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Hamiltonian s-properties and eigenvalues of k-connected graphs

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A B S T R A C T

Chvátal and Erdös (1972) [5] proved that, for a k-connected graph G, if the stability number $\alpha(G) \le k - s$, then G is Hamilton-connected ($s = 1$) or Hamiltonian ($s = 0$) or traceable ($s = -1$). Motivated by the result, we focus on tight sufficient spectral conditions for k-connected graphs to possess Hamiltonian s-properties. We say that a graph possesses Hamiltonian s-properties, which means that the graph is Hamilton-connected if $s = 1$, Hamiltonian if $s = 0$, and traceable if $s = -1$.

For a real number $a \ge 0$, and for a k-connected graph G with order n, degree diagonal matrix $D(G)$ and adjacency matrix $A(G)$, we have identified best possible upper bounds for the spectral radius $\lambda_1(aD(\Gamma) + A(\Gamma))$, where Γ is either G or the complement of G, to warrant that G possesses Hamiltonian s-properties. Sufficient conditions for a graph G to possess Hamiltonian s-properties in terms of upper bounds for the Laplacian spectral radius as well as lower bounds of the algebraic connectivity of G are also obtained. Other best possible spectral conditions for Hamiltonian s-properties are also discussed.

1. Introduction

We consider simple, undirected and connected graphs with undefined terms and notation reference to [3]. As in [3], $\overline{G}$, $\alpha(G)$, $\kappa(G)$, $\delta(G)$ and $d(v)$ denote the complement, the **stability number** (also call the **independence number**), **the connectivity**, the minimum degree of a graph G and the degree of vertex v in G, respectively. Let $K_{a,b}$ denote complete bipartite graphs on n vertices, where $a + b = n$.

A well-known result of Whitney [28] states that $\kappa(G) \le \delta(G)$ for any graph G. A graph G is **k-connected** if $\kappa(G) \ge k$. A cycle (path, respectively) passing through all the vertices of a graph is called a Hamilton cycle (Hamilton path, respectively). A graph G is called **Hamilton-connected** if every two vertices of G are connected by a Hamilton path. A graph containing a Hamilton cycle is called a **Hamiltonian graph**. It is known that all Hamilton-connected graphs are Hamiltonian. A graph containing a Hamilton path is said to be **traceable**.

For any graph G with the adjacency matrix $A(G)$ and the diagonal degree matrix $D(G)$, we define $\lambda_1(aD(G) + bA(G))$ to be the spectral radius of $aD(G) + bA(G)$, where $a \ge 0$ and $b > 0$ are two real numbers. When $a = 0$ and $b = 1$, the value $\lambda_1(aD(G) + bA(G))$ is called the **spectral radius** of a graph G, denoted by $\rho_1(G)$. If $a = 1$ and $b = 1$, then $\lambda_1(aD(G) + bA(G))$ is called the **Q-index** of a graph G, and is denoted by $q_1(G)$. Furthermore, for a real number $\alpha \in [0, 1)$, $\lambda_1(A_\alpha(G)) =$

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$\lambda_1(\alpha D(G)+(1-\alpha)A(G))$ is called the A_α**-spectral radius** of G, formerly introduced by Nikiforov in [26]. We denote by $\lambda_n(G)$ the **least eigenvalue** of G. The matrix $L(G)=D(G)-A(G)$ is known as the Laplacian matrix of G. Let $\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_{n-1}(G)\geq\mu_n(G)$ be the Laplacian eigenvalues of G. It is known that $\mu_n(G)=0$. The values $\mu_1(G)$ and $\mu_{n-1}(G)$ are called the **Laplacian spectral radius** of G and the **algebraic connectivity** of G, respectively.

The investigation on sufficient spectral conditions which warrant Hamiltonian s-properties of a graph was initiated by Fielder and Nikiforov [11]. However, the results in the literature mainly focus on the spectral radius and the Q-index of dense graphs. Hardly any of them involve graphs with uniform edge density and the Laplacian eigenvalues. Recently, Li [17] initially proved sufficient conditions of $\rho_1(G)$ based on the connectivity to assure a connected graph to be Hamiltonian and traceable. Inspired by a well-known theorem of Chvatál and Erdös [5], we present tight sufficient spectral conditions on certain matrices arisen from graphs by taking a unified approach to assure a k-connected graph to possess Hamiltonian s-properties. For an integer s with $s\in\{1,0,-1\}$, we say that a graph G possesses **Hamiltonian s-properties** if each of the following holds: if $s=1$, then G is Hamilton-connected; if $s=0$, then G is Hamiltonian; and if $s=-1$, then G is traceable.

Our ideas are also motivated by the literatures [6,8,13,19,18,20–24]. One of our goals is to investigate the relationship between Hamiltonian s-properties, the spectral radius $\lambda_1(aD(G)+bA(G))$ and $\lambda_1(aD(\overline{G})+bA(\overline{G}))$ of a k-connected graph G. This provides a mechanism to take a unified approach to the adjacency spectral radius, the signless Laplacian spectral radius, and the A_α-spectral radius of G. Another goal of this research is to initiate studies to find tight bounds of $\mu_1(G)$, $\mu_{n-1}(G)$ and $\mu_1(G)+\lambda_n(G)$ to predict k-connected graphs to possess Hamiltonian s-properties. The main results are as follows.

