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Multigraphic Degree Sequences and Hamiltonian-connected Line Graphs

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Abstract Let G be a multigraph. Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then $G(e, e')$ is the graph obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. A graph G is strongly spanning trailable if for any $e, e' \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail.

The design of n processor network with given number of connections from each processor and with a desirable strength of the network can be modelled as a degree sequence realization problem with certain desirable graphical properties. A sequence $d = (d_1, d_2, \cdots, d_n)$ is multigraphic if there is a multigraph G with degree sequence d, and such a graph G is called a realization of d. A multigraphic degree sequence d is strongly spanning trailable if d has a realization G which is a strongly spanning trailable graph, and d is line-hamiltonian-connected if d has a realization G such that the line graph of G is hamiltonian-connected. In this paper, we prove that a nonincreasing multigraphic sequence $d = (d_1, d_2, \cdots, d_n)$ is strongly spanning trailable if and only if either $n = 1$ and $d_1 = 0$ or $n \geq 2$ and $d_n \geq 3$. Applying this result, we prove that for a nonincreasing multigraphic sequence $d = (d_1, d_2, \cdots, d_n)$, if $n \geq 2$ and $d_n \geq 3$, then d is line-hamiltonian-connected.

Keywords strongly spanning trailable graphs; multigraphic degree sequence; hamiltonian-connected graphs; line graph
2000 MR Subject Classification 05C45; 05C75

1 Introduction

This paper studies finite and undirected graphs without loops, but multiple edges are allowed. When we say graph in this paper, it always means multigraph, unless otherwise stated. Undefined terms can be found in [2]. Let G be a graph. Denote by $\kappa(G)$ and $\kappa'(G)$ the connectivity and edge-connectivity of a graph G, respectively. If X is a set of vertices, then $G - X$ denotes the graph obtained from G by deleting X. If Y is a set of edges, then $G - Y$ and $G + Y$ denote the graphs obtained from G by deleting edges in Y and adding edges in Y, respectively. Particularly, if $Y = \{e\}$, we write $G - e$ for $G - \{e\}$ and $G + e$ for $G + \{e\}$.

A graph is hamiltonian if it has a spanning cycle, and is hamiltonian-connected if for any distinct vertices u and v, G contains a spanning (u, v)-path. It is well known that every hamiltonian-connected graph must be 3-connected. The line graph of a graph G, denoted by

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$L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common.

Let $O(G)$ denote the set of odd degree vertices in G. If G is connected with $O(G) = \emptyset$, then G is eulerian. If G has a spanning eulerian subgraph, then G is supereulerian. A graph G is collapsible if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H such that $O(H) = R$. If G is collapsible, then by definition, G is supereulerian and so $\kappa'(G) \geq 2$.

For $u, v \in V(G)$, a (u, v)-trail is a trail of G from u to v. If $u = v$, a (u, v)-trail is an eulerian subgraph of G. For $e, e' \in E(G)$, an (e, e')-trail is a trail of G having end-edges e and e'. An (e, e')-trail is dominating if each edge of G is incident with at least one internal vertex of the trail; it is spanning if it is a dominating trail and it contains all the vertices of G. A graph G is spanning trailable if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail.

Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then $G(e, e')$ is the graph obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. A graph G is strongly spanning trailable if for any $e, e' \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. Since $e = e'$ is possible, strongly spanning trailable graphs are a special class of supereulerian graphs. By definition, a strongly spanning trailable graph is also spanning trailable. Supereulerian graphs and strongly spanning trailable graphs are closely related to the study of hamiltonian line graphs and hamiltonian-connected line graphs, respectively.

