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Modulo orientations and matchings in graphs

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ABSTRACT

The modulo orientation problem seeks a so-called mod $(2t+1)$-orientation of an undirected graph, in which the indegree is equal to outdegree under modulo $2t+1$ at each vertex. Jaeger's circular flow conjecture states that every graph G with edge connectivity $\kappa'(G) \geq 4t$ has a mod $(2t+1)$-orientation. Lovász et al. (2013) verified it for $\kappa'(G) \geq 6t$, and later Han et al. (2018) disproved Jaeger's conjecture with infinitely many counterexamples for $t \geq 3$. In this paper, we show there are essentially finitely many exceptions for graphs with a bounded matching number. More generally, for any positive integers t and s, there exists a finite family $\mathcal{G}(t,s)$ of graphs not admitting any mod $(2t+1)$-orientations, such that any graph G with $\kappa'(G) \geq 2t+2$ and matching number $\alpha'(G) \leq s$ has a mod $(2t+1)$-orientation if and only if G cannot be contracted to an element of $\mathcal{G}(t,s)$. This immediately implies a Chvátal-Erdős type theorem and we additionally characterize all infinitely many graphs with $\kappa' \geq \alpha'$ but without a nowhere-zero 3-flow. Our results also indicate that the problem of seeking mod orientations for planar graphs with bounded matching number belongs to P, while for general planar graphs it is a known NP-complete problem.

1. Introduction

This paper studies loopless finite graphs in which multiple edges may be permitted. For many standard notation and terminologies, we follow the textbook of Bondy and Murty [1]. Specifically, in a graph G we use notations minimum degree $\delta(G)$, edge connectivity $\kappa'(G)$ and matching number $\alpha'(G)$, respectively. If in a graph G two vertices u and v are adjacent, then we write $u \sim v$. Given disjoint vertex subsets $A, I \subseteq V(G)$, denote $[A, I]_G = \{uv \in E(G) : u \in A, v \in I\}$. Sometimes we use easier notations $[u, I]_G$ or $[A, v]_G$ if $A = \{u\}$ or $I = \{v\}$, and we also use $\partial_G(A) = [A, V(G) - A]_G$, which may often omit subscript for convenience.

If we assign an orientation to all the edges of an undirected graph G, then G is said to possess an orientation. Fixed a graph G with orientation D, let $E_D^-(v)$ ($E_D^+(v)$, resp.) be the set of all ingoing (outgoing, resp.) arcs at vertex v and let $d_D^-(v) = |E_D^-(v)|$, $d_D^+(v) = |E_D^+(v)|$. If f is a mapping assign each $e \in E(G)$ to an integer in $\mathbb{Z}$ satisfying $\sum_{e\in E_D^+(v)} f(e) = \sum_{e\in E_D^-(v)} f(e)$ at each vertex $v \in V(G)$, then we call (D, f) an *integer flow*. We call it a *nowhere-zero k-flow* (abbreviated as k-NZF) if it holds additionally that $0 < |f(e)| < k$, for any $e \in E(G)$. A *nowhere-zero modular k-flow* of G is an ordered pair (D, f) where D is an orientation of $E(G)$ and f is a function: $E(G) \to \mathbb{Z}_k - \{0\}$ such that $\sum_{e\in E_D^+(v)} f(e) \equiv \sum_{e\in E_D^-(v)} f(e)$ (mod k) at each vertex $v \in V(G)$. Tutte [22] gives the following fundamental theorem that a graph G admits a k-NZF if and

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only if G admits a nowhere-zero modular k-flow. A nowhere-zero modular k-flow is also called $\mathbb{Z}_k$-NZF. For a 3-NZF of the graph G, by choosing reversed orientation of certain edges of value 2 in G, we may obtain a $\mathbb{Z}_3$-NZF (D', f_1) such that $f_1(e) = 1$ for each $e \in E(G)$. This orientation D' is called a *mod 3-orientation*, which satisfies $|E^+_{D'}(x)| \equiv |E^-_{D'}(x)| \pmod 3$ for each $x \in V(G)$. So a graph has a 3-NZF if and only if it has a mod 3-orientation. In general, for a graph G, a *mod* $(2t+1)$-*orientation* is an orientation D such that the outdegree $d^+_D(x)$ is congruent to indegree $d^-_D(x)$ modulo $2t+1$ for every vertex x. Denote by $\mathcal{M}_{2t+1}$ the family of all mod $(2t+1)$-orientation-admissible graphs. The authors in [13,14,17] studied a more general concept of strongly $\mathbb{Z}_{2t+1}$-connected graphs, allowing orientation with prescribed boundaries at each vertex. A graph G is *strongly* $\mathbb{Z}_{2t+1}$-*connected* if for every $\alpha : V(G) \to \mathbb{Z}_{2t+1}$ with $\sum_{u \in V(G)} \alpha(u) \equiv 0 \pmod{2t+1}$, there exists an orientation D with $d^+_D(u) - d^-_D(u) \equiv \alpha(u) \pmod{2t+1}, \forall u \in V(G)$. Let $\langle \mathcal{SZ}_{2t+1} \rangle$ be the graph family consisting of all strongly $\mathbb{Z}_{2t+1}$-connected graphs.

Tutte [23] showed that a planar graph H has a proper vertex 3-coloring if and only if its planar dual H^* admits a 3-NZF (or equivalently $H^* \in \mathcal{M}_3$). The 3-vertex-coloring planar graph problem is NP-complete, and thus the mod 3-orientation problem is NP-complete by duality. In [18], MacGillivray and Siggers further proved that the homomorphism problem to odd cycle C_{2t+1} on planar graph is NP-complete. By the duality of circular flow and circular coloring, this also gives the NP-completeness of mod $(2t+1)$-orientation problem for fixed $t > 0$.

On the other hand, every triangle-free planar graph is vertex 3-colorable from the classical Grötzsch's 3-coloring theorem, which equivalently provides a 3-NZF for every 4-edge-connected planar graph by duality. Bill Tutte in 1970s suggested that the later statement maybe hold for nonplanar graphs as well. This is now known as the celebrated 3-flow conjecture.

Conjecture 1.1. (Tutte's 3-flow conjecture, see [1]) *Every* 4-*edge-connected graph has a* 3-*NZF.*

Tutte's flow conjectures were further extended by Jaeger [10] and Lai [13] to highly connected graphs for general mod $(2t+1)$-orientations. In 2012, Thomassen [21] proved the weak versions of these conjectures for high edge-connectivity $2(2t+1)^2 + 2t + 1$. It was further improved to $6t$-edge-connected graphs by Lovász et al. [17].

Theorem 1.2. (Lovász, Thomassen, Wu and Zhang [17]) *Every* $6t$-*edge-connected graph is strongly* $\mathbb{Z}_{2t+1}$-*connected, and therefore admits a mod* $(2t+1)$-*orientation.*

However, the original problems of Jaeger [10] and Lai [13] were answered negatively in [9] recently, for larger values of t.