For real number a, integers k and δ with $a\geq 0$, $1\leq k\leq\delta$, and $s\in\{1,0,-1\}$, define

$$f(a,n,k,\delta,s)=\begin{cases}\delta\sqrt{\frac{k-s+1}{n-k+s-1}} & \text{if } a=0,\\ \frac{\delta n}{n-k+s-1} & \text{if } a=1,\\ \frac{a\delta n}{n-k+s-1} & \text{if } a\in(0,1),\\ \max\{a\delta,\frac{a\delta(k-s+1)}{n-k+s-1}\} & \text{if } a\in(1,+\infty).\end{cases} \tag{1}$$

Theorem 1.1. *Let G be a k-connected graph of order $n\geq 10$ and minimum degree $\delta=\delta(G)$. Suppose that*

$$\lambda_1(aD(G)+A(G))\leq f(a,n,k,\delta,s). \tag{2}$$

Then each of the following holds.
(i) If $a\in\{0,1\}$, then G possesses Hamiltonian s-properties if and only if $G\ncong K_{k,k-s+1}$.
(ii) If $0<a<1$ or if $1<a<+\infty$, then G possesses Hamiltonian s-properties.

It can be seen that when $a=0$ or 1, the upper bound on $\lambda_1(aD(G)+A(G))$ is tight in some sense in Theorem 1.1. For graphs G and H, we use $H\subseteq G$ to denote the fact that H is a subgraph of G. Let $\mathcal{F}(p,q)=\{G: K_{p,q}\subseteq G\subseteq K_p\vee qK_1\}$ be a family of graphs.

Theorem 1.2. *Let G be a k-connected graph of order $n\geq 10$. If*

$$\lambda_1(aD(\overline{G})+A(\overline{G}))\leq(a+1)(k-s),$$

then G possesses Hamiltonian s-properties if and only if $G\notin\mathcal{F}(k,k-s+1)$.

For the Laplacian matrix, tight bounds on $\mu_1(G)$ and $\mu_{n-1}(G)$ to assume a k-connected graph to possess Hamiltonian s-properties are proved as follows.

Theorem 1.3. *Let G be a k-connected graph of order $n\geq 3$ and minimum degree $\delta=\delta(G)$. Each of the following holds.*
(i) If $\mu_1(G)<\dfrac{n\delta}{n-k+s-1}$, then G possesses Hamiltonian s-properties.
(ii) If $\delta=n-k+s-1$ and $\mu_1(G)\leq\dfrac{n\delta}{n-k+s-1}$, then G possesses Hamiltonian s-properties if and only if $G\notin\mathcal{F}(k,k-s+1)$.

Theorem 1.4. *Let G be a k-connected graph of order $n\geq 3$. If*

$$\mu_{n-1}(G)\geq n-k+s-1,$$

then G possesses Hamiltonian s-properties if and only if $G\notin\mathcal{F}(k,k-s+1)$.

Theorem 1.5. *Let G be a k-connected graph of order $n\geq 3$ and minimum degree $\delta=\delta(G)$. If*

$$\mu_1(G)+\lambda_n(G)\leq\frac{n\delta}{n-k+s-1}-\sqrt{\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil},$$

then G possesses Hamiltonian s-properties if and only if $G \ncong K_{k,k-s+1}$ *for* $s \in \{1, 0\}$.

It can be observed that the upper bound on $\mu_1(G) + \lambda_n(G)$ in Theorem 1.5 is tight for $s \in \{1, 0\}$ in some sense. In the next section, we display some tools to be employed in our arguments. The proofs of the main results are in the subsequent section.

2. Preliminaries

We in this section will present some important results that will be used in our arguments. Recall that a bipartite graph is called balanced if its two partite sets A and B have equal number of vertices.

Theorem 2.1. *(Moon and Moser [25]) Let G be a balanced bipartite graph of order* $2n$ *with bipartition* (A, B)*. If* $d(x) + d(y) > n$ *for every pair of nonadjacent vertices* $x \in A$ *and* $y \in B$*, then G is Hamiltonian.*

Theorem 2.2. *(Jackson [16]) Let G be a 2-connected bipartite graph with bipartition* (A, B)*, where* $|A| \geq |B|$*. If each vertex in A has degree at least k and each vertex in B has degree at least l, then G contains a cycle of length at least* $2\min(|B|, k + l - 1, 2k - 2)$*. Moreover, if* $|A| = |B|$ *and* $k = l$*, then G contains a cycle of length at least* $2\min(|B|, 2k - 1)$*.*

Theorem 2.3. *(Dirac [9], Ore [27]) Let G be a graph of order* $n \geq 3$ *and minimum degree* $\delta(G)$*. If*

$$\delta(G) \geq \frac{n+s}{2},$$

then G possesses Hamiltonian s-properties.

Theorem 2.4. *(Chvatál and Erdös [5]) Let G be a k-connected graph of order* $n \geq 3$*. If*

$$\alpha(G) \leq k - s,$$

then G possesses Hamiltonian s-properties.

Note that $k \geq 2$ is a trivial condition in Theorem 2.4 for $s \in \{1, 0\}$.

Theorem 2.5. *(Anderson and Morely [1]) Let G be a graph of order* $n \geq 2$*. Then* $\mu_1(G) \leq n$ *with equality if and only if* $\overline{G}$ *is disconnected.*

Theorem 2.6. *(Fiedler [10]) Let G with n vertices contain an independent set of size t. Then* $\mu_{n-1}(G) \leq n - t$*.*

Theorem 2.7. *(Godsil and Newman [12]) Let G be a loopless graph, and* $\mu_1(G)$ *be the Laplacian spectral radius. For any independent set I of size t, we have* $t \leq n\frac{\mu_1 - \overline{d}_I}{\mu_1}$*, where* $\overline{d}_I = \frac{1}{t}\sum_{i \in I} d_i$*.*

Theorem 2.8. *(Constantine [7]) If G is a graph of order n, then*

$$\lambda_n(G) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil},$$

with equality if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$*.*

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. Define $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i and j, and $A < B$ if $A \leq B$ and $A \neq B$.