Theorem 1.1[5]**.** *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph H with $E(G - V(H)) = \emptyset$.*

A graph G is nontrivial if $E(G) \neq \emptyset$. An edge cut X of G is essential if $G - X$ has at least two nontrivial components. A graph G is essentially k-edge-connected if G does not have an essential edge cut of size less than k. Let $V_1(G)$ be the set of vertices with degree 1 in G. The core of a graph G, denoted by G_0, is obtained from $G - V_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Theorem 1.2[8]**.** *Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G.*
(i) *G_0 is uniquely determined by G with $\kappa'(G_0) \geq 3$.*
(ii) *(see also Lemma 2.9 of [6]) If any $e, e' \in E(G_0)$, $G_0(e, e')$ has a spanning $(v_e, v_{e'})$-trail, then $L(G)$ is hamiltonian-connected.*
(iii) *(see also Proposition 2.2 of [6]) $L(G)$ is hamiltonian-connected if and only if for any pair of edges $e, e' \in E(G)$, G has a dominating (e, e')-trail.*

The design of n processor network with given number of connections from each processor and with a desirable strength of the network can be modelled as a degree sequence realization problem with certain desirable graphical properties. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of G, and $d(v_i)$ be the degree of v_i in G. Then the sequence $(d(v_1), d(v_2), \cdots, d(v_n))$ is called the degree sequence of G. A sequence $d = (d_1, d_2, \cdots, d_n)$ is nonincreasing if $d_1 \geq d_2 \geq \cdots \geq d_n$. A sequence is multigraphic if there is a multigraph G with degree sequence d. This graph G is called a realization of d. A multigraphic degree sequence d is supereulerian (resp., strongly spanning trailable) if d has a realization G which is a supereulerian graph (resp., strongly spanning trailable graph), and d is line-hamiltonian (resp., line-hamiltonian-connected) if d has a realization G such that the line graph of G is hamiltonian (resp., hamiltonian-connected).

Hakimi[4] gave a characterization for multigraphic degree sequences as follows.

Theorem 1.3[4]**.** *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing degree sequence with $n \geq 2$ and nonnegative integer d_i for $1 \leq i \leq n$. Then the sequence $d = (d_1, d_2, \cdots, d_n)$ is multigraphic if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.*

In [1], Boesch and Harary presented the following theorem which is due to Butler.

Theorem 1.4[1]**.** *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing degree sequence with $n \geq 2$ and nonnegative integer d_i for $1 \leq i \leq n$. Let j be an index with $2 \leq j \leq n$. Then the sequence $(d_1, d_2, \cdots, d_n)$ is multigraphic if and only if the sequence $(d_1 - 1, d_2, \cdots, d_{j-1}, d_j - 1, d_{j+1}, \cdots, d_n)$ is multigraphic.*

Gu et al.[3] characterized supereulerian degree sequences and line-hamiltonian degree sequences.

Theorem 1.5[3]**.** *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence. Then d is supereulerian if and if either $n = 1$ and $d_1 = 0$, or $n \geq 2$ and $d_n \geq 2$.*

Theorem 1.6[3]**.** *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence with $n \geq 3$. Then the following are equivalent.*
(i) *d is line-hamiltonian.*
(ii) *either $d_1 \geq n - 1$, or $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2)$.*
(iii) *d has a realization G such that $G - V_1(G)$ is supereulerian.*

In this paper, we investigate strongly spanning trailable sequences and line-hamiltonian-connected sequences.

Theorem 1.7. *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence. Then d is strongly spanning trailable if and only if either $n = 1$ and $d_1 = 0$ or $n \geq 2$ and $d_n \geq 3$.*

Theorem 1.8. *Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence. If $n \geq 2$ and $d_n \geq 3$, then d is line-hamiltonian-connected.*

The rest of this paper is organized as follows. In Section 2, we present some useful lemmas. In Section 3, using these lemmas, we prove Theorems 1.7 and 1.8.

