_i}) \cap Z_{a,b} = \emptyset$. Thus for each $v'_i \in V(H_{z_i})$, we have $d_G(v'_i) \geq an + b$. By (14), $n \geq N \geq \frac{c+t+|b|}{a} \geq \frac{c+t-b}{a}$, and so $d_G(v'_i) \geq an + b > c + t \geq |\partial_G(H_{z_i})|$, It follows by Lemma 2.4 that $|V(H_{z_i})| \geq c + t + 1$. It follows that there must be a vertex $v_i \in V(H_{z_i})$ such that $N_G[v_i] \subseteq V(H_{z_i})$. This proves (ii).

We may assume that $n'' = |Z''| > 1$. Let $v_1, \ldots, v_{n''}$ be the vertices as defined in (ii). For each $i \in \{1, 2, \ldots, n''\}$, as $v_i \notin Z_{a,b}$, $|V(H_{z_i})| \geq |N_G[v_i]| \geq an + b + 1$, and so

$$n \geq \sum_{i=1}^{n''} |V(H_{z_i})| \geq n''(an + b + 1), \text{ or } n'' \leq \frac{n}{an+b+1}.$$

By (14), $n > N \geq \frac{(1+|b|)(1+a)}{a^2}$, and so $-\frac{b+1}{a(an+b+1)} \leq \frac{|b|+1}{(an+b+1)} < 1$. It follows that

$$n'' \leq \frac{n}{an+b+1} = \frac{an+b+1-(b+1)}{a(an+b+1)} = \frac{1}{a} - \frac{b+1}{a(an+b+1)} \leq \frac{1}{a} + 1. \tag{20}$$

By (20) and by Claim 3.2(ii), we have $|Z_c| = |Z'| + |Z''| + |Z_c \cap V'_X| \leq s + t + 5 + \frac{1}{a} + 1 + s = \frac{1+a(2s+t+6)}{a}$. This proves (iii).

To prove (iv), we observe that for any vertex $z \in V(J') - Z_c$, $d_{J'}(z) \geq c+1$, and so by (18),

$$(c+1)|V(J') - Z_c| \leq \sum_{v \in V(J')} d_{J'}(v) = 2|E(J')| \leq 4n' - 10.$$

It follows that $|V(J') - Z_c| \leq \frac{4n'-10}{c+1}$, and so by Claim 3.2(iii),

$$\frac{1 + a(2s+t+6)}{a} \geq |Z_c| = n' - |V(J') - Z_c| \geq n' - \frac{4n' - 10}{c+1} = n'\left(1 - \frac{4}{c+1}\right) + \frac{10}{c+1}. \tag{21}$$

By algebraic manipulations and by (21), (13) and (14), we have

$$n' \leq \left(\frac{1 + a(2s+t+6)}{a} - \frac{10}{c+1}\right)\left(1 - \frac{4}{c+1}\right)^{-1} = \frac{(c+1)[a(2s+t)+1] + 2a(3c-2)}{a(c-3)} \leq N.$$

Thus (iv) holds, and so the claim is justified.

By Claim 3.2(iv), and by Lemma 2.2, the (s,t)-reduction of G is a member in $\mathcal{F}$, contrary to Observation 3.1(ii). This completes the proof of Theorem 1.1. ∎

4. Proof of Theorem 1.2

Throughout this section, let s and t be given nonnegative integers, and let $n = |V(G)| \geq 51$. Assume that G is a counterexample to Theorem 1.2. As G is a counterexample, we have the following observation.

Observation 4.1. *Each of the following holds for this graph G.*

(i) G satisfies (5) and $\kappa'(G) \geq j_0(s,t)$.

(ii) G is not contractible to a member in $\{2K_3^{--}, 2K_3^{-}, 2K_3, K_4, K_{2,2}^1, K_{2,3}\}$.

(iii) There exist disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$ such that $G - Y$ does not have a spanning closed trail that contains all edges in X. By Observation 2.1(iv), $(G - Y)(X)$ is not supereulerian.