Theorem 2.9. *(Berman and Plemmons [2], Horn and Johnson [15]) Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two* $n \times n$ *matrices with the spectral radii* $\lambda_1(A)$ *and* $\lambda_1(B)$*. If* $0 \leq A \leq B$*, then* $\lambda_1(A) \leq \lambda_1(B)$*. Furthermore, if B is irreducible and* $0 \leq A < B$*, then* $\lambda_1(A) < \lambda_1(B)$*.*

The main tool in our paper is the eigenvalue interlacing technique described below. Given two non-increasing real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to **interlace** the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is **tight** if exists an integer $k \in [0, m]$ such that $\theta_i = \eta_i$ for $1 \leq i \leq k$ and $\theta_{n-m+i} = \eta_i$ for $k + 1 \leq i \leq m$.

Consider an $n \times n$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The **quotient matrix** $R(M)$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is **equitable** if each block $M_{i,j}$ of M has constant row (and column) sum.

Theorem 2.10. *(Brouwer and Haemers [4,14]) Let M be a real symmetric matrix. Then the eigenvalues of every quotient matrix of M interlace the ones of M. Furthermore, if the interlacing is tight, then the partition is equitable.*

3. Proofs

In the proofs of Theorems 1.1-1.5, we say that a graph possesses Hamiltonian s-properties, which means that the graph is Hamilton-connected if $s = 1$, Hamiltonian if $s = 0$, and traceable if $s = -1$. Before proceeding further, we present a technical lemma for the spectral radius of nonnegative matrices of bipartite graphs.

Lemma 3.1. *Let H be a bipartite graph with bipartition (X, Y). If $|X| = x$, $|Y| = y$ and $|E(H)| = r$, then*

$$\lambda_1(aD(H) + A(H)) \geq \frac{1}{2}\left(a(\frac{r}{x} + \frac{r}{y}) + \sqrt{(a^2 - 1)(\frac{r}{x} - \frac{r}{y})^2 + (\frac{r}{x} + \frac{r}{y})^2}\right).$$

Proof. Let $R(aD(H) + A(H))$ be the quotient matrix of $aD(H) + A(H)$ with respect to the partition (X, Y). One can see that

$$R(aD(H) + A(H)) = \begin{pmatrix} \frac{ar}{x} & \frac{r}{x} \\ \frac{r}{y} & \frac{ar}{y} \end{pmatrix}.$$

A direct computation shows that the characteristic polynomial of $R(aD(H) + A(H))$ is

$$\lambda^2 - a(\frac{r}{x} + \frac{r}{y})\lambda + (a^2 - 1)\frac{r^2}{xy} = 0,$$

which yields

$$\begin{aligned}\lambda_1(R(aD(H) + A(H))) &= \frac{1}{2}\left(a(\frac{r}{x} + \frac{r}{y}) + \sqrt{a^2(\frac{r}{x} + \frac{r}{y})^2 - 4(a^2 - 1)\frac{r^2}{xy}}\right) \\ &= \frac{1}{2}\left(a(\frac{r}{x} + \frac{r}{y}) + \sqrt{(a^2 - 1)(\frac{r}{x} - \frac{r}{y})^2 + (\frac{r}{x} + \frac{r}{y})^2}\right).\end{aligned}$$

The result follows from Theorem 2.10. □

We are now in a position to present the proofs of Theorems 1.1-1.5.

Proof of Theorem 1.1. (i) It is routine to verify that $G \cong K_{k,k-s+1}$ does not possess Hamiltonian s-properties. Therefore, it suffices to prove the sufficiency. We argue by contradiction and assume that

$$G \ncong K_{k,k-s+1} \text{ and } G \text{ does not possess Hamiltonian } s\text{-properties.} \tag{3}$$

We shall justify two claims below.

Claim 1. $n \geq 2k - s + 1$.

In fact, if $n \leq 2k - s$, then $\delta(G) \geq \kappa(G) \geq k \geq \frac{n+s}{2}$. By Theorem 2.3, G possesses Hamiltonian s-properties, a contradiction. Claim 1 holds.

By Theorem 2.4, $\alpha(G) \geq k - s + 1$, and then there exists an independent set $X = \{u_i \in V(G) | 1 \leq i \leq k - s + 1\}$. Let $Y = V(G) \setminus X = \{v_j | 1 \leq j \leq n - k + s - 1\}$. Consider the bipartite spanning subgraph H of G with the partitions X and Y. Let r be the number of edges with one end-vertex in X and the other in Y. Then $r \geq \delta(k - s + 1)$. For simplicity, define $\xi = k - s + 1$. By Theorem 2.9 and Lemma 3.1, we have

$$\begin{aligned}\lambda_1(aD(G) + A(G)) &\geq \lambda_1(aD(H) + A(H)) \\ &\geq \frac{1}{2}\left(a(\frac{r}{\xi} + \frac{r}{n-\xi}) + \sqrt{(a^2 - 1)(\frac{r}{\xi} - \frac{r}{n-\xi})^2 + (\frac{r}{\xi} + \frac{r}{n-\xi})^2}\right).\end{aligned} \tag{4}$$

Claim 2. If $a \in \{0, 1\}$, then $2k - s + 1 \leq n \leq 2(k - s + 1)$.