Theorem 1.3. ([9])

(1) *For every integer* $t \geq 3$, *there exist infinitely many* $4t$-*edge-connected graphs without a mod* $(2t+1)$-*orientation.*
(2) *For every integer* $t \geq 5$, *there exist infinitely many* $(4t+1)$-*edge-connected graphs without a mod* $(2t+1)$-*orientation.*

Pushing further on the edge connectivity condition to warrant mod $(2t+1)$-orientation seems to be very challenged in either direction. It remains widely open seeking other types of nice sufficient conditions for mod orientations.

In this paper, we prove a relatively positive result that if a graph family has a bounded matching number, then after certain reduction operations, there are only finitely many $(2t+2)$-edge-connected graphs without mod $(2t+1)$-orientations in this family. To state our theorem formally, we shall first introduce graph contraction operation and the concept of $\langle \mathcal{SZ}_{2t+1} \rangle$-reduction below.

For an edge e of the graph G, *edge contraction*, denoted by G/e, is an operation which removes edge e from the graph while simultaneously merging the vertices of e into a single vertex and then delete the generating loops. More generally, the operation may be performed on a set of edges by contracting each edge (in any order).

Fixed a graph G, each vertex contains in a maximal strongly $\mathbb{Z}_{2t+1}$-connected subgraph, since the singleton $K_1 \in \langle \mathcal{SZ}_{2t+1} \rangle$. By Proposition 2.2 in [13], every vertex lies in one unique maximal strongly $\mathbb{Z}_{2t+1}$-connected subgraph of G. Select all the maximal strongly $\mathbb{Z}_{2t+1}$-connected subgraph of G, denoted by $G_1, G_2, \cdots, G_c$. Define $G' = G/(\cup_{i=1}^{c} E(G_i))$ as the $\langle \mathcal{SZ}_{2t+1} \rangle$-*reduction* of G, or saying that G is $\langle \mathcal{SZ}_{2t+1} \rangle$-reduced to G'. Thus, for any graph G, its $\langle \mathcal{SZ}_{2t+1} \rangle$-*reduction* G' is unique. A graph G is called $\langle \mathcal{SZ}_{2t+1} \rangle$-*reduced* if $G = G'$ (i.e. it contains no strongly $\mathbb{Z}_{2t+1}$-connected subgraph G with $|V(G)| > 1$). The construction in [9] indicates that for every $t \geq 5$, there are infinitely many $\langle \mathcal{SZ}_{2t+1} \rangle$-reduced graphs without a mod $(2t+1)$-orientation in the family of all $(4t+1)$-edge-connected graphs. For $G \in \langle \mathcal{SZ}_{2t+1} \rangle$, it is proved in [13] that, for every supergraph Γ of G, $\Gamma \in \mathcal{M}_{2t+1}$ if and only if Γ/G does. Therefore, seeking mod $(2t+1)$-orientations of a graph G is equivalent to seeking mod $(2t+1)$-orientations of the $\langle \mathcal{SZ}_{2t+1} \rangle$-reduction of G. Our first main result is formally stated as follows.

Theorem 1.4. *For a fixed integer* $s > 0$, *there exists a graph family* $\mathcal{G}(t, s)$ *of finite cardinality such that every graph* G *with* $\kappa'(G) \geq 2t+2$ *and* $\alpha'(G) \leq s$ *has a mod* $(2t+1)$-*orientation if and only if its* $\langle \mathcal{SZ}_{2t+1} \rangle$-*reduction* $G' \notin \mathcal{G}(t, s)$.

As the $\langle \mathcal{SZ}_{2t+1} \rangle$-reduction operation is a special type of contraction, Theorem 1.4 is still valid with contraction replacing $\langle \mathcal{SZ}_{2t+1} \rangle$-reduction. That is, for every graph G with $\kappa'(G) \ge 2t+2$ and $\alpha'(G) \le s$, $G \in \mathcal{M}_{2t+1}$ if and only if G is not contractible to a graph in the finite graph family $\mathcal{G}(t,s)$.

The contractibility problem, asking to decide whether a graph is contractible to G, is NP-complete for any triangle-free graph G other than star [2], and it is polynomial-time solvable when G is a star [2], a clique [15,19], and some other graphs. It is proved by Kamiński, Paulusma, and Thilikos [11] that for any graph G, there is a polynomial-time algorithm to decide whether a planar graph is contractible to G. Since there are constant many graphs in the family $\mathcal{G}(t,s)$ in Theorem 1.4, we can check whether a planar G is contractible to a member of $\mathcal{G}(t,s)$ in polynomial-time. Furthermore, each member of $\mathcal{G}(t,s)$ has size bounded by some constant $N(t,s)$ (see Section 2), which is constant-time determined. Therefore, we obtain a polynomial-time algorithm from Theorem 1.4 to decide whether a planar graph G with $\kappa'(G) \ge 2t+2$ and $\alpha'(G) \le s$ has a mod $(2t+1)$-orientation for fixed s and t. In contrast, the mod orientation problem remains NP-complete for planar graph G with $\kappa'(G) \ge 2t+2$ when $t \ge 2$ as shown by Esperet, Montassier, Ochem, Pinlou [4].

Corollary 1.5. *For fixed integers $s > 0$ and $t > 0$, there exists a polynomial-time algorithm to decide whether a planar graph G with $\kappa'(G) \ge 2t+2$ and $\alpha'(G) \le s$ has a mod $(2t+1)$-orientation. Moreover, the running time is $\mathcal{O}(2^{(2t-1)N(t,s)^2}|V(G)|^{\mathcal{O}(N(t,s))})$.*

Note that the polynomial-time algorithm of Corollary 1.5 extends to graphs embeddable on surfaces as well, since Kamiński et al. [11] also provides a polynomial-time algorithm for contractibility of graphs embeddable on a given surface.

Theorems with conditions involving the relationship between $\kappa(G)$ (or $\kappa'(G)$) and $\alpha(G)$ (or $\alpha'(G)$) are often called *Chvátal-Erdős* type theorem, see [5]. Theorem 1.4, together with Theorem 1.2, immediately implies a Chvátal-Erdős type result: if G satisfies a Chvátal-Erdős type condition $\kappa'(G) \ge \max\{\alpha'(G), 2t+2\}$, then G admits a mod $(2t+1)$-orientation with essentially finitely many exceptions.