2 Lemmas

Lemma 2.1 (Theorem 3.3 of [6])**.** *Let G be a graph with $\kappa'(G) \geq 3$. If every 3-edge-cut of G has at least one edge in a 2-cycle or 3-cycle of G, then $G(e, e')$ is collapsible for any $e, e' \in E(G)$.*

Lemma 2.2 (Theorem 2.3 (iii) of [6])**.** *Let G be a graph. If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u,v)-trail.*

Lemma 2.3 (Proposition 1.1.3 of [8])**.** *If $L(G)$ is k-connected, then G is essentially k-edge-connected. Moreover, when $L(G)$ is not complete, $L(G)$ is k-connected if and only if G is essentially k-edge-connected.*

In the following, we shall define some strongly spanning trailable graphs.

Definition 2.4. *Let $s, l \geq 0$,$t \geq 3$ and $n \geq 4$ be integers.*
(i) *Let $K_2^{(3)}$ be the graph with 2 vertices and 3 multiple edges. For convienience, we call one vertex of $K_2^{(3)}$ the center vertex.*
(ii) *Let P_t be a path $v_1 v_2 \cdots v_t$ with t vertices. Denote by P'_t the graph obtained from P_t by*

adding a new vertex v and $t-2$ edges $v_2v, v_3v, \cdots, v_{t-1}v$, and adding two parallel edges between v and v_i for $i = 1, t$, where v is called the center vertex of P'_t.

(iii) *Let $P_{s,t}$ be the graph obtained from s vertex disjoint P'_2 and one P'_t by identifying their center vertices.*

(iv) *Let $P'_{s,l}$ be the graph obtained from s vertex disjoint P'_2 and l vertex disjoint $K_2^{(3)}$ by identifying their center vertices (P_t, P'_t, $P_{s,t}$ and $P'_{s,l}$ are depicted in Figure 2.1).*

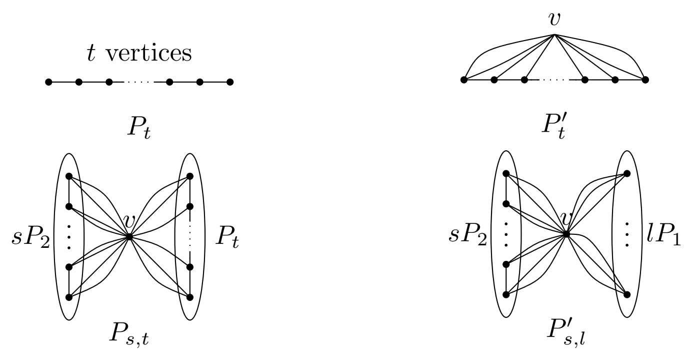

Figure 2.1. Graphs P_t, P'_t, $P_{s,t}$ and $P'_{s,l}$

Lemma 2.5. *Let K_4 be a complete graph on 4 vertices. For integers $s, l \geq 0$ and $t \geq 3$, let $K_2^{(3)}$, P'_t, $P_{s,t}$ and $P'_{s,l}$ be the multigraphs defined in Definition 2.4. Then $K_2^{(3)}$, K_4, P'_t, $P_{s,t}$ and $P'_{s,l}$ are strongly spanning trailable. Moreover, each of the following results holds:*

(i) *$K_2^{(3)}$ has 2 vertices and the degree sequence of $K_2^{(3)}$ is $(3, 3)$;*

(ii) *K_4 has 4 vertices and the degree sequence of K_4 is $(3, 3, 3, 3)$;*

(iii) *P'_t has $t+1$ vertices and the degree sequence of P'_t is $(t+2, 3, \cdots, 3)$;*

(iv) *$P_{s,t}$ has $2s+t+1$ vertices and the degree sequence of $P_{s,t}$ is $(4s+t+2, 3, \cdots, 3)$;*

(v) *$P'_{s,l}$ has $2s+l+1$ vertices and the degree sequence of $P'_{s,l}$ is $(4s+3l, 3, \cdots, 3)$.*