Assume first that $a=0$. By (4) and the assumption of Theorem 1.1, we have

$$\begin{aligned}\lambda_1(aD(G)+A(G)) &\geq \lambda_1(aD(H)+A(H)) \geq \lambda_1(R(aD(H)+A(H)))\\ &\geq r\sqrt{\frac{1}{\xi(n-\xi)}} \geq \delta\sqrt{\frac{\xi}{n-\xi}} = \delta\sqrt{\frac{k-s+1}{n-k+s-1}}\\ &\geq \lambda_1(aD(G)+A(G)).\end{aligned} \tag{5}$$

It follows that all the inequalities in (5) must be equalities. Hence $G \cong H$ and $r=\delta(k-s+1)$. Furthermore, $\lambda_n(aD(H)+A(H)) = -\lambda_1(aD(H)+A(H)) = -r\sqrt{\frac{1}{\xi(n-\xi)}} = \lambda_2(R(aD(H)+A(H)))$, and thus the interlacing is tight. By Theorem 2.10, the partition is equitable. That is to say, each vertex v_j of Y in G has the same number of neighbors in X, and thus $\frac{r}{n-k+s-1} \geq \delta(G)$. Then we have $n \leq 2(k-s+1)$. By Claim 1, we have $2k-s+1 \leq n \leq 2(k-s+1)$.

Next, we assume that $a=1$. By (4) and assumption of Theorem 1.1, we have

$$\begin{aligned}\lambda_1(aD(G)+A(G)) &\geq \lambda_1(aD(H)+A(H)) \geq \lambda_1(R(aD(H)+A(H)))\\ &\geq \frac{r}{\xi}+\frac{r}{n-\xi} \geq \frac{\delta n}{n-\xi} = \frac{\delta n}{n-k+s-1}\\ &\geq \lambda_1(aD(G)+A(G)).\end{aligned} \tag{6}$$

It follows that all the inequalities in (6) must be equalities. Hence $G \cong H$ and $r=\delta(k-s+1)$. Furthermore, $\lambda_n(aD(H)+A(H)) = 0 = \lambda_2(R(aD(H)+A(H)))$, and hence the interlacing is tight. By Theorem 2.10, the partition is equitable. That is, each vertex v_j of Y in G has the same number of neighbors in X, and thus $\frac{r}{n-k+s-1} \geq \delta(G)$. Then we have $n \leq 2(k-s+1)$. By Claim 1, we have $2k-s+1 \leq n \leq 2(k-s+1)$. This proves Claim 2.

By the assumption of Theorem 1.1, $s \in \{1,0,-1\}$. If $s=1$, then $n=2k$, and so both $n-(k-s+1)=k$ and $d(v_j)=\delta(G)$. As $d(u_i)=d(v_j)=\delta(G) \geq \kappa(G) \geq k$, we observe that $G \cong K_{k,k} = K_{k,k-s+1}$, contrary to (3).

Assume that $s=0$. Then by Claim 2, $n \in \{2k+1, 2k+2\}$. If $n=2k+1$, then $n-(k-s+1)=k$ and $d(v_j) > \delta(G) \geq k$. As $d(u_i)=\delta(G) \geq \kappa(G) \geq k$, we have $G \cong K_{k,k+1} = K_{k,k-s+1}$, contrary to (3). If $n=2k+2$, then $n-(k-s+1)=k+1$ and $d(v_j)=\delta(G)$. As $d(u_i)=d(v_j)=\delta(G) \geq \kappa(G) \geq k$, it follows that G is a balanced bipartite graph of order $2k+2$ such that, as $k \geq 2$, $d(u_i)+d(v_j) \geq 2k > k+1$ for any $u_iv_j \notin E(G)$. By Theorem 2.1, G is Hamiltonian, which is contrary to (3).

Finally we assume that $s=-1$. Then by Claim 2, $n \in \{2k+2, 2k+3, 2k+4\}$. If $n=2k+2$, then $n-(k-s+1)=k$ and $d(v_j) > \delta(G)$, and in this case, $d(u_i)=\delta(G) \geq \kappa(G) \geq k$. Then $G \cong K_{k,k+2} = K_{k,k-s+1}$, contrary to (3). If $n=2k+3$, then $n-(k-s+1)=k+1$ and $d(v_j) > \delta(G) \geq k$ with $d(u_i)=\delta(G) \geq \kappa(G) \geq k$. Since $n \geq 9$, we have $k \geq 3$, and so $k+1 \leq 2k-2$. By Theorem 2.2, G contains a cycle of length $2k+2$. Since $n=2k+3$ and $k \geq 3$, it follows that G has a path containing all the vertices of G, and therefore G is traceable, contrary to (3). Assume that $n=2k+4$. Then $n-(k-s+1)=k+2=k-s+1$ and $d(v_j)=\delta(G)$. By $n \geq 10$, we have $k \geq 3$. As $d(u_i)=d(v_j)=\delta(G) \geq \kappa(G) \geq k$, it follows by Theorem 2.2 that G contains a cycle of length $2k+4$ which implies that G is traceable, contrary to (3). This completes the proof of Theorem 1.1(i).