Theorem 1.6. *For fixed integer $t > 0$, there is a finite family of non-modulo $(2t+1)$-orientation-admissible graphs $\mathcal{F}_1(t)$ with the following property: A graph G with $\kappa'(G) \ge \max\{\alpha'(G), 2t+2\}$ has a mod $(2t+1)$-orientation if and only if it is not contracted to a graph in $\mathcal{F}_1(t)$.*

Theorems 1.4 and 1.6 are best possible results in a sense that the edge connectivity $2t+2$ cannot be replaced by $2t+1$. In fact, there are infinitely many $(2t+1)$-edge-connected $\langle \mathcal{SZ}_{2t+1} \rangle$-reduced graphs with fixed matching number admitting no mod $(2t+1)$-orientation. Let $n \ge s \ge 3$ be integers. The graph $K_{s,n}^{+k}$ is constructed from complete bipartite graph $K_{s,n}$ by adding k extra edges connecting vertices of degree n arbitrarily. The obtained graph $K_{s,n}^{+k}$ may have parallel edges. We write $K_{s,n}^{+}$ as an abbreviation of $K_{s,n}^{+1}$ in this paper. It is easy to observe that $K_{3,n}^{+}$ admits no mod 3-orientation for any $n \ge 3$. This example can be generalized to mod $(2t+1)$-orientations for any t. For example, let $\mathcal{K}(t)$ be a graph family defined by $\mathcal{K}(t) = \{K_{2t+1,c}^{+} : c \ge 2t+1\}$. Then each member in $\mathcal{K}(t)$ is a $(2t+1)$-edge-connected $\langle \mathcal{SZ}_{2t+1} \rangle$-reduced graph without a mod $(2t+1)$-orientation, providing infinitely many examples of connectivity $2t+1$ respected to Theorems 1.4 and 1.6.

The 3-Flow Conjecture of Tutte has been intensively studied. For example, Grünbaum in [7] proved that if G is planar with $\kappa'(G) \ge 2$ and has at most three 3-edge-cuts, then G has a 3-NZF. Steinberg and Younger in [20] showed that if G is a projective planar graph with $\kappa'(G) \ge 2$ and at most one 3-edge-cut, then G has a 3-NZF.

While Theorems 1.4 and 1.6 with $t = 1$ address the 3-flow problem, the next result of this research is an improvement to Theorem 1.6 by characterizing all the (infinitely many) $\langle \mathcal{SZ}_3 \rangle$-reduced graphs G with $\kappa'(G) \ge \alpha'(G)$. Denote $\mathcal{G}_1$ to be a class of graphs in Fig. 1.

Theorem 1.7. *Every bridgeless graph G with $\kappa'(G) \ge \alpha'(G)$ has a 3-NZF, unless G belongs to one of the following exceptional cases.*

(1) *G can be contracted (or $\langle \mathcal{SZ}_3 \rangle$-reduced) to a graph in $\mathcal{G}_1$.*
(2) *G is the graph K_4 or $K_{3,n}^{+}$ for a positive integer $n \ge 3$.*

In 2002, Kochol [12] showed that Conjecture 1.1 is equivalent to a seemly stronger form that every bridgeless graph with at most three 3-edge-cuts admits a 3-NZF. The corollary below is an immediate application of Theorem 1.7.

Corollary 1.8. *Every bridgeless graph G with $\kappa'(G) \ge \alpha'(G)$ admits a 3-NZF provided that G has at most three 3-edge-cuts.*

In the next two sections, we shall give the proofs of our main results, Theorems 1.4 and 1.7.

2. Modulo orientations and matchings

We start with some needed lemmas. Let mH be the graph constructed from H by replacing each edge of H with m parallel edges. We summarize some fundamental properties of mod $(2t+1)$-orientation and strongly $\mathbb{Z}_{2t+1}$-connectedness from [13] and [14].

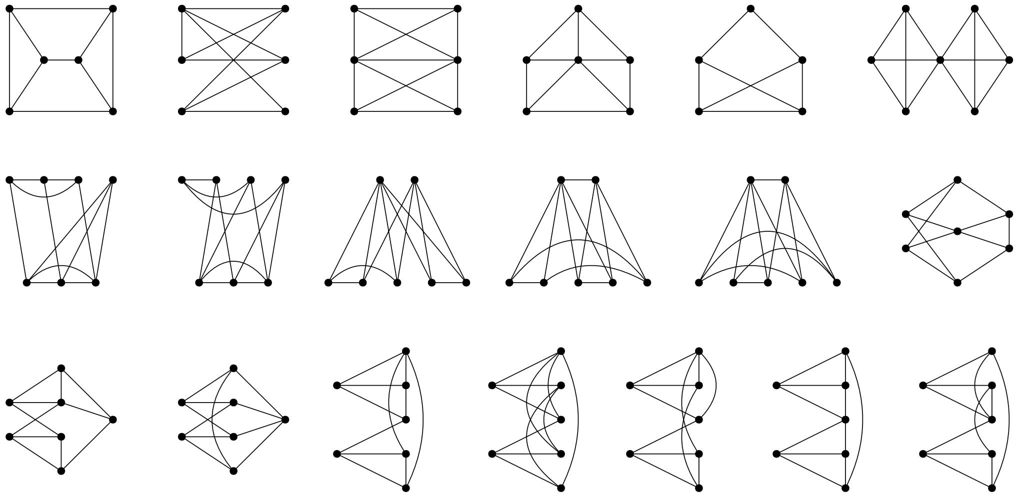

Fig. 1. The graph family $\mathcal{G}_1$ in Theorem 1.7 consists of 19 graphs. There are four graphs with 6 vertices, one graph with 5 vertices, and the rest fourteen graphs have 7 vertices.

Lemma 2.1. ([13,14]) *For fixed integers $m, t > 0$ and fixed graph H, we have properties below:*

(1) *For $H \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ and $e \in E(H)$, $H/e \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$.*
(2) *For $H \subseteq G$, if both $G/H \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ and $H \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$, then $G \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$.*
(3) *The graph $mK_2 \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ if and only if $m \geq 2t$.*
(4) *A nontrivial complete graph $K_m \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ if and only if $m \geq 4t+1$.*
(5) *The graph $H \in \mathcal{M}_{2t+1}$ if and only if its $\langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$-reduction $H' \in \mathcal{M}_{2t+1}$.*
(6) *The graph $H \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ if and only if its $\langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$-reduction $H' = K_1$.*

The lifting lemma below can be easily obtained from the definition of strongly $\mathbb{Z}_{2t+1}$-connected graphs.

Lemma 2.2 *(Lifting). For a graph H with $v_1v_2, v_1v_3 \in E(H)$, construct a graph $H_{[v_1,v_2v_3]}$ from H by removing v_1v_2, v_1v_3 and adding a new edge v_2v_3. If $H_{[v_1,v_2v_3]} \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$, then $H \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$.*

For notational convenience, we always use $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ to denote the two parts of a complete bipartite graph $K_{m,n}$. For any subset $\{t_1, t_2, \ldots, t_\ell\}$ of $\mathbb{Z}_m$, form a graph $K_{m,n}(t_1, t_2, \ldots, t_\ell)$ from $K_{m,n}$ by identifying $u_1, \ldots, u_{t_1}$, identifying $u_{t_i+1}, \ldots, u_{t_{i+1}}$ for each $1 \leq i \leq \ell - 1$ and identifying $u_{t_\ell+1}, \ldots, u_m$, respectively. Define a family of graphs $\mathcal{B}^*(m,n)$ to be

$$\mathcal{B}^*(m,n) = \{K_{m,n}(t_1, t_2, \ldots, t_\ell) : \{t_1, t_2, \ldots, t_\ell\} \subseteq \mathbb{Z}_m\}.$$