Proof. It is routine to verify (i)-(v). It is sufficient to show that for any graph $G \in \{K_2^{(3)}, K_4, P'_t, P_{s,t}, P'_{s,l}\}$, G is strongly spanning trailable. For any graph $G \in \{K_2^{(3)}, K_4, P'_t, P_{s,t}, P'_{s,l}\}$, $\kappa'(G) \geq 3$ and every 3-edge-cut of G has at least one edge in a 2-cycle or 3-cycle of G. By Lemma 2.1, for any $e, e' \in E(G)$, $G(e, e')$ is collapsible. Then by Lemma 2.2, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. By definition, G is strongly spanning trailable. This completes the proof of Lemma 2.5. □

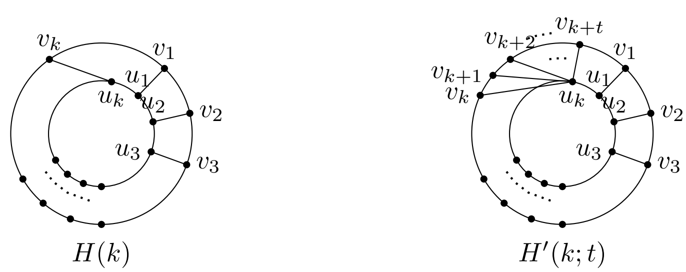

Figure 2.2. Graphs $H(k)$ and $H'(k;t)$

Definition 2.6. *Let $C_1 = v_1v_2 \cdots v_kv_1$ and $C_2 = u_1u_2 \cdots u_ku_1$ be two cycles, where k is an integer and $k \geq 3$.*

(i) *Let $H(k)$ be the graph obtained from C_1 and C_2 by adding all the edges in $E(C_1, C_2) = \{v_1u_1, v_2u_2, \cdots, v_ku_k\}$.*
(ii) *Let $H'(k;t)$ be the graph obtained from $H(k+t)$ by identifying the vertices $u_k, u_{k+1}, u_{k+2}, \cdots, u_{k+t}$ to a new vertex (which is also denoted by u_k) and deleting the resulting loops.*

Lemma 2.7. *Let $H(k)$ and $H'(k;t)$ be the graphs defined in Definition 2.6. Then $H(k)$ and $H'(k;t)$ are strongly spanning trailable simple graphs, and each of the following results holds:*
(i) *$H(k)$ has $2k$ vertices and the degree sequence of $H(k)$ is $(3, 3, \cdots, 3)$.*
(ii) *$H'(k;t)$ has $2k+t$ vertices and the degree sequence of $H'(k;t)$ is $(t+3, 3, \cdots, 3)$.*

Proof. It is easy to verify (i) and (ii). Firstly we show that $H(k)$ is strongly spanning trailable. By definition, it suffices to show that for any $e, e' \in E(H(k))$, $H(k)(e, e')$ has a spanning $(v_e, v_{e'})$-trail T. In the following proof, if $1 \le m < n \le k$, we denote $\overset{+}{\overline{v_mv_n}} = v_mv_{m+1}\cdots v_n$ and $\overset{-}{\overline{u_nu_m}} = u_nu_{n-1}\cdots u_m$. If $1 \le n \le k$ and $k-n$ is odd, we denote $\overset{-}{\widetilde{u_ku_n}} = u_kv_kv_{k-1}u_{k-1}u_{k-2}v_{k-2}v_{k-3}u_{k-3}\cdots u_{n-1}v_{n-1}v_nu_n$; if $1 \le n \le k$ and $k-n$ is even, we denote $\overset{-}{\widetilde{u_ku_n}} = u_kv_kv_{k-1}u_{k-1}u_{k-2}v_{k-2}v_{k-3}u_{k-3}\cdots v_{n-2}v_{n-2}v_{n-1}u_{n-1}u_n$ and $\overset{-}{\widetilde{u_kv_n}} = \overset{-}{\widetilde{u_ku_n}}v_n$.