(ii) We argue by contradiction and assume that G does not possess Hamiltonian s-properties. If $0<a<1$, then $a^2-1<0$, and so by (4),

$$\begin{aligned}\lambda_1(aD(H)+A(H)) &\geq \frac{1}{2}\left(a(\frac{r}{\xi}+\frac{r}{n-\xi})+\sqrt{(a^2-1)(\frac{r}{\xi}-\frac{r}{n-\xi})^2+(\frac{r}{\xi}+\frac{r}{n-\xi})^2}\right)\\ &> \frac{1}{2}\left(a(\frac{r}{\xi}+\frac{r}{n-\xi})+\sqrt{(a^2-1)(\frac{r}{\xi}+\frac{r}{n-\xi})^2+(\frac{r}{\xi}+\frac{r}{n-\xi})^2}\right)\\ &= a(\frac{r}{\xi}+\frac{r}{n-\xi}).\end{aligned}$$

It follows that

$$\begin{aligned}\lambda_1(aD(G)+A(G)) &\geq \lambda_1(aD(H)+A(H))\\ &> a(\frac{r}{\xi}+\frac{r}{n-\xi}) \geq \frac{a\delta n}{n-\xi} = \frac{a\delta n}{n-k+s-1},\end{aligned}$$

contrary to (2).

If $1<a<+\infty$, then $a^2-1>0$, and so by (4),

$$\begin{aligned}\lambda_1(aD(H)+A(H)) &\geq \frac{1}{2}\left(a(\frac{r}{\xi}+\frac{r}{n-\xi})+\sqrt{(a^2-1)(\frac{r}{\xi}-\frac{r}{n-\xi})^2+(\frac{r}{\xi}+\frac{r}{n-\xi})^2}\right)\\ &> \frac{1}{2}\left(a(\frac{r}{\xi}+\frac{r}{n-\xi})+\sqrt{(a^2-1)(\frac{r}{\xi}-\frac{r}{n-\xi})^2+(\frac{r}{\xi}-\frac{r}{n-\xi})^2}\right)\\ &= \max\{\frac{ar}{\xi}, \frac{ar}{n-\xi}\}.\end{aligned}$$

It follows that

$$\lambda_1(aD(G)+A(G)) \geq \lambda_1(aD(H)+A(H)) > \max\{\frac{ar}{\xi}, \frac{ar}{n-\xi}\} \geq \max\{a\delta, \frac{a\delta(k-s+1)}{n-k+s-1}\},$$

contrary to (2). We complete the proof of Theorem 1.1(ii). □

By definition, any graph $G \in \mathcal{F}(k, k-s+1)$ is not Hamilton-connected if $s=1$, not Hamiltonian if $s=0$, and not traceable if $s=-1$. Thus we have the following observation.

$$\text{If } G \in \mathcal{F}(k, k-s+1), \text{ then } G \text{ does not possess Hamiltonian } s\text{-properties.} \tag{7}$$

Proof of Theorem 1.2. By (7), it suffices to prove the sufficiency. We assume that

$$G \notin \mathcal{F}(k, k-s+1) \text{ and } G \text{ does not possess Hamiltonian } s\text{-properties.} \tag{8}$$

By Theorem 2.4, $\alpha(G) \geq k-s+1$, and thus there exists an independent set $X = \{u_i \in V(G) | 1 \leq i \leq k-s+1\}$ in G. Let $Y = V(G) \setminus X = \{v_j | 1 \leq j \leq n-k+s-1\}$.

If $G[Y]$ is a clique in G, then $K_{k-s+1} \cup (n-k+s-1)K_1$ is a spanning subgraph of $\overline{G}$. It follows by the hypothesis of Theorem 1.2 and Theorem 2.9 that

$$\begin{aligned}(a+1)(k-s) &\geq \lambda_1(aD(\overline{G})+A(\overline{G})) \\ &\geq \lambda_1(aD(K_{k-s+1} \cup (n-k+s-1)K_1) + A(K_{k-s+1} \cup (n-k+s-1)K_1)) \\ &= (a+1)(k-s). \end{aligned} \tag{9}$$

Thus all the inequalities in (9) must be equalities. Hence $\overline{G} \cong K_{k-s+1} \cup (n-k+s-1)K_1$, and so $G \cong K_{n-k+s-1} \vee (k-s+1)K_1$. Since G does not possess Hamiltonian s-properties, by Theorem 2.3, we must have $n-k+s-1 = \delta(G) < \frac{n+s}{2}$, and so $2(n-k+s-1) \leq n+s-1$. This implies $n-k+s-1 \leq k$. Note that G is k-connected. Then $n-k+s-1 = \delta(G) \geq k$. Thus $G \cong K_k \vee (k-s+1)K_1$, which contradicts (8).

If $G[Y]$ is not a clique in G, then $K_{k-s+1} \cup \overline{G[Y]}$ is a spanning subgraph of $\overline{G}$. By the assumption of Theorem 1.2 and Theorem 2.9, we have

$$\begin{aligned}(a+1)(k-s) &\geq \lambda_1(aD(\overline{G})+A(\overline{G})) \\ &\geq \lambda_1(aD(K_{k-s+1} \cup \overline{G[Y]}) + A(K_{k-s+1} \cup \overline{G[Y]})). \end{aligned} \tag{10}$$

Note that (10) holds for any $G[Y]$. Then we have $|Y| = n-k+s-1 \leq k-s+1$, and so $n \leq 2(k-s+1)$. By Claim 1 in the proof of Theorem 1.1, we obtain that $n \geq 2k-s+1$. Therefore, $2k-s+1 \leq n \leq 2(k-s+1)$.