Lemma 2.3. *Every member of $\mathcal{B}^*(2p+2, 4p^2+2p)$ is strongly $\mathbb{Z}_{2p+1}$-connected.*

Proof. We first apply the lifting lemma to show the complete bipartite graph $K_{2p+2,4p^2+2p}$ is strongly $\mathbb{Z}_{2p+1}$-connected. Recall that $U = \{u_1, u_2 \ldots, u_{2p+2}\}$ and $V = \{v_1, v_2, \ldots, v_{4p^2+2p}\}$ are the set of all degree $4p^2+2p$ vertices and all degree $2p+2$ vertices in $K_{2p+2,4p^2+2p}$, respectively. Obtain a graph K' from $K_{2p+2,4p^2+2p}$ by lifting $v_{2pi+1}u_{i+1}, v_{2pi+1}u_{i+2}$, lifting $v_{2pi+2}u_{i+1}, v_{2pi+2}u_{i+2}, \ldots$, and lifting $v_{2pi+2p}u_{i+1}, v_{2pi+2p}u_{i+2}$ to obtain $2p$ parallel edges between u_{i+1} and u_{i+2}, for each $0 \leq i \leq 2p$. Then $K'[U] \in \langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$ by Lemma 2.1(3). Notice that $|[v_j, U]_{K'}| = 2p$ for each $1 \leq j \leq 4p^2+2p$. By Lemma 2.1(2)(3), we have $K' \in \langle \mathcal{S}\mathbb{Z}_{2p+1} \rangle$. Therefore, $K_{2p+2,4p^2+2p} \in \langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$ by Lemma 2.2. Since the strongly $\mathbb{Z}_{2p+1}$-connectedness is preserved under identifying vertices and every member in $\mathcal{B}^*(2p+2, 4p^2+2p)$ is obtained from identifying vertices of $K_{2p+2,4p^2+2p}$, we conclude that every member of $\mathcal{B}^*(2p+2, 4p^2+2p)$ is strongly $\mathbb{Z}_{2p+1}$-connected. ■

The following elementary counting fact is also needed in our proof.

Fact 1. *For fixed positive integers ℓ, n, there are exactly $\binom{n+\ell-1}{\ell-1}$ non-negative integral solutions $\langle x_1, x_2, \ldots, x_\ell \rangle$ for the equation $x_1 + x_2 + \cdots + x_\ell = n$.*

Define $N(t,s) = (4t^2+2t)\binom{2s+2t+1}{2s-1} + 2s$. Let $\mathcal{F}(t,s)$ be the family of all $(2t+2)$-edge-connected $\langle \mathcal{S}\mathbb{Z}_{2t+1} \rangle$-reduced graphs of order between 2 and $N(t,s)$ with matching number at most s. Then the edge multiplicity of each graph in $\mathcal{F}(t,s)$ is at most $2t-1$ by Lemma 2.1(3). So there are indeed finitely many graphs in $\mathcal{F}(t,s)$. Define $\mathcal{G}(t,s)$ as the family of graphs in $\mathcal{F}(t,s)$ which are not in $\mathcal{M}_{2t+1}$. We will prove a stronger theorem below, which implies Theorem 1.4 by Lemma 2.1(5)(6).

Theorem 2.4. *For any $(2t+2)$-edge-connected graph G with $\alpha'(G) \le s$, we have $G \in \langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$ if and only if G cannot be $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced to a member in $\mathcal{F}(t,s)$.*

Proof. If $G \in \langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$, then G is $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced to $K_1 \notin \mathcal{F}(t,s)$ by Lemma 2.1(6). We shall show the converse that if G cannot be $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced to a member in $\mathcal{F}(t,s)$, then $G \in \langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$.

Let G be a counterexample and let G' be its $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduction. Then $G' \notin \mathcal{F}(t,s)$ and this leads to

$$|V(G')| > N(t,s) = (4t^2+2t)\binom{2s+2t+1}{2s-1} + 2s. \tag{1}$$

Since G' is the $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduction of G, we have $\alpha'(G') \le \alpha'(G) \le s$. Let $M = \{w_1w_2, w_3w_4, \ldots, w_{2d-1}w_{2d}\}$ be a maximum matching of G', where $d \le s$. Denote $W = \{w_1, \ldots, w_{2d}\}$. Then $Z = V(G') - W$ is an independent set of G' by the maximality of M. Since $\kappa'(G') \ge \kappa'(G) \ge 2t+2$, we have $|[z,W]_{G'}| \ge 2t+2$ for any $z \in Z$. Pick arbitrary $2t+2$ edges from $[z,W]_{G'}$, denoted by $H(z)$, for each $z \in Z$. Let $G'_1 = \cup_{z\in Z} H(z)$ be an edge subset as well as the edge-induced subgraph of G'.

We claim that there exists a member of $\mathcal{B}^*(2t+2, 4t^2+2t)$ contained in G'_1, therefore in G'. This will contradict to the assumption that G' is a $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced graph by Lemma 2.3.

For any $w \in W$ and $z \in Z$, we use $x(w,z) = |[w,z]_{G'_1}|$ to denote the number of edges in $H(z)$ between w and z. We also define $x(w,z) = 0$ if w is not in the graph $H(z)$. Since $H(z)$ consists of $2t+2$ edges, we have, for each $z \in Z$,

$$x(w_1,z) + x(w_2,z) + \cdots + x(w_{2d},z) = 2t+2.$$

Since $d \le s$ and by (1), we have

$$|Z| = |V(G')| - 2d > N(t,s) - 2s \ge (4t^2+2t)\binom{2s+2t+1}{2s-1}.$$

It follows from Fact 1 and Pigeon-Hole Principle that there exists a subset $Z_1 \subset Z$ of size $4t^2+2t$ such that, for any $a, b \in Z_1$,

$$\langle x(w_1,a), x(w_2,a), \ldots, x(w_{2d},a)\rangle = \langle x(w_1,b), x(w_2,b), \ldots, x(w_{2d},b)\rangle.$$

Denote $x_1, \ldots, x_{\ell+1}$ to be all the nonzero coordinates in $\langle x(w_1,a), x(w_2,a), \ldots, x(w_{2d},a)\rangle$, where we have $x_1 + \ldots + x_{\ell+1} = 2t+2$. Then the graph $[S_1, Z_1]_{G'_1} \cong K_{2t+2,4t^2+2t}(t_1, t_2, \ldots, t_\ell)$ is a member of $\mathcal{B}^*(2t+2, 4t^2+2t)$, where $t_1 = x_1$, $x_{\ell+1} = (2t+2) - t_\ell$, $t_i - t_{i-1} = x_i$ for $2 \le i \le \ell$. Here in the graph $[S_1, Z_1]_{G'_1}$, S_1 denotes the set of all vertices w_i such that $x(w_i,a) \ne 0$ in $\langle x(w_1,a), x(w_2,a), \ldots, x(w_{2d},a)\rangle$ for $1 \le i \le 2d$. This proves the claim as well as the theorem. ■

Remark. By [15], the running time of the algorithm deciding whether a planar graph G can be contracted to H is $|V(G)|^{\mathcal{O}(|V(H)|)}$. As each graph in $\mathcal{F}(t,s)$ (or $\mathcal{G}(t,s)$) has at most $N(t,s)^2$ vertices and multiplicity at most $2t-1$, there are at most $2^{(2t-1)N(t,s)^2}$ graphs in this family. Therefore, the running time of the algorithm from Theorem 1.4 to decide whether a planar graph G with $\kappa'(G) \ge 2t+2$ and $\alpha'(G) \le s$ has a mod $(2t+1)$-orientation for fixed s and t is then $\mathcal{O}(2^{(2t-1)N(t,s)^2}|V(G)|^{\mathcal{O}(N(t,s))})$, which is polynomial in $|V(G)|$.