For the edge subsets X and Y defined in Observation 4.1(ii), let $J = (G - Y)(X)$, J' be the reduction of J. In the following, we continue adopting the notation in Observation 2.1 for the definition of V_X and E_X, and define h and h_1 as in (8), and V'_X as in (11). By Observation 4.1(iii) and Theorem 2.1(i),

$$J' \text{ is not supereulerian, and so } J' \neq K_1. \tag{22}$$

Since $\kappa'(G) \geq t+2$, we conclude that J' is a 2-edge-connected nontrivial reduced graph. As before, for each $z \in V(J')$, we let H'_z denote the preimage of z in J and H_z denote the restoration of H'_z in G. Define h and h_1 as in (8), and set

$$\begin{aligned} Z_0 &= Z_0(G) = \{v \in V(G) : |N_G(v)| < \frac{n}{2} - 2\}, \\ Z' &= \{v \in V(J') : \partial_G(V(H_v)) \leq s+5, V(H_v) \cap Z_0 = \emptyset\}, \end{aligned} \tag{23}$$

We shall justify the following claims.

Claim 4.1. *Each of the following holds.*

(i) $|V'_X| \leq s$ and $|Z' - V'_X| \leq 2$.

(ii) If $J' - V'_X$ has at least 3 vertices z_1, z_2, z_3 of degree 2 such that for each $i \in \{1,2,3\}$, $\partial_G(H_{z_i}) \cap Y = \emptyset$ and there exist a vertex $v_i \in V(H_{z_i})$ with v_1, v_2, v_3 being a stable set of G, then $|V(J' - V'_X)| \leq 4$.

By the definition of V'_X, $|V'_X| \leq |X| \leq s$. By (23), for any $z \in Z' - V_X$ and for any $v \in H_z$, we have $d_G(v) \geq \frac{n}{2} - 2$. Hence by Lemma 2.4, $|V(H_z)| \geq \frac{n}{2} - 2$. It follows that $|Z'|(\frac{n}{2} - 2) \leq \sum_{z \in Z'} |V(H_z)| \leq n$, and so as $n > 12$, we have

$$|Z' - V'_X| \leq \frac{2n}{n-4} = 2 + \frac{8}{n-4} < 3.$$

Thus $|Z' - V'_X| \leq 2$, which proves Claim 4.1(i).

Assume that $J' - V'_X$ has 3 vertices z_1, z_2, z_3 of degree 2 such that for each $i \in \{1,2,3\}$, there exist a vertex $v_i \in V(H_{z_i})$ with v_1, v_2, v_3 being a stable set of G. By Lemma 3.1(i) with $a = \frac{1}{2}$ and $b = -2$, we conclude that at least two of these three vertices are in $Z' - V'_X$, and so $|Z' - V'_X| \geq 2$. By Claim 4.1(i), we conclude that $|Z' - V'_X| = 2$ and so we may assume that $z_1, z_2 \in Z' - V'_X$. By (5) and (23), and as $\partial_G(H_{z_i}) \cap Y = \emptyset$, for $i \in \{1,2\}$, every vertex $v \in V(H_{z_i})$ has degree $d_G(v) \geq \frac{n}{2} - 2$, and so by Lemma 2.4, we have $|V(H_{z_i})| \geq \frac{n}{2} - 1$. It follows that

$$n = |V(G)| \geq |V(J' - V'_X) - \{z_1, z_2\}| + \sum_{i=1}^{2} |V(H_{z_i})| \geq |V(J' - V'_X)| - 2 + 2\left(\frac{n}{2} - 1\right) = |V(J' - V'_X)| - 4 + n,$$

and so $|V(J' - V'_X)| \leq 4$. This completes the proof of the claim.

Claim 4.2. $F(J') \geq 3$.

We assume by contradiction that $F(J') = 2$, and so by Theorem 2.2(ii), we have $J' = K_{2,h}$. By Observation 4.1(iii), J is not supereulerian. By Theorem 2.1(i), J' is not supereulerian. Thus h must be an odd integer and so $h \geq 3$. If $h - h_1 \leq 1$, then by Lemma 3.1(iii), G is contractible to a member in $\{2K_3^{--}, 2K_3^-, 2K_3, K_4, K_{2,3}\}$, contrary to Observation 4.1(ii). Hence $h - h_1 \geq 2$, and so Lemma 3.1(iv), either $t \in \{1, 2\}$ and $h - h_1 = 2$, or $t = 0$ and $0 \leq s \leq 2$. Let $D_h(J') = \{u_1, u_2\}$, and let H'_{u_1}, H'_{u_2} be the preimages of u_1 and u_2 in J, respectively.