Assume that $s=1$. Then $n=2k$, that is, $|Y|=|X|=k$. As $d(u_i) \geq \delta(G) \geq k$ for any $u_i \in X$, we obtain that $G \in \mathcal{F}(k,k) = \mathcal{F}(k, k-s+1)$, which is contrary to (8).

Next we assume that $s=0$. Then $n \in \{2k+1, 2k+2\}$. If $n=2k+1$, then $|Y|=k$ and $|X|=k+1$. Since $d(u_i) \geq \delta(G) \geq k$, then we have $G \in \mathcal{F}(k, k+1) = \mathcal{F}(k, k-s+1)$, contrary to (8). If $n=2k+2$, then $|Y|=|X|=k+1$. For any $G[Y]$, we assume that G is not Hamiltonian in (8). However, when $G[Y]$ is an independent set, we observe that G is a balanced bipartite graph of order $2k+2$ such that, as $k \geq 2$, $d(u_i)+d(v_j) \geq 2k > k+1$ for any $u_iv_j \notin E(G)$. By Theorem 2.1, G is Hamiltonian, a contradiction.

Finally we suppose that $s=-1$. Then $n \in \{2k+2, 2k+3, 2k+4\}$. If $n=2k+2$, then $|Y|=k$ and $|X|=k+2$. Note that $d(u_i) \geq \delta(G) \geq k$. Then $G \in \mathcal{F}(k, k+2) = \mathcal{F}(k, k-s+1)$, contrary to (8). If $n=2k+3$, then $|Y|=k+1$ and $|X|=k+2$. As $n \geq 9$, then we have $k \geq 3$, and therefore $k+1 \leq 2k-2$. For any $G[Y]$, we always assume that G is not traceable in (8). However, when $G[Y]$ is an independent set, we observe that G is a 3-connected bipartite graph with bipartition (X, Y) and $|X| > |Y|$. Note that $d(u_i) \geq \delta(G) \geq k$ and $d(v_j) \geq \delta(G) \geq k$. By Theorem 2.2, G contains a cycle of length $2k+2$. Note that $n=2k+3$ and $k \geq 3$, it follows that G has a path containing all the vertices of G, and therefore G is traceable, a contradiction. If $n=2k+4$, then $|Y|=|X|=k+2$. By $n \geq 10$, we have $k \geq 3$ and $k+2 \leq 2k-1$. For any $G[Y]$, we always assume that G is not traceable in (8). However, when $G[Y]$ is an independent set, G is a 3-connected bipartite graph with bipartition (X, Y) and $|X|=|Y|$. Note that $d(u_i) \geq \delta(G) \geq k$ and $d(u_j) \geq \delta(G) \geq k$. It follows by Theorem 2.2 that G contains a cycle of length $2k+4$, then G is traceable, a contradiction. This completes the proof of Theorem 1.2. □

Before proving Theorem 1.3, we shall indicate that it suffices to consider graphs satisfying the inequality $\delta \leq n-k+s-1$ when discussing Theorem 1.3. In fact, assume that G is a graph with $\delta > n-k+s-1$. Then as $\delta \geq \kappa(G) \geq k$, we have $2\delta \geq n+s$, and so $\delta \geq \frac{n+s}{2}$. By Theorem 2.3, G possesses Hamiltonian s-properties. Thus we only need to consider $\delta \leq n-k+s-1$, or equivalently, $\frac{n\delta}{n-k+s-1} \leq n$. By Theorem 2.5, the upper bound on $\mu_1(G)$ in Theorem 1.3 is well-defined.

Proof of Theorem 1.3. Suppose, to the contrary, that G does not possess Hamiltonian s-properties. By Theorems 2.3 and 2.4, then $n \geq 2\delta - s + 1$ and $\alpha(G) \geq k - s + 1$. Let X be an independent set in G such that $|X| = k - s + 1$. Let r be the number of edges between X and $V(G)\backslash X$. Then $r \geq \delta|X|$. Accordingly, the quotient matrix $R(L)$ of $L(G)$ on the partition $(X, V(G)\backslash X)$ becomes:

$$R(L) = \begin{pmatrix} \frac{r}{k-s+1} & -\frac{r}{k-s+1} \\ -\frac{r}{n-k+s-1} & \frac{r}{n-k+s-1} \end{pmatrix}.$$

Let $\mu_1(R(L)), \mu_2(R(L))$ be the eigenvalues of $R(L)$. Then $\mu_1(R(L)) \geq \mu_2(R(L)) = 0$. By algebraic manipulations, we have

$$\mu_1(R(L)) = \frac{r}{k-s+1} + \frac{r}{n-k+s-1}.$$

By Theorem 2.10, then

$$\begin{aligned} \mu_1(G) \geq \mu_1(R(L)) &= \frac{r}{k-s+1} + \frac{r}{n-k+s-1} \\ &\geq (\frac{1}{k-s+1} + \frac{1}{n-k+s-1})(k-s+1)\delta = \frac{n\delta}{n-k+s-1}, \end{aligned} \tag{11}$$

which contradicts the assumption of this theorem.