If $e = e' \in E(C_1)$ or $E(C_2)$, without loss of generality, we assume that $e = e' = v_kv_1 \in E(C_1)$, and let $T = v_ev_1u_1\overset{-}{\overline{u_ku_2}}\ \overset{+}{\overline{v_2v_k}}v_{e'}(= v_e)$. If $e = e' \in E(C_1, C_2)$, we assume that $e = e' = v_1u_1$, and let $T = v_eu_1\overset{-}{\overline{u_ku_2}}\ \overset{+}{\overline{v_2v_k}}v_1v_{e'}(= v_e)$. For any case, T is a spanning $(v_e, v_{e'})$-trail of $H(k)(e, e')$.

If $e \ne e'$, by symmetry, we distinguish the following four cases (see Table 2.1). For convenience, if $s = k$, let $v_{s+1} = v_1$ and $u_{s+1} = u_1$ in Table 2.1. For any case, we can find a spanning $(v_e, v_{e'})$-trail T of $H(k)(e, e')$. Hence $H(k)$ is strongly spanning trailable.

Now we show that $H'(k;t)$ is also strongly spanning trailable. Recall that $H'(k;t)$ is the graph obtained from $H(k+t)$ by identifying the vertices $u_k, u_{k+1}, u_{k+2}, \cdots, u_{k+t}$ and deleting the resulting loops. For any $e, e' \in E(H'(k;t))$, we have $e, e' \in E(H(k+t))$. Since $H(k+t)$ is strongly spanning trailable, $H(k+t)(e, e')$ has a spanning $(v_e, v_{e'})$-trail T. Then we obtain a spanning trail of $H'(k;t)(e, e')$ from T by by identifying the vertices $v_k, v_{k+1}, v_{k+2}, \cdots, v_{k+t}$ and deleting the resulting loops. Hence $H'(k;t)$ is strongly spanning trailable. This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let G be a graph on $n \ge 2$ vertices and e_0 be an edge of G. If $G^* = G - e_0$ is strongly spanning trailable, then G is also strongly spanning trailable.*

Proof. Assume that G and $e_0 \in E(G)$ satisfy the hypothesis of Lemma 2.8. Then

$$G^* = G - e_0 \text{ is strongly spanning trailable.} \tag{2.1}$$

By definition, it suffices to show that for any $e, e' \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. According to the location of e and e', we distinguish the following cases.

Case 1. $e, e' \in E(G^*)$.

By (2.1), $G^*(e, e')$ has a spanning $(v_e, v_{e'})$-trail T. Then T is also a spanning $(v_e, v_{e'})$-trail of $G(e, e')$.

Case 2. $e \in E(G^*)$ and $e' = e_0 \notin E(G^*)$.

Assume that $e' = e_0 = ab$. By (2.1), G^* is 2-edge-connected. Let $c \in V(G^*)$ and $e_1 = bc \in E(G^*)$. By (2.1), for e and e_1, $G^*(e, e_1)$ has a spanning (v_e, v_{e_1})-trail T. If $v_{e_1}b \in E(T)$, let $T' = T - \{v_{e_1}b\} + \{v_{e'}b\}$. If $v_{e_1}b \notin E(T)$, then $cv_{e_1} \in E(T)$, and let $T' = T - \{cv_{e_1}\} + \{cb, bv_{e'}\}$. Then T' is a spanning $(v_ev_{e'})$-trail of $G(e, e')$.

Table 2.1. Spanning $(v_e, v_{e'})$-trails of $H(k)(e, e')$.