Case 4.1. *Both $t \in \{1, 2\}$ and $h - h_1 = 2$.*

Let z_1, z_2 be the two vertices in $D_2(J') - V'_X$. By Lemma 3.1(iv), $Y = E_G[V(H_{z_1}), V(H_{z_2})]$. If $t = 1$, then $|\partial_G(V(H_{z_1}))| = |\partial_G(V(H_{z_2}))| = 3$, and so $j_0(s, 1) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 3$. Thus by (2), $s \in \{0, 1\}$. As $h_1 \leq s \leq 1$ and $h \geq 3$, we conclude that $h_1 = s = 1$, $t = 1$ and $h = 3$. It follows that $(X \cup Y) \cap E(H_{z_1} \cup H_{z_2} \cup H'_{u_1} \cup H'_{u_2}) = \emptyset$, and $G/(H_{z_1} \cup H_{z_2} \cup H'_{u_1} \cup H'_{u_2}) \cong K_4$, contrary to Observation 4.1 (ii).

Therefore, we assume that $t = 2$, and so $|\partial_G(V(H_{z_1}))| = |\partial_G(V(H_{z_2}))| = 4$. Thus $j_0(s, 1) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 4$. By (2), $s \in \{0, 1, 2\}$. Since $h \geq 3$ is odd, and $1 \leq h_1 \leq s$, we conclude that $h_1 = 1$, $t = 2$ and $h = 3$. If $s = 1$, then $H_{z_1} \cup H_{z_2} \cup H'_{u_1} \cup H'_{u_2}$ is a subgraph of G; if $s = 2$, then one of the restoration of $H_{z_1}, H_{z_2}, H'_{u_1}, H'_{u_2}$ in G may contain an edge in X. We continue using $H_{z_1}, H_{z_2}, H'_{u_1}, H'_{u_2}$ to denote their restorations in G. Then $G/(H_{z_1} \cup H_{z_2} \cup H'_{u_1} \cup H'_{u_2}) \cong 2K_3^-$, contrary to Observation 4.1 (ii). Case 4.1 is proved.

Case 4.2. *Both $t = 0$ and $0 \leq s \leq 2$.*

Hence $Y = \emptyset$ in this case. Denote $D_2(J') - V'_X = \{z_1, z_2, \ldots, z_{h-h_1}\}$. Then $\{z_1, z_2, \ldots, z_{h-h_1}\}$ is a stable set of J'. By symmetry, we may assume for any $1 \leq i < j \leq h - h_1$, there exists a vertex $z'_i \in V(H_{z_i})$ such that $d_G(z'_i) \leq d_G(z'_j)$. As for each $i \in \{1, 2, \ldots, h - h_1\}$, we have $|\partial_G(H_{z_i})| = 2$, it follows by (5) and Lemma 2.5(ii) that

$$n - 3 \geq n - (|V(H_{u_1})| - |V(H_{u_2})| - |V(H_{z_1})|) = \sum_{i=2}^{h-h_1} |V(H_{z_i})| \geq (h - h_1 - 1)(\frac{n}{2} - 3). \tag{24}$$