Furthermore, if $\mu_1(G) \leq \frac{n\delta}{n-k+s-1}$, then all the inequalities in (11) must be equalities. So $r = (k-s+1)\delta$, and the partition is equitable. That is to say, each vertex of X has δ neighbors in $V(G)\backslash X$, and each vertex of $V(G)\backslash X$ has $\frac{(k-s+1)\delta}{n-k+s-1}$ neighbors in X. Note that $\delta = n-k+s-1$. As G is k-connected, we have $\delta \geq k$. Thus $2\delta - s + 1 \geq \delta + k - s + 1 = n \geq 2\delta - s + 1$, forcing $\delta = k$ and $n = 2k - s + 1$. Hence $G \in \mathcal{F}(k, k-s+1)$, a contradiction. Conversely, it is obvious that $G \in \mathcal{F}(k, k-s+1)$ does not possess Hamiltonian s-properties. $\square$

Proof of Theorem 1.4. By (7), it suffices to prove the sufficiency. Let $G \notin \mathcal{F}(k, k-s+1)$ be a graph. We assume that G does not possess Hamiltonian s-properties. As $n \geq 3$, G cannot be a complete graph. By Theorems 2.3 and 2.4, then $n \geq 2\delta - s + 1$ and $\alpha(G) \geq k - s + 1$. Let I be an independent set with size $\alpha(G)$ in G. By Theorem 2.6 and the hypothesis of Theorem 1.4, we have

$$n - k + s - 1 \leq \mu_{n-1}(G) \leq n - \alpha(G) \leq n - k + s - 1.$$

Hence

$$n - k + s - 1 = n - \alpha(G) = \mu_{n-1}(G) \leq \kappa(G) \leq \delta,$$

and so

$$\delta + k - s + 1 \geq n \geq 2\delta - s + 1 \geq \delta + \kappa(G) - s + 1 \geq \delta + k - s + 1.$$

Therefore $n = 2\delta - s + 1$, $\delta = \kappa(G) = k$ and $\alpha(G) = k - s + 1$, implying that $|V(G)\backslash I| = k = \delta$. Note that for each vertex $v \in I$, $d(v) \geq \delta$. Then $G \in \mathcal{F}(k, k-s+1)$, a contradiction. $\square$

Proof of Theorem 1.5. It is routine to verify that $G \cong K_{k,k-s+1}$ does not possess Hamiltonian s-properties, where $s \in \{1, 0\}$ be an integer. Therefore, it suffices to prove the sufficiency. We assume that

$$G \ncong K_{k,k-s+1}, \text{ where } s \in \{1, 0\}, \tag{12}$$

and

$$G \text{ does not possess Hamiltonian } s\text{-properties, where } s \in \{1, 0, -1\}. \tag{13}$$

By Theorems 2.3 and 2.4, then $n \geq 2\delta - s + 1 \geq 2k - s + 1$, and $\alpha(G) \geq k - s + 1$. Let I be an independent set with size $\alpha(G)$ in G. By Theorem 2.7, $\alpha(G) \leq n\frac{\mu_1 - \overline{d}_I}{\mu_1}$, and hence $\mu_1 \geq \frac{n\overline{d}_I}{n-\alpha(G)}$. Note that $\overline{d}_I \geq \delta$ and $\alpha(G) \geq k - s + 1$. Then $\mu_1 \geq \frac{n\overline{d}_I}{n-\alpha(G)} \geq \frac{n\delta}{n-k+s-1}$. Combining Theorem 2.8 and the condition of Theorem 1.5, we have

$$\frac{n\delta}{n-k+s-1} - \sqrt{\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil} \leq \frac{n\overline{d}_I}{n-\alpha(G)} - \sqrt{\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil} \leq \mu_1 + \lambda_n \leq \frac{n\delta}{n-k+s-1} - \sqrt{\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil},$$

and therefore all the inequalities must be equalities, that is,

$$\mu_1 = \frac{n\overline{d}_I}{n-\alpha(G)} = \frac{n\delta}{n-k+s-1}, \ \lambda_n = -\sqrt{\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil}.$$

So $\alpha(G)=k-s+1$, $G\cong K_{\frac{n}{2},\frac{n}{2}}$ if n is even, and $G\cong K_{\frac{n-1}{2},\frac{n+1}{2}}$ if n is odd. Next we consider three different values of s, respectively.

- $s=1$. Note that G is not Hamiltonian-connected by (13). Then $G\cong K_{\frac{n}{2},\frac{n}{2}}$ if n is even, and $G\cong K_{\frac{n-1}{2},\frac{n+1}{2}}$ if n is odd. Consider even n. Note that $\alpha(G)=k$. Then $\frac{n}{2}=k$, and hence $G\cong K_{k,k}$, contrary to (12). For odd n. Note that $\alpha(G)=k$. Then $\frac{n+1}{2}=k$, and hence $n=2k-1$, which contradicts $n\geq 2k$.
- $s=0$. Note that G is not Hamiltonian by (13). Then $G\cong K_{\frac{n-1}{2},\frac{n+1}{2}}$. Note that $\alpha(G)=k+1$. Then $\frac{n+1}{2}=k+1$, and hence $n=2k+1$. So $G\cong K_{k,k+1}$, which is contrary to (12).
- $s=-1$. It is obvious that $K_{\frac{n}{2},\frac{n}{2}}$ and $G\cong K_{\frac{n-1}{2},\frac{n+1}{2}}$ are traceable, which contradicts (13). $\square$

4. Corollaries of Theorems 1.1 and 1.2

Throughout this section, we assume that a and b are real numbers with $a\geq 0$ and $b>0$, k, s and δ are integers with $1\leq k\leq\delta$ and $s\in\{1,0,-1\}$. Next we consider the nonnegative matrix $aD(G)+bA(G)$. Theorems 1.1 and 1.2 have the following more general forms.