In the following section, we will focus on the special important case of $t=1$, which is Theorem 1.7 concerning mod 3-orientation. However, Theorem 1.7 seems not possible to extend to general mod $(2t+1)$-orientations. One may obverse that for $k \le t$ and $c \ge 2t+2$, the graph $K^{+k}_{2t+1,c}$ and some graphs obtained by identifying some large degree vertices of $K^{+k}_{2t+1,c}$ are still $(2t+1)$-edge-connected $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced graphs without mod $(2t+1)$-orientations. Also, applying 2-sum operations on some of those graphs results in more $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced graphs without mod $(2t+1)$-orientations and with a small matching number. Thus the structures of all such exceptional graphs are much more complicated, which seems far from being characterized. As we can see from the proof of Theorem 1.7 below, the arguments require to characterize all $\langle \mathcal{S}\mathbb{Z}_3\rangle$-reduced graphs of small order. However, it seems hopeless to characterize all $\langle \mathcal{S}\mathbb{Z}_{2t+1}\rangle$-reduced graphs without mod $(2t+1)$-orientations on at most $4t+3$ vertices by hand for general t.

3. Nowhere-zero 3-Flows and matchings

Note that a graph G admits a mod 3-orientation if and only if G admits a 3-NZF; and the concept of strongly $\mathbb{Z}_3$-connected is the same as the so-called "$\mathbb{Z}_3$-connected" in some literature [3,16]. In this section we characterize all the exceptions of Theorem 1.4 when $t=1$, which is the content of Theorem 1.7.

Proof Outline of Theorem 1.7. To prove Theorem 1.7, we shall only need to focus on $\langle\mathcal{S}\mathbb{Z}_3\rangle$-reduced graphs by Lemma 2.1. We will apply a similar but more structural argument as in the proof of Theorem 1.4 to show every such graph has restricted structure and order. Then we analyze the matching number to rule out the graphs $K_{3,n}^{+}$ and further restrict to graphs with order at most 7. Then we apply various tools below to handle all those small graphs to characterize all the exceptions.

First, we display some more needed lemmas.

Lemma 3.1. ([16]) *If G is $\langle\mathcal{S}\mathbb{Z}_3\rangle$-reduced and $|V(G)| = 7$, then $|E(G)| \le 13$.*

Lemma 3.2. ([16]) *If $\kappa'(G) \ge 4$ and $|V(G)| \le 13$, then G admits a 3-NZF.*

Lemma 3.3. (Hakimi [8]) *Given a graph G, let $\ell\colon V(G) \to \mathbb{Z}$ be a function with $\sum_{u\in V(G)} \ell(u) = 0$ and $\ell(u) \equiv d_G(u) \pmod 2$ for any $u \in V(G)$. Then G admits an orientation D with $\ell(u) = d_D^+(u) - d_D^-(u)$ for any $u \in V(G)$ if and only if*

$$|\sum_{u\in A} \ell(u)| \le |\partial_G(A)| \text{ for any vertex subset } A \subseteq V(G). \tag{2}$$

The ***k*-wheel** W_k is a graph constructed by adding a new center vertex connecting to each vertex of the k-cycle.

Lemma 3.4. ([3]) *For any integer $s \ge 1$, the even wheel W_{2s} is strongly $\mathbb{Z}_3$-connected.*

Lemma 3.5. ([16]) *Assume that an odd wheel W_{2s+1} is a proper subgraph of a graph G. Let A, B be a bipartition of the vertex set $V(W_{2s+1})$. Form a graph $G_{[A,B]}$ from G by removing all edges of $E(W_{2k+1})$, contracting the sets A and B into two single vertices u and v, respectively, and then connecting a new edge uv.*
(i) If $G_{[A,B]}$ has a 3-NZF, then so does G.
(ii) If $G_{[A,B]} \in \langle\mathcal{S}\mathbb{Z}_3\rangle$, then $G \in \langle\mathcal{S}\mathbb{Z}_3\rangle$.

Lemma 3.6. *Let $k \ge 0$ and $n \ge 3$ be integers. Then $K_{3,n}^{+k}$ admits a 3-NZF if and only if $k \ne 1$.*

Proof. First, assume $k \ne 1$ and we shall show $K_{3,n}^{+k}$ admits a 3-NZF. If $k = 0$, then $K_{3,n}^{+k} = K_{3,n}$ is a complete bipartite graph which obviously has a 3-NZF. Now assume $k > 0$. Let $V(K_{3,n}^{+k}) = \{v_1, \ldots, v_{n+3}\}$, and denote the three vertices with degree more than 3 to be v_1, v_2, v_3. Since $2K_2 \in \langle\mathcal{S}\mathbb{Z}_3\rangle$ by Lemma 2.1(3), if $K_{3,n}^{+k}$ has parallel edges, then $K_{3,n}^{+k} \in \langle\mathcal{S}\mathbb{Z}_3\rangle$ after contracting all 2-cycles by Lemma 2.1(2). Hence $K_{3,n}^{+k}$ admits a 3-NZF if $k \ge 4$. Now it just needs to show the cases of $k = 2, 3$, where the new added edges are non-parallel. If $k = 3$, then edges of $K_{3,n}^{+3}$ can be partitioned to $K_{3,n}$ and K_3, which both have a 3-NZF and so does $K_{3,n}^{+3}$. Thus assume $k = 2$ in the following. Define a function $\ell\colon V(K_{3,n}) \to \mathbb{Z}$ by $\ell(v_i)$ are 3 and -3 alternately for $i \in \{4, \ldots, n+3\}$, $\ell(v_2) = -3$, $\ell(v_1) = \ell(v_3) = 0$ when n is odd and $\ell(v_2) = 0$, $\ell(v_1) = 3$, $\ell(v_3) = -3$ when n is even. Then we can verify statement (2) of Lemma 3.3 for $K_{3,n}^{+k}$, and so $K_{3,n}^{+k}$ admits a mod 3-orientation D, or equivalently it admits 3-NZF. Conversely, if $k = 1$, then it is routine to check that $K_{3,n}^{+}$ does not admit a 3-NZF. ■

Fan and Zhou [6] in 2008 characterized 3-NZF of simple graphs under Ore-condition.