As $n > 50$, (24) implies that $1 \leq h - h_1 \leq 3$. Since $0 \leq h_1 \leq s$, we also have $0 \leq h_1 \leq 2$. If $h_1 = 0$, then since $h \geq 3$, we have $h = 3$, and so G is contractible to $K_{2,3}$, contrary to Observation 4.1 (ii). Hence we assume that $h_1 \in \{1, 2\}$. Since $Y = \emptyset$, we observe that for each $i \in \{1, 2, \ldots, h - h_1\}$, $\partial_G(V(H_{z_i})) \subseteq \partial_G(V(H_{u_1})) \cup \partial_G(V(H_{u_2}))$. If $h_1 = 1$, then as $h \geq 3$ is odd and $h - h_1 \leq 3$, we have $h - h_1 = 2$, and so G is contractible to $K^1_{2,2}$, contrary to Observation 4.1 (ii). If $h_1 = 2$, then as $h \geq 3$ is odd and $h - h_1 \leq 3$, either $h = 3$ or $h = 5$. If $h_1 = 2$ and $h = 3$, then G is contractible to $2K_3^{--}$, contrary to Observation 4.1 (ii). Hence we assume that $h = 5$ and so $h - h_1 = 3$. Then G is contractible to $K^2_{2,3}$. In this case, the pair of parallel edges in $K^2_{2,3}$ must be edges in X, and the 3 vertices of degree 2 in this $K^2_{2,3}$ are the three vertices in $D_2(J')$ and $V(J' - V'_X) = V(K^2_{2,3})$. As $Y = \emptyset$, by Claim 4.1(ii), we must have $5 = |V(K^2_{2,3})| = |V(J' - V'_X)| \leq 4$, a contradiction. Hence the possibility of $h = 5$ does not occur. This implies that we cannot have $F(J') = 2$, and so we must have $F(J') \geq 3$, validating Claim 4.2.

Let $n'' = |V(J') - V'_X|$. By Claim 4.2 and Theorem 2.2(iii), $2|E(J')| \leq 4n' - 10$. By (23), if $z \in V(J') - Z'$, then $d_{J'}(z) \geq s + 6$. Hence by $V'_Z \subset Z'$ and by Claim 4.1(ii),

$$\begin{aligned}(s + 6)(n'' - 2) &\leq (s + 6)(|V(J' - V'_X)| - |Z'|) \leq \sum_{v \in V(J' - V'_X)} d_{J'}(v) \leq 2|E(J')| \\ &\leq \sum_{v \in V(J')} d_{J'}(v) = 2|E(J')| \leq 4|V(J')| - 10 = 4n'' + 4|V'_X| - 10.\end{aligned}$$

Hence $(s + 2)(n'' - 2) + 4n'' - 8 \leq 4n'' + 4|V'_X| - 10$, or $(s + 2)(n'' - 2) \leq 4|V'_X| - 2$. Thus by $|V'_X| \leq s$,

$$n'' \leq 2 + \frac{4|V'_X| - 2}{s + 2} \leq 2 + \frac{4(s + 2) - 10}{s + 2} = 6 - \frac{10}{s + 2}. \tag{25}$$

Recall that $\kappa'(J') \geq 2$. If $1 \leq s \leq 2$, then $|V(J')| \leq 5$ with at most 2 vertices of degree 2. By Theorem 2.2(iii), $F(J') \leq 1$, contrary to Claim 4.2. Assume that $s = 3$. Then by (25), we have $n'' \leq 4$, and so $|V(J')| \leq 7$ and $|V'_X| \leq 3$. It follows by Theorem 2.2(iii) and Claim 4.2 that $F(G) = 3$, and so by Theorem 2.2(iv), $J' \cong K_{1,3}(1, 1, 1)$ with $|V'_X| = s = 3$. This implies that $G - Y$ can be contracted to K_4, and so $\kappa'(G - Y) \leq \kappa'(K_4) = 3$. By (2) with $s = 3$, $\kappa'(G) \geq t + 4$, and so $3 \geq \kappa'(G - Y) \geq 4$, a contradiction.

Hence we may assume that $s \geq 4$ and $n'' \in \{4, 5\}$. By (2) and by assumption of Theorem 1.2, we have $\kappa'(G) \geq j_0(s, t) \geq s + t \geq 4$. As $|Y| \leq t$, we have $\kappa'(G - Y) \geq s \geq 4$. Let $X_1 \subseteq X$ such that $|X - X_1| = \min\{2, |X|\}$. Then by Lemma 2.1 with $s = 2\ell + \epsilon$ and $k = 2$, we conclude that $G - (X_1 \cup Y)$ has 2 edge-disjoint spanning trees. Since adding a new vertex with two edges joining the new vertex to a graph with two edge-disjoint spanning trees will also result in a graph with two edge-disjoint spanning trees, we conclude that $(G - Y)(X_1)$ also has two edge-disjoint spanning trees.