(e, e')	s	$k-s$	T
(v_kv_1, v_sv_{s+1})	$\lceil \frac{k}{2} \rceil \le s \le k-1$	odd	$v_e\overset{+}{\overline{v_1v_s}}\ \overset{-}{\overline{u_su_1}}\ \overset{-}{\widetilde{u_kv_{s+1}}}v_{e'}$
	$\lceil \frac{k}{2} \rceil \le s \le k-2$	even	$v_e\overset{+}{\overline{v_1v_{s-1}}}\ \overset{-}{\overline{u_{s-1}u_1}}\ \overset{-}{\widetilde{u_kv_s}}v_{e'}$
(v_1u_1, v_sv_{s+1})	$\lceil \frac{k+1}{2} \rceil \le s \le k$	odd	$v_e\overset{+}{\overline{v_1v_s}}\ \overset{-}{\overline{u_su_1}}\ \overset{-}{\widetilde{u_kv_{s+1}}}v_{e'}$
	$3 \le \lceil \frac{k+1}{2} \rceil \le s \le k$	even	$v_e\overset{+}{\overline{v_1v_{s-1}}}\ \overset{-}{\overline{u_{s-1}u_1}}\ \overset{-}{\widetilde{u_kv_s}}v_{e'}$
(v_kv_1, u_su_{s+1})	$\lceil \frac{k}{2} \rceil \le s \le k-1$	odd	$v_e\overset{+}{\overline{v_1v_{s-1}}}\ \overset{-}{\overline{u_{s-1}u_1}}\ \overset{-}{\widetilde{u_ku_s}}v_{e'}$
	$\lceil \frac{k}{2} \rceil \le s \le k-2$	even	$v_e\overset{+}{\overline{v_1v_s}}\ \overset{-}{\overline{u_su_1}}\ \overset{-}{\widetilde{u_ku_{s+1}}}v_{e'}$
	$s = k$	0	$v_e\overset{+}{\overline{v_1v_k}}\ \overset{-}{\overline{u_ku_1}}v_{e'}$
(v_1u_1, v_su_s)	$4 \le \lceil \frac{k}{2} \rceil + 1 \le s \le k-1$	odd	$v_e\overset{+}{\overline{v_1v_{s-2}}}\ \overset{-}{\overline{u_{s-2}u_1}}\ \overset{-}{\widetilde{u_kv_{s+1}}}v_sv_{s-1}u_{s-1}u_sv_{e'}$
	$s = 3, k = 4$	1	$v_ev_1v_4v_3v_2u_2u_1u_4u_3v_{e'}$
	$4 \le \lceil \frac{k}{2} \rceil + 1 \le s \le k$	even	$v_e\overset{+}{\overline{v_1v_{s-2}}}\ \overset{-}{\overline{u_{s-2}u_1}}\ \overset{-}{\widetilde{u_ku_{s+1}}}u_su_{s-1}v_{s-1}v_sv_{e'}$
	$s = k = 3$	0	$v_ev_1v_3v_2u_2u_1u_3v_{e'}$

Case 3. $e = e' = e_0 \notin E(G^*)$.

Assume that $e = e' = e_0 = ab$. By (2.1), G^* is 2-edge-connected. Let $c, d \in V(G^*)$, $e_1 = bc \in E(G^*)$ and $e_2 = ad \in E(G^*)$. If $c = a$, by (2.1), $G^*(e_1)$ has a supereulerian subgraph T. Let $T' = T - \{av_{e_1}, v_{e_1}b\} + \{av_e, v_eb\}(v_e = v_{e'})$, and then T' is a spanning $(v_e, v_{e'})$-trail in $G(e, e')$. If $d = b$, we prove the result similarly. Hence we assume that $c \neq a$ and $d \neq b$.

By (2.1), $G'(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail T. If $v_{e_1}b \in E(T)$, let $T' = T - \{v_{e_1}b\} + \{v_eb\}$; if $v_{e_1}b \notin T$, let $T' = T - \{cv_{e_1}\} + \{cb, bv_e\}$. If $av_{e_2} \in E(T')$, let $T'' = T' - \{av_{e_2}\} + \{av_e\}$; if $av_{e_2} \notin E(T')$, let $T'' = T' - \{dv_{e_2}\} + \{da, av_e\}$. Then T'' is a spanning $(v_ev_{e'})$-trail in $G(e, e')$. This completes the proof of Lemma 2.8. □

3 Proofs of Theorems 1.7 and 1.8

Proof of Theorem 1.7. Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence. The case when $n = 1$ is trivial and so we shall assume that $n \ge 2$. If d is strongly spanning trailable, then d has a strongly spanning trailable realization G. If $d_n \le 2$, G has a vertex v with degree 1 or 2. If G has a vertex v with degree 2, let $N_G(v) = \{u_1, u_2\}$ and $e = u_1v$, $e' = u_2v$. Obviously, $G(e, e')$ has no spanning $(v_e, v_{e'})$-trails. If G has a vertex u with degree 1, let $N_G(u) = v_1$ and $e = e' = v_1u$. Then $G(e, e')$ has no spanning $(v_e, v_{e'})$-trails. So $d_n \ge 3$.