Theorem 3.7. (Fan and Zhou [6]) *Given a simple graph G with $|V(G)| \ge 3$, if G satisfies the Ore-condition that $d(s) + d(t) \ge |V(G)|$ for any pair of non-adjacent vertices s, t, then G admits a 3-NZF if and only if G is not isomorphic to a graph in Fig. 2.*

For a matching M of a graph G, a path P is called an *M-augmenting path* if both end vertices of P are not in $V(M)$, and the edges of P are alternately in $E(G) - M$ and in M. It is well-known that a matching M is maximum if and only if there is no M-augmenting path.

Lemma 3.8. *If a simple graph G satisfies $|V(G)| \ge 3k$ and $\kappa'(G) \ge \alpha'(G) = k$, then G contains $K_{k,k}$ as a subgraph.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. Since $\alpha'(G) = k$, we may assume that $M = \{v_{2i-1}v_{2i} : 1 \le i \le k\}$ is a maximum matching of G. Hence there is no M-augmenting path in G, and moreover, $\{v_{2k+1}, \ldots, v_n\}$ is an independent vertex-set. Assume that v_{2k+1} is adjacent to both end vertices of an edge of M, say $v_{2k+1} \sim v_1$ and $v_{2k+1} \sim v_2$ without loss of any generality. Then each of $v_{2k+2}, \ldots, v_n$ is adjacent to neither v_1 nor v_2. Otherwise, it would cause an M-augmenting path, a contradiction. Since $\kappa'(G') \ge k$, we know degree of v_{2k+2} is at least k, and so v_{2k+2} is also adjacent to both end vertices of another edge of M, say $v_{2k+2} \sim v_3$ and $v_{2k+2} \sim v_4$ without loss of any generality. Then each of $v_{2k+3}, \ldots, v_n$ is adjacent to neither v_3 nor v_4 for the same reason. Repeat this argument again and again, we would have that v_{2k+i} is not adjacent to $v_1, v_2, \ldots, v_{2i-1}, v_{2i}$

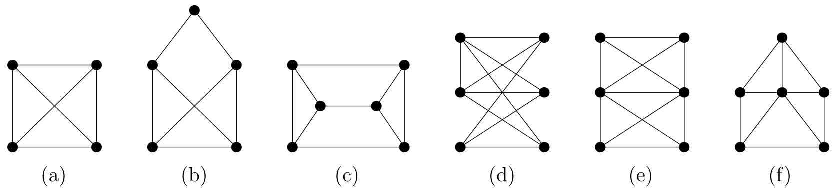

Fig. 2. The graphs presented in Theorem 3.7. All graphs satisfy Ore-condition but do not admit a 3-NZF.

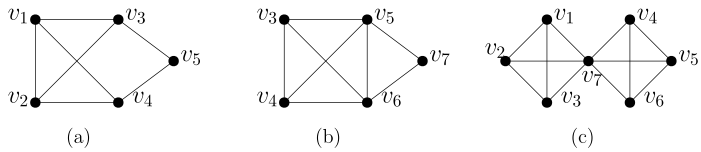

Fig. 3. The graph (a) is for Claim 1, and the graphs (b) and (c) are for Claim 5.

for any $1 \le i \le k$. But this implies v_{3k} is adjacent to none of the vertices in $V(M)$. Since $\{v_{2k+1}, \dots, v_n\}$ is an independent vertex set, this shows that v_{3k} is an isolated vertex, contradicting to $\kappa'(G') \ge k$.

Now we assume instead that v_{2k+1} is adjacent to precisely one end vertex of each edge of M, say $v_{2k+1} \sim v_{2i-1}$ for each $1 \le i \le k$. As there is no M-augmenting path, for any $j \ge 2$, v_{2k+j} is adjacent to none of $\{v_2, v_4, \dots, v_{2k}\}$. Since $\kappa'(G') \ge k$, v_{2k+j} must be adjacent to v_{2i-1} for any $1 \le i, j \le k$. Therefore, the set $\{v_1, v_3, \dots, v_{2k-1}, v_{2k+1}, \dots, v_{2k+k}\}$ induces a complete bipartite graph $K_{k,k}$ as required. ■

Now we shall prove Theorem 1.7, restated as the following equivalent version.

Theorem 3.9. *Let G be a bridgeless graph with $\kappa'(G) \ge \alpha'(G)$. Then either G has a 3-NZF, or G can be $\langle \mathcal{S}\mathbb{Z}_3 \rangle$-reduced to a graph in $\mathcal{G}_1 \cup K_4 \cup \{K^+_{3,t} : t \ge 3\}$.*

Proof. When $\alpha'(G) = 1$, the simplification of G is spanned by a $K_{1,n-1}$. As G is bridgeless, it implies that G consists of a branch of parallel edges joining to the center vertex. Hence by Lemma 2.1 G has a 3-NZF. If $\alpha'(G) \ge 6$, then by Theorem 1.2 G admits a 3-NZF. It remains to show the cases of $2 \le \alpha'(G) \le 5$. We use G' to represent the $\langle \mathcal{S}\mathbb{Z}_3 \rangle$-reduction of G as above. Then we have

$$5 \ge \kappa'(G') \ge \kappa'(G) \ge \alpha'(G) \ge \alpha'(G') \ge 2.$$

Claim 1. *If $G' \notin \mathcal{M}_3$ and $\alpha'(G') = 2$, then G' is K_4 or (a) in Fig. 3.*

Proof. When $|V(G')| = 4$, we have that G' is either C_4 or C_4 adding chords. They all have a 3-NZF except K_4. If $|V(G')| = 5$, then G' is one of $K_{2,3}$, C_5 and C_5 adding chords. It is easy to verify they all have a 3-NZF except the graph (a) in Fig. 3. Next assume that $|V(G')| = n \ge 6$. Since $\alpha(G') = 2$ and $\kappa'(G') \ge 2$, we get that G' contains $K_{2,2}$ as a subgraph by Lemma 3.8. Assume a maximal matching of G is $M = \{v_1v_2, v_3v_4\}$. Then $v_5, \dots, v_n$ induce an independent set. Using a similar argument, one can justify that $G' \cong K_{2,n-2}$ when $v_1 \nsim v_3$ and $G' \cong K^+_{2,n-2}$ when $v_1 \sim v_3$. In either case, G' has a 3-NZF. □

Claim 2. *If $\alpha'(G') = 4, 5$, then G admits a 3-NZF.*

Proof. Since $\kappa'(G') \ge \alpha'(G') \ge 4$, one has $|V(G')| = n \ge 14$ by Lemma 3.2. By Lemma 3.8, G' contains a subgraph which is isomorphic to $K_{4,4}$. Since $K_{4,4}$ is $\mathbb{Z}_3$-connected, this is a contradiction to G' is $\langle \mathcal{S}\mathbb{Z}_3 \rangle$-reduced. □

Claim 3. *If $G' \notin \mathcal{M}_3$ and $|V(G')| \ge 8$, then $G' \cong K^+_{3,t}$ for some integer $t \ge 5$.*