Since $J = (G - Y)(X) = ((G - Y)(X_1))(X - X_1)$ and since $|X - X_1| = \min\{2, |X|\} \leq 2$, it follows that $F(J) \leq 2$, and so $F(J') \leq F(J) \leq 2$, contrary to Claim 4.2. Hence J' must be isomorphic to K_1, contrary to (22). This completes the proof of the theorem. ■

In the arguments to prove Corollary 1.1 and Theorem 1.3, we define the following notation for a graph G, and for an integer $k \geq 2$:

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(v_i) : v_1, v_2, \ldots, v_k \in V(G) \text{ and } E(G[\{v_1, v_2, \ldots, v_k\}]) = \emptyset\}. \tag{26}$$

We are to apply Theorem 1.2 to prove Corollary 1.1. By Proposition 1.1, the necessity of Corollary 1.1 holds. Hence we assume that $\kappa'(G) \geq j_0(s,t)$ to show that G is (s,t)-supereulerian. Arguing by contradiction, we assume that G satisfies (6) with $\kappa'(G) \geq j_0(s,t)$ but G is not (s,t)-supereulerian. By (3) and (26), we have $\sigma_2(G) \leq 2m_2(G)$. Thus that $\sigma_2(G) \geq n - 4$ implies that $m_2(G) \geq \frac{n}{2} - 2$. It follows, by Theorem 1.2 and by the assumption that G is not (s,t)-supereulerian, that G must be contractible to a graph H, which is isomorphic to a member in $\{2K_3^{--}, 2K_3^{-}, 2K_3, K_4, K_{2,2}^{1}, K_{2,3}\}$. Let $n' = |V(H)|$ and $V(H) = \{z_1, z_2, \ldots, z_{n'}\}$, and let H_{z_i} be the preimage of z_i in G.

Since $\delta(G) \geq 5 > 4 \geq \Delta(H)$, it follows by Lemma 2.4 that for any vertex $z_i \in V(H)$, we have $|V(H_{z_i})| \geq 6 > 4 \geq |\partial_G(V(H_{z_i}))|$. Therefore, there must be a vertex $v_i \in V(H_{z_i})$ such that $N_G(v_i) \subseteq V(H_{z_i})$, which implies that $E(G[\{v_1, v_2, v_3\}]) = \emptyset$. By (6), we have

$$2n \geq 2\sum_{i=1}^{3} |V(H_{z_i})| \geq 2\sum_{i=1}^{3} |N_G[v_i]| \geq 3(n-4) + 6 = 3n - 6.$$

This leads to $51 \leq n \leq 9$, a contradiction, and so the sufficiency of Corollary 1.1 is also proved. ■

There exist infinitely many examples to show that if we relax the condition $\delta(G) \geq 5$ in Corollary 1.1, then Corollary 1.1 will not hold.

Example 4.1. Let $m > 4$ be an arbitrary integer. Let z_1, z_2, z_3 denote the vertices of a $2K_3$, which is defined in Fig. 1. Let $2K_3(m)$ denote a simple graph formed from $2K_3$ by replacing z_1 by a complete subgraph K_m, and each of z_2 and z_3 replaced by a complete graph K_4 in such a way that $\delta(2K_3(m)) \geq 4$. Then $2K_3(m)$ satisfies (6). As $2K_3(m)$ is contractible to $2K_3$, and by Example 1.1, $2K_3$ is not $(2,2)$-supereulerian, we conclude that $2K_3(m)$ is not $(2,2)$-supereulerian with $\kappa'(2K_3(m)) = 4 = j_0(2,2)$.

5. Proof of Theorem 1.3

By Proposition 1.1, the necessity of Theorem 1.3 holds. Hence we assume that $\kappa'(G) \geq j_0(s,t)$ to show that G is (s,t)-supereulerian.

Proof of the sufficiency of Theorem 1.3. We also argue by contradiction to prove the sufficiency of Theorem 1.3, and assume that G is a counterexample to Theorem 1.3 with $n = |V(G)| \geq 51$. This assumption leads to the following observation.