Corollary 4.1. *Let G be a k-connected graph of order $n\geq 10$ and minimum degree $\delta(G)$. If*

$$\lambda_1(aD(G)+bA(G))\leq bf(\frac{a}{b},n,k,\delta,s),$$

then each of the following holds.
(i) $a=0$ or $a=b$. G possesses Hamiltonian s-properties if and only if $G\ncong K_{k,k-s+1}$.
(ii) $0<a<b$ or $a>b$. G possesses Hamiltonian s-properties.

Proof. Suppose that G does not possess Hamiltonian s-properties.

(i) $a=0$ or $a=b$. That is, $\frac{a}{b}=0$ or 1. Notice that $K_{k,k-s+1}$ does not possess Hamiltonian s-properties. Therefore, it suffices to prove the sufficiency. Assume that $G\ncong K_{k,k-s+1}$. By Theorem 1.1(i), $\lambda_1(aD(G)+A(G))>f(a,n,k,\delta,s)$. Note that $aD(G)+bA(G)=b(\frac{a}{b}D(G)+A(G))$. It follows that $\lambda_1(aD(G)+bA(G))=b\lambda_1(\frac{a}{b}D(G)+A(G))>bf(\frac{a}{b},n,k,\delta,s)$, a contradiction.

(ii) $0<a<b$ or $a>b$, i.e., $\frac{a}{b}\in(0,1)$ or $(1,+\infty)$. Similar to the last part of proof of (i), (ii) follows immediately. $\square$

As one of main results of this paper, Corollary 4.1 can be applied to obtain sufficient condition in terms of the A_α-spectral radius $\lambda_1(A_\alpha(G))$ for a k-connected graph G to possess Hamiltonian s-properties.

Corollary 4.2. *Let G be a k-connected graph of order $n\geq 10$ and minimum degree $\delta(G)$, and let α be a real number with $\alpha\in[0,1)$. If*

$$\lambda_1(A_\alpha(G))\leq(1-\alpha)f(\frac{\alpha}{1-\alpha},n,k,\delta,s),$$

then each of the following holds.
(i) $\alpha=0$ or $\alpha=\frac{1}{2}$. G possesses Hamiltonian s-properties if and only if $G\ncong K_{k,k-s+1}$.
(ii) $\alpha\in(0,\frac{1}{2})$ or $\alpha\in(\frac{1}{2},1)$. G possesses Hamiltonian s-properties.

Particularly, sufficient conditions on $\rho_1(G)$ and $q_1(G)$ are as follows.

Corollary 4.3. *Let G be a k-connected graph of order $n\geq 4$ with minimum degree $\delta(G)$. If*

$$\rho_1(G)\leq\delta\sqrt{\frac{k-s+1}{n-k+s-1}},$$

then each of the following holds.
(i) G is Hamiltonian-connected if and only if $G\ncong K_{k,k-s+1}$ for $s=1$.
(ii) (R. Li [17]) G is Hamiltonian if and only if $G\ncong K_{k,k-s+1}$ for $s=0$, where $n\geq 6$.
(iii) (R. Li [17]) G is traceable if and only if $G\ncong K_{k,k-s+1}$ for $s=-1$, where $n\geq 10$.

Corollary 4.4. *Let G be a k-connected graph of order $n\geq 4$ with minimum degree $\delta(G)$. If*

$$q_1(G)\leq\frac{\delta n}{n-k+s-1},$$

then each of the following holds.
(i) G is Hamiltonian-connected if and only if $G\ncong K_{k,k-s+1}$ for $s=1$.

(ii) G is Hamiltonian if and only if $G \ncong K_{k,k-s+1}$ *for* $s = 0$*, where* $n \geq 6$.
(iii) G is traceable if and only if $G \ncong K_{k,k-s+1}$ *for* $s = -1$*, where* $n \geq 10$.

At the end of this paper, tight upper bound on $\lambda_1(aD(\overline{G}) + bA(\overline{G}))$ are proposed.

Corollary 4.5. *Let G be a k-connected graph of order* $n \geq 10$ *and minimum degree* $\delta(G)$*. If* $\lambda_1(aD(\overline{G}) + bA(\overline{G})) \leq (a+b)(k-s)$*, then G possesses Hamiltonian s-properties if and only if* $G \notin \mathcal{F}(k, k-s+1)$.

Proof. Note that $\mathcal{F}(k, k-s+1)$ does not possess Hamiltonian s-properties. Therefore, it suffices to prove the sufficiency. For $G \notin \mathcal{F}(k, k-s+1)$. Suppose that G does not possess Hamiltonian s-properties. By Theorem 1.2, $\lambda_1(aD(\overline{G}) + A(\overline{G})) > (a+1)(k-s)$. Since $aD(\overline{G}) + bA(\overline{G}) = b(\frac{a}{b}D(\overline{G}) + A(\overline{G}))$, it follows that $\lambda_1(aD(\overline{G}) + bA(\overline{G})) = b\lambda_1(\frac{a}{b}D(\overline{G}) + A(\overline{G})) > b(\frac{a}{b} + 1)(k-s) = (a+b)(k-s)$, a contradiction. $\square$

By choosing different values of a and b, sufficient conditions in terms of $\rho_1(\overline{G})$, $q_1(\overline{G})$ and $\lambda_1(A_\alpha(\overline{G}))$ for a k-connected graph G to possess Hamiltonian s-properties are easily obtained.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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