Proof. From the above claims, one has $\alpha'(G') = 3$. Assume that $v_6 \sim v_5$, $v_4 \sim v_3$ and $v_2 \sim v_1$, and the rest vertices form an independent set. Suppose that v_7 is adjacent to both v_2 and v_1. As $\kappa(G') \ge 3$, the degree of v_7 is at least 3, and so, without loss of generality, assume that $v_7 \sim v_3$. Then v_8 is not adjacent to v_1, v_2, v_4 as $\alpha'(G') = 3$. Hence v_8 is adjacent to v_3, v_5, v_6. Since v_4 has degree at least 3 and cannot be adjacent to $v_9, \dots, v_n$, we have that v_4 is adjacent to at least two of v_1, v_2, v_5, v_6. But in each case it results that $\alpha'(G') > 3$. So, for each $k \in \{7, \dots, n\}$, v_k is adjacent to each of v_5, v_3, v_1 and is adjacent to none of v_6, v_4, v_2. Thus G' must be one of the graphs $K_{3,n-3}$, $K^+_{3,n-3}$, $K^{+2}_{3,n-3}$ or $K^{+3}_{3,n-3}$. Among them, only the graph $K^+_{3,n-3}$ does not have a 3-NZF by Lemma 3.6. □

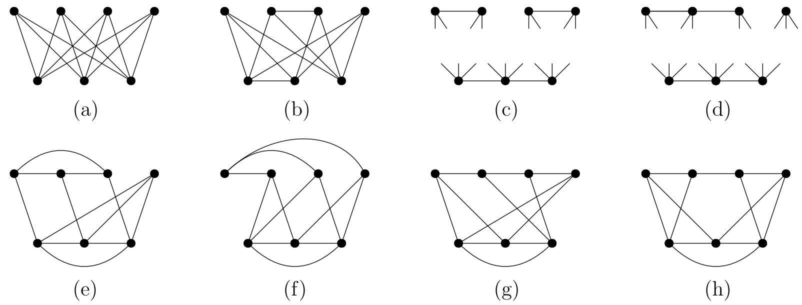

Fig. 4. The graphs for case (3) in Claim 5.

Claim 4. *If $G' \notin \mathcal{M}_3$ and $|V(G')| = 6$, then G' is one of (c), (d), (e) and (f) in Fig. 2.*

Proof. Since $|V(G')| = 6$ and $\delta(G') \geq \kappa'(G') \geq 3$, it satisfies the Ore-condition as in Theorem 3.7, and so G' is one of (c), (d), (e) and (f) in Fig. 2. □

A vertex is called a k**-vertex** if it is a vertex of degree k.

Claim 5. *If $G' \notin \mathcal{M}_3$ and $|V(G')| = 7$, then G' is one of the following graphs: (c) in Fig. 3; (e), (f) in Fig. 4; (a), (c), (e) in Fig. 5; (a), (b), (d)–(j) in Fig. 6; or $K^+_{3,4}$.*

Proof. Note that $\alpha'(G') = 3$ by $|V(G')| = 7$. First, assume that G' contains a 6-vertex, say v_7. Let $H = G[\{v_1, \ldots, v_6\}]$ be an induced subgraph of G. As $\kappa'(G') \geq \alpha'(G') = 3$, one has that the degree of each vertex of H is at least 2. Thus H contains a cycle. If it has an even length circle, then G' has a subgraph which is an even wheel graph. However, the even wheel graph is strongly $\mathbb{Z}_3$-connected by Lemma 3.4, which contradicts to G' is $\langle S\mathbb{Z}_3\rangle$-reduced. So H has two circles of length 3 or one circle of length 5. In the latter case, one can find an even length cycle in H as well. Thus H must have two circles of length 3. We may assume that v_1, v_2, v_3, v_7 induce a K_4, say H_1; and v_4, v_5, v_6, v_7 induce the other K_4, say H_2. If there exists no edge between H_1 and H_2, then G' is exactly (c) in Fig. 3. Clearly, this graph does not have a 3-NZF. If there exist edges between H_1 and H_2, then apply Lemma 3.5 to contract these two K_4's into a K_2, and the resulting graph is bridgeless with 3 vertices, which admits a 3-NZF. This implies G has a 3-NZF by Lemma 3.5.

Now assume instead, G' does not have any 6-vertex in the following. By Ore-condition, there exists a pair of nonadjacent 3-vertices; otherwise G' admits a 3-NZF by Theorem 3.7. We have 5 cases depending on the number of 3-vertices.

(1) There are exactly two 3-vertices. Assume G' has a 5-vertex, then it has even number of 5-vertices, and so $|E(G')| \geq (2 \cdot 5 + 2 \cdot 3 + 3 \cdot 4)/2 = 14$. It follows by Lemma 3.1 that G' is not $\langle S\mathbb{Z}_3\rangle$-reduced, a contradiction. Hence G' has no 5-vertex, namely, G' has five 4-vertices. Next we apply lifting operations on those 4-vertices. First split each 4-vertex into two 2-vertices, and then shrink the corresponding 2-vertices. After splitting all the 4-vertices, G' becomes a $3K_2$, which has a 3-NZF. Thus G' admits a 3-NZF.

(2) There are exactly three 3-vertices. Thus there are odd number 5-vertices. If G' has more than one 5-vertices, then we also have $|E(G')| \geq (3 \cdot 3 + 3 \cdot 5 + 4)/2 = 14$. Thus G' is not a $\langle S\mathbb{Z}_3\rangle$-reduced graph by Lemma 3.1, a contradiction. Then G' has exactly one 5-vertex and three 4-vertices. After splitting all 4-vertices as before, one can get a graph of order 4 with parallel edges, which has a 3-NZF. So G' also admits a 3-NZF.

(3) There are exactly four 3-vertices.

(3.1) Assume there are exactly three 4-vertices, say v_3, v_2, v_1. Let $H = G[\{v_3, v_2, v_1\}]$ be the graph induced by v_3, v_2, v_1.

(3.1.1) Assume H has no edges. Then G has just one realization $K_{3,4}$, see graph (a) in Fig. 4, which has a 3-NZF.

(3.1.2) Assume H has exactly one edge. Then G has just one realization (b) shown in Fig. 4, which has a 3-NZF as well.

(3.1.3) Assume H has exactly two edges. Consider the graph K induced by other vertices v_4, v_5, v_6, v_7. Then K is either two 2-paths or one 3-path together with an isolate vertex, see (c) and (d) in Fig. 4. Define a function $\ell\colon V(G) \to \mathbb{Z}$ with $\ell(v_4) = \ell(v_6) = 3$, $\ell(v_7) = \ell(v_5) = -3$, $\ell(v_1) = \ell(v_2) = \ell(v_3) = 0$. It is routine to justify that $|\partial_G(A)| \geq |\sum_{u \in A} \ell(u)|, \forall A \subset V(G)$. By Lemma 3.3, G admits an orientation D with $\ell(s) = d^+_G(s) - d^-_G(s)$, for any $s \in V(G)$. So this gives a mod 3-orientation of G, thus a 3-NZF in each case.