Observation 5.1. *Each of the following holds for this graph G.*

(i) G satisfies (7) and $\kappa'(G) \geq j_0(s,t)$.

(ii) There exist disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$ such that $G - Y$ does not have a spanning closed trail that contains all edges in X. By Observation 2.1(iv), $(G - Y)(X)$ is not supereulerian.

Fix a pair of disjoint edge subsets X and Y of $E(G)$ whose existence is assured by Observation 5.1(ii). As before, we let $J = (G - Y)(X)$, J' be the reduction of J, and adopt the notation in Observation 2.1 for the definition of V_X and E_X. Define V_X' as in (11). Once again, by Observation 2.1(iv) and (v) and by $\kappa'(G) \geq j_0(s,t)$, we have $\kappa'(J') \geq \kappa'(J) \geq 2$, J' is not supereulerian, and $F(J') \geq 2$. For each vertex $z \in V(J')$, let H_z' be the preimage of z in J and let H_z denote the restoration of H_z' in $G - Y$. In the arguments below, we let $D_2(J')$ be the set of vertices of degree 2 in J'. Then by (11), $V_X' \subseteq D_2(J')$. Then by (8)$h = |D_2(J')|$ and $h_1 = |V_X'|$.

Claim 5.1. *J' does not have three vertices z_1, z_2, z_3 such that there exist three vertices $v_1, v_2, v_3 \in V(G)$ with the property that for each $i \in \{1, 2, 3\}$, $v_i \in V(H_{z_i})$ with $N_G(v_i) \subseteq V(H_{z_i})$.*

By contradiction, assume that such vertices exist. Then, as $N_G(v_i) \subseteq V(H_{z_i})$, we must have $E(G[\{v_1, v_2, v_3\}]) = \emptyset$ and so by (7),

$$n = |V(G)| \geq \sum_{i=1}^{3} |V(H_{z_i})| \geq \sum_{i=1}^{3} (|N_G(v_i)| + 1) > (n-3) + 3 = n, \tag{27}$$

a contradiction. This proves the claim.

Claim 5.2. *G cannot be contractible to a graph L with $\Delta(L) \leq 4$ and $|V(L)| \geq 3$.*

By contradiction, we assume that G is contractible to a graph L with $\Delta(L) \leq 4$ and $|V(L)| \geq 3$. Let w_1, w_2, w_3 be three distinct vertices of $V(L)$ and let H_{w_i} be the preimage of w_i in G. As $\delta(G) \geq 5 > 4 \geq \Delta(L)$, it follows by Lemma 2.4 that $|V(H_{w_i})| \geq 6$ for any i with $1 \leq i \leq 3$. As $|V(H_{w_i})| > 4 \geq |\partial_G(H_{w_i})|$, there must be a vertex $v_i \in V(H_{w_i})$ such that $N_G(v_i) \subseteq V(H_{w_i})$. Thus by (7), we obtain the same contradiction as in (27). This proves Claim 5.2.

Claim 5.3. *J' is not isomorphic to a member in $\{K_{2,2p+1} : p \geq 1\}$.*

We assume that for some integer $p \geq 1$, $J' = K_{2,2p+1}$. Thus $h = |D_2(J')| = 2p+1$ is an odd integer. Let $D_h(J') = \{u_1, u_2\}$, and let $D_2(J) = \{z_1, z_2, \ldots, z_h\}$. For each vertex $w \in V(J')$, let H'_w denote the preimages of w in J, and H_w denote the restoration of H'_w in $G - Y$. Since $\kappa'(G) \geq j_0(s,t)$, it follows by Lemma 3.1 that either Lemma 3.1(iii) or Lemma 3.1(iv) must hold. If Lemma 3.1(iii) holds, then G is contracted to a member in $\{2K_3^{--}, 2K_3^{-}, 2K_3\}$, leading to a contradiction to Claim 5.2. Hence we assume that Lemma 3.1(iv) must hold, and so $h - h_1 \geq 2$ and either $t \in \{1,2\}$, $h - h_1 = 2$ and edges in Y are joining vertices in the preimages of the two vertices in $D_2(J') - V'_X$, or $t = 0$ and $0 \leq s \leq 2$.