We prove the sufficiency by induction on $m = \sum_{i=1}^{n} d_i$. If $n = 2$ and $d = (3, 3)$, then $m = 6$ and by Lemma 2.5, graph $K_2^{(3)}$ is a strongly spanning trailable realization of d.

Suppose that theorem holds for all such multigraphic sequences with smaller value of m. Next we shall prove the theorem holds for multigraphic sequences with value of m. We distinguish the following cases.

Case 1. $d_1 = d_2 = 3$.

Then $d = (3, 3, \cdots, 3)$. By Theorem 1.3, $\sum_{i=1}^{n} d_i = 3n$ is even, and so n is even. If $n = 2$, let

$G = K_2^{(3)}$. If $n = 4$, let $G = K_4$. If $n \geq 6$, let $G = H(n/2)$. By Lemmas 2.5 and 2.7, G is a strongly spanning trailable realization of d.

Case 2. $d_1 \geq 4,\ d_2 = 3$.

Then $d = (d_1, 3, \cdots, 3)$. By Theorem 1.3, $d_1 + 3n - 3$ is even and $4 \leq d_1 \leq 3n - 3$. Since $d_1 + 3n - 3$ is even, $d_1 \neq n - 2$ or n. If $4 \leq d_1 \leq n - 3$, let $G = H'(\frac{n-d_1+3}{2}; d_1 - 3)$. If $d_1 = n - 1$, let $G = W_n$. If $n + 1 \leq d_1 \leq 2n - 3$, let $s = \frac{d_1-(n+1)}{2} = \frac{d_1-n-1}{2}$, $t = n - 1 - 2s = 2n - d_1$, and $G = P_{s,t}$. If $2n - 2 \leq d_1 \leq 3n - 3$, let $s = \frac{n-1-(d_1-2n+2)}{2} = \frac{3n-d_1-3}{2}$, $l = d_1 - 2n + 2$ and $G = P'_{s,l}$. Then by Lemmas 2.5, 2.1 and 2.7, G is a strongly spanning trailable graph realization of d.

Case 3. $d_1 \geq d_2 \geq 4$.

Since $(d_1, d_2, d_3, \cdots, d_n)$ is multigraphic degree sequence, by Theorem 1.4, $(d_1 - 1, d_2 - 1, d_3, \cdots, d_n)$ is also a multigraphic degree sequence. Since $d_1 - 1 \geq d_2 - 1 \geq 3$ and $\sum_{i=1}^{n} d_i - 2 = m - 2$, by induction, $(d_1 - 1, d_2 - 1, d_3, \cdots, d_n)$ has a strongly spanning trailable realization G^*. By Lemma 2.8, we can obtain a strongly spanning trailable realization of d from G^* by adding an edge. This completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing multigraphic sequence with $n \geq 2$ and $d_n \geq 3$. By Theorem 1.7, d has a strongly spanning trailable realization G. If $L(G)$ is a complete graph, then $L(G)$ is hamiltonian-connected. Now we assume that $L(G)$ is not a complete graph. Since $d_n \geq 3$, the core of G is isomorphic to G, and so $\kappa'(G) \geq 3$ by Theorem 1.2. By Lemma 2.3, $\kappa(L(G)) \geq 3$. Since G is strongly spanning trailable, by Theorem 1.2(ii), $L(G)$ is hamiltonian-connected. This completes the proof of Theorem 1.8. □

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