(3.1.4) Assume H has exactly 3 edges. Then G has 4 realizations (e), (f) (g) and (h) as in Fig. 4. Thus we easily get that each of the graphs (g) and (h) has a 3-NZF, while the graphs (e) and (f) not.

(3.2) Assume there are exactly two 5-vertices and one 4-vertex. Suppose that there are two adjacent 3-vertices, and say that $v_2 \sim v_1$ by symmetry. Since $|E(G')| = (2 \cdot 5 + 4 \cdot 3 + 4)/2 = 13$, one has that v_7, v_6, v_5, v_4, v_3 induce a graph with 8 edges, which is (b) in Fig. 3 or W_4. But $W_4 \in \langle S\mathbb{Z}_3\rangle$ by Lemma 3.4, then we have G' does not contain

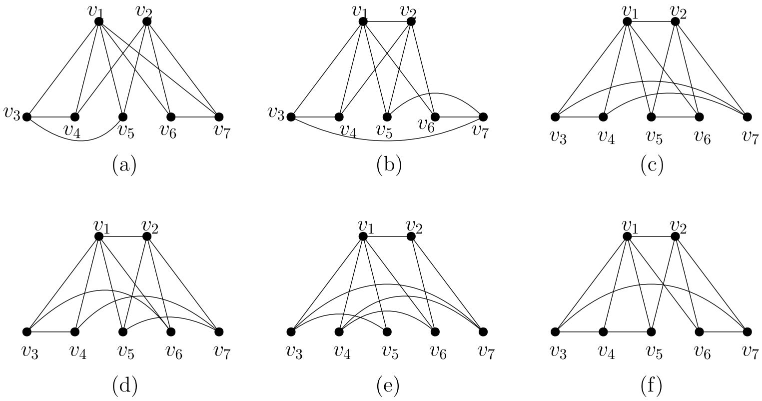

Fig. 5. The graphs for case (4) in Claim 5.

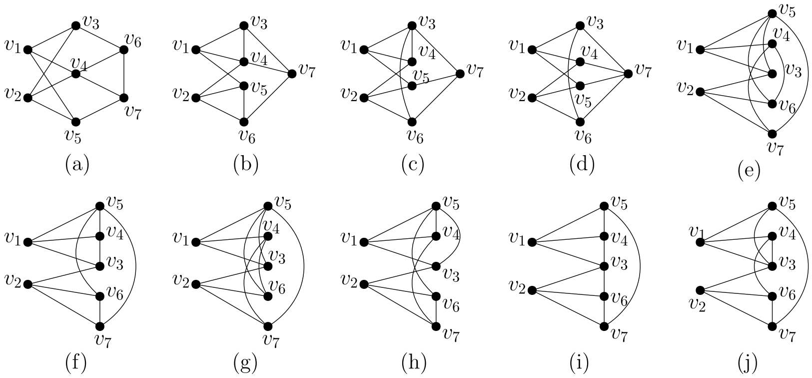

Fig. 6. The graphs for case (5) in Claim 5.

W_4 as it is $\langle S\mathbb{Z}_3\rangle$-reduced. Thus we obtain that v_3, v_4, v_5, v_6, v_7 precisely induce the graph (b) in Fig. 3. Notice also that v_3, v_4, v_5, v_6 induce a K_4, which is also an odd wheel W_3. Denote two partitions of $\{v_6, v_5, v_4, v_3\}$ by $\mathcal{P}_1 = \{v_5, v_3\} \cup \{v_6, v_4\}$ and $\mathcal{P}_2 = \{v_4, v_5\} \cup \{v_3, v_6\}$, respectively. By Lemma 3.5, with a careful analysis we can choose an appropriate partition from $\mathcal{P}_i$ for some $i \in \{1, 2\}$, say $\mathcal{P}_i = A \cup B$, such that $G'_{[A,B]}$ admits a 3-NZF. Hence G' has a 3-NZF, a contradiction. This shows that there is no adjacent 3-vertices. Hence $G' \cong K^+_{3,4}$.

(4) There are five 3-vertices. Then there is a 5-vertex, say v_1, and a 4-vertex, say v_2.
   (4.1) Assume v_1 is not adjacent to v_2. Then v_2 and v_1 have 4 common neighbor vertices. Hence G' is isomorphic to the graph (a) in Fig. 5, which does not have a 3-NZF.
   (4.2) Assume v_1 is adjacent to v_2, and that v_2 and v_1 have exactly 3 common neighbor vertices. Then the graph must be (b) in Fig. 5. It is straightforward to check that it has a 3-NZF.
   (4.3) Assume v_1 is adjacent to v_2, and that v_2 and v_1 have exactly 2 common neighbor vertices. Then there are four such graphs, (c), (d), (e) or (f) in Fig. 5. We can check one by one that the graphs (c) and (e) do not have 3-NZF.

(5) There are six 3-vertices. Then G' has exactly one 4-vertex. By Ore condition, there exist two 3-vertices, say v_2 and v_1, such that $v_2 \nsim v_1$. We have 3 subcases dividing by the number of common neighbor vertices of v_1 and v_2.
   (5.1) Assume v_1 and v_2 have 3 common neighbor vertices, say v_5, v_4, v_3. If v_7 or v_6 is a 4-vertex, then such a graph does not exist. So assume instead that one of v_5, v_4, v_3 is a 4-vertex, say v_3. If $v_3 \sim v_4$, then there is no graph satisfied above condition. Hence, G' must be the graph (a) in Fig. 6, which does not have a 3-NZF.
   (5.2) Assume v_1 and v_2 have exactly 2 common neighbor vertices, say v_3, v_4. Assume the other neighbor vertex of v_1 and of v_2 is v_5 and v_6, respectively. If v_4 or v_5 is a 4-vertex, then G' is (b) in Fig. 6, which does not admit a 3-NZF. If v_3 or v_6 is a 4-vertex, then G' is (c) in Fig. 6, which admits a 3-NZF. If v_7 is the 4-vertex, then G' is (d) in Fig. 6, which does not admit a 3-NZF.
   (5.3) Assume v_1 and v_2 have exactly one common neighbor vertex, say v_3. Assume the extra two neighbor vertices of v_1 and v_2 are $\{v_4, v_5\}$ and $\{v_6, v_7\}$, respectively. If one of v_7, v_6, v_5, v_4 is a 4-vertex, without loss of any generality, say v_5, then G' is one of the graphs (e), (f), (g) or (h) in Fig. 6. Hence they all do not have a 3-NZF with an easy one by one verification. Now assume v_3 is the 4-vertex. Then G' is one of the graphs (i) and (j) in Fig. 6, which does not have a 3-NZF in each case. $\square$

By Claims 1-5, we conclude that if $G' \notin \mathcal{M}_3$, then $G' \in \mathcal{G}_1 \cup K_4 \cup \{K_{3,t}^+ : t \geq 3\}$ as desired. ■

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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