Case 5.1. *Both $t \in \{1,2\}$ and $h - h_1 = 2$.*

Without loss of generality, we assume that z_1, z_2 are the two vertices in $D_2(J') - V'_X$. By Lemma 3.1(iv), $Y = E_G[V(H_{z_1}), V(H_{z_2})]$. If $t = 1$, then $|\partial_G(V(H_{z_1}))| = |\partial_G(V(H_{z_2}))| = 3$, and so $j_0(s,1) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 3$. Thus by (2), $s \in \{0,1\}$. As $h_1 \leq s \leq 1$ and $h \geq 3$, we conclude that $h_1 = s = 1, t = 1$ and $h = 3$, and so $(X \cup Y) \cap E(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2}) = \emptyset$. Let $L = G/(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2})$. Then $|V(L)| = 5$ with $\Delta(L) = 3$, contrary to Claim 5.2.

Therefore, we assume that $t = 2$, and so $|\partial_G(V(H_{z_1}))| = |\partial_G(V(H_{z_2}))| = 4$. Thus $j_0(s,1) \leq \kappa'(G) \leq |\partial_G(V(H_{z_1}))| = 4$. By (2), $s \in \{0,1,2\}$. Since $h \geq 3$ is odd, and $1 \leq h_1 \leq s$, we conclude that $h_1 = 1$, $t = 2$ and $h = 3$. Once again, $(X \cup Y) \cap E(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2}) = \emptyset$. Let $L' = G/(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2})$. Then $|V(L')| = 5$ with $\Delta(L') = 3$, contrary to Claim 5.2. This completes the prove for Case 5.1.

Case 5.2. *Both $t = 0$ and $0 \leq s \leq 2$.*

$D_2(J') - V'_X = \{z_1, z_2, \ldots, z_{h-h_1}\}$. Then $\{z_1, z_2, \ldots, z_{h-h_1}\}$ is a stable set of J'. If $h - h_1 \geq 3$, and by symmetry we assume $D_2(J') - V'_X = \{z_1, z_2, z_3\}$. It follows from $\delta(G) \geq 5 > 2 = |\partial_G(H_{z_i})|$ and by Lemma 2.4 that for each $i \in \{1,2,3\}$, $|V(H_{z_i})| \geq 6$. As $|\partial_G(H_{z_i})| = 2$, there must be a vertex $v_i \in V(H_{z_i})$ with $N_G(v_i) \subseteq V(H_{z_i})$, leading to a violation of Claim 5.1. Hence we must have $h - h_1 \leq 2$. Since h is an odd number, we have either $s = 1, h - h_1 = 2$ or $s = 2, h - h_1 = 1$. In either case, $h = 3$ and $X \cap E(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2}) = \emptyset$. Define $L'' = G/(H_{z_1} \cup H_{z_2} \cup H_{z_3} \cup H_{u_1} \cup H_{u_2})$. Then $|V(L'')| = 5$ with $\Delta(L'') = 3$, contrary to Claim 5.2. This completes the prove for Case 5.2, as well as Claim 5.3.

Claim 5.4. *Each of the following holds.*
(i) $F(J') \geq 3$.
(ii) $s \leq 3$.

If $F(J') \leq 2$, then as J is not supereulerian, it follows by Theorem 2.2(i) that $J' \in \{K_{2p+1} : p \geq 1\}$, contrary to Claim 5.3. This proves (i).

Now assume that $s \geq 4$. Then by assumption of Corollary 1.1, $\kappa'(G) \geq j_0(s,t)$. Without loss of generality, we assume that $|X| = s$. Let X' be a subset of X such that $|X'| = 2$, and let $X'' = X - X'$. Then by Lemma 2.1, $G - (X'' \cup T)$ has 2 edge-disjoint spanning trees T_1 and T_2 (say). For each edge $e_i = u_i v_i \in X''$, subdividing e_i means to add a new vertex v_{e_i} and to replace the edge e_i by the path $u_i v_{e_i} v_i$, thus $T_1 + \{u_i v_{e_i}\}$ and $T_2 + \{v_{e_i} v_i\}$ are two edge-disjoint spanning trees of $G - ((X'' - \{e_i\}) \cup T)$. This indicates that $(G - Y)(X - X')$ also have two edge-disjoint spanning trees, and so as $|X'| = 2$, we have $F(J) = F((G - Y)(X)) \leq 2$. Since J is not supereulerian and $\kappa'(J') \geq \kappa'(J) \geq 2$, by Theorem 2.1(i), J' is a member in $\{K_{2p+1} : p \geq 1\}$, contrary to Claim 5.3. This proves (ii), as well as Claim 5.4.

Denote $d_i = |D_i(J')|$ and $n' = |V(J')|$. By Claim 5.4, $F(J') \geq 3$. By Theorem 2.2(iii), we have $2n' - |E(J')| \geq 5$. This leads to

$$4\sum_{i\geq 2} d_i - \sum_{i\geq 2} i d_i \geq 10, \text{ or equivalently, } 2d_2 + d_3 \geq 10 + \sum_{i\geq 5}(i-4)d_i.$$

It follows that $2(d_2 + d_3) \geq 10$, or $d_2 + d_3 \geq 5$, where equality holds if and only if $d_3 = 0$, $d_2 = 5$ and $\sum_{i\geq 5}(i-4)d_i = 0$. By Claim 5.4(ii), $|D_2(J') \cup D_3(J') - V'_X| \geq 2$, where equality holds if and only if $d_3 = 0$, $|V'_X| = s = 3$, $d_2 = 5$ and $\sum_{i\geq 5}(i-4)d_i = 0$. Thus if $|D_2(J') \cup D_3(J') - V'_X| = 2$, then $|V(J')| = |D_2(J')| = 5$, implying that J' is a cycle of 5 vertices, and so J' must be supereulerian, a contradiction to Observation 5.1(ii). Hence we must have $|D_2(J') \cup D_3(J') - V'_X| > 2$, and so $D_2(J') \cup D_3(J') - V'_X$ contains three distinct vertices z_1, z_2, z_3. Let H'_{z_i} denote the preimage of z_i in J and H_{z_i} the restoration of H'_{z_i} in $G - Y$. Then by $\delta(G) \geq 5$ and by Lemma 2.4, we have $|V(H_{z_i})| \geq 6 > 3 \geq |\partial_G(H_{z_i})|$, and so for each $i \in \{1,2,3\}$, there must be a vertex $v_i \in V(H_{z_i})$, such that $N_G(v_i) \subseteq V(H_{z_i})$. This is a contradiction to Claim 5.1, and so the theorem is proved. ■

6. Concluding remarks

In applications, the must avoid edges Y may represent the connections at fault and the must included edges in X may represent the routing constrains, the notion of (s, t)-supereulerian would also be a suitable model in interconnection network studies for appropriate problems. Theoretically, determining if a graph is $(0, 0)$-supereulerian is known as an NP-complete problem, it would be of interests to seek new best possible sufficient conditions to warrant the property of being (s, t)-supereulerian. This paper presents a sharp strengthened Ore Type degree condition in Theorem 1.2, whose corollary implies an Ore Type degree conditions for (s, t)-supereulerian graphs. In particular, it generalizes the Dirac degree condition for (s, t)-supereulerian graphs obtained in [16]. A common density condition, called the neighborhood union condition of a graph G, is defined as

$$U_2(G) = \min\{|N_G(u) \cup N_G(v)| : u, v \in V(G) \text{ and } uv \notin E(G)\}.$$

As $m_2(G) \geq \frac{1}{2}U_2(G)$, our main results can also imply certain sufficient conditions for (s, t)-supereulerian graphs using neighborhood union conditions. Thus it will be of interests to seek other sufficient conditions that are independent of this strengthened Ore Type degree condition, and other structural or extremal conditions for a graph to be (s, t)-supereulerian. In particular, in view of (26), it is also of interest to investigate sufficient conditions for a graph G with bounded stability number $\alpha(G)$ to be (s, t)-supereulerian, and sharp conditions in terms of $\sigma_k(G)$ for large values of k for a graph G to be (s, t)-supereulerian.

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