Supereulerian Digraph Strong Products

Hongjian Lai¹, Omaema Lasfar², Juan Liu³

¹Department of Mathematics, West Virginia University, Morgantown, USA
²College of Mathematics Sciences, Xinjiang Normal University, Urumqi, China
³College of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang, China

Email: hongjianlai@gmail.com, oal0001@mix.wvu.edu, liujuan1999@126.com


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Abstract

A vertex cycle cover of a digraph $H$ is a collection $C = \{C_1, C_2, \ldots, C_k\}$ of directed cycles in $H$ such that these directed cycles together cover all vertices in $H$ and such that the arc sets of these directed cycles induce a connected subdigraph of $H$. A subdigraph $F$ of a digraph $D$ is a circulation if for every vertex in $F$, the indegree of $v$ equals its outdegree, and a spanning circulation if $F$ is a cycle factor. Define $f(D)$ to be the smallest cardinality of a vertex cycle cover of the digraph obtained from $D$ by contracting all arcs in $F$, among all circulations $F$ of $D$. A digraph $D$ is supereulerian if $D$ has a spanning connected circulation. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if $D_1$ and $D_2$ are nontrivial strong digraphs such that $D_1$ is supereulerian and $D_2$ has a cycle vertex cover of order $\|V(D_1)\|$ with $\|C\| \leq \|V(D_1)\|$, then the Cartesian product $D_1$ and $D_2$ is also supereulerian. In this paper, we prove that for strong digraphs $D_1$ and $D_2$, if for some cycle factor $F_1$ of $D_1$, the digraph formed from $D_1$ by contracting arcs in $F_1$ is hamiltonian with $f(D_1) \not> \|V(D_1)\|$, then the strong product $D_1$ and $D_2$ is also supereulerian.

Keywords

Supereulerian Digraph, Direct Product, Strong Product, Cycle Factors, Eulerian Digraph

1. Introduction

We consider finite graphs and digraphs. Undefined terms and notation will follow [1] for graphs and [2] for digraphs. We will often write $D = (V(D), A(D))$ with $V(D)$ and $A(D)$ denoting the vertex set and arc set of $D$, respectively. As we are to discuss products, for digraphs $D_1$ and $D_2$ with $u \in V(D_1)$ and
Thus, throughout this article, for vertices $v \in V(D)$ of a digraph $D$, we use the notation $\overrightarrow{vu}$ to denote the arc oriented from $u$ to $v$ in $D$, where $u$ is the tail and $v$ is a head of the arc, and use $\overrightarrow{uv}$ to denote either $\overrightarrow{uu}$ or $\overrightarrow{vu}$. When $\overrightarrow{uv} \in A(D)$, we say that $u$ and $v$ are adjacent. Using the terminology in [2], digraphs do not have parallel arcs (arcs with the same tail and the same head) or loops (arcs with same tail and head). If $D$ is a digraph, we often use $G(D)$ to denote the underlying undirected graph of $D$, obtained from $D$ by erasing all orientation on the arcs of $D$.

For a positive integer $n$, we define $\{1, 2, \ldots, n\} = \{1, 2, \ldots, n\}$. Throughout this paper, we use paths, cycles and trails as defined in [1] when the discussion is on an undirected graph $G$, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph $D$. A walk in $D$ is an alternating sequence $W = x_1, a_1, x_2, \ldots, x_k, a_k, x_{k+1}$ of vertices $x_i$ and arcs $a_j$ from $D$ such that $a_j = x_i x_{i+1}$ for every $i \in [k]$ and $j \in [k-1]$. A walk $W$ is closed if $x_k = x_1$, and is open otherwise. We use $V(W) = \{x_i : i \in [k]\}$ and $A(W) = \{a_j : j \in [k-1]\}$. We say that $W$ is a walk from $x_1$ to $x_k$ or an $(x_1, x_k)$-walk. If $x_1 = x_k$, then we say that the vertex $x_1$ is the initial vertex of $W$, the vertex $x_k$ is the terminal vertex of $W$, and $x_1$ and $x_k$ are end-vertices of $W$. The length of a walk is the number of its arcs. When the arcs of $W$ are understood from the context, we will denote $W$ by $x_1, x_2, \ldots, x_k$. A trail in $D$ is a walk in which all arcs are distinct. Always we use a trail to denote an open trail. If the vertices of $W$ are distinct, then $W$ is a path. If the vertices $x_1, x_2, \ldots, x_k$ of the path $W$ are distinct satisfying $k \geq 3$ and $x_1 = x_k$, then $W$ is a cycle.

A digraph $D$ is strong if, for every pair $x, y$ of distinct vertices in $D$, there exists an $(x, y)$-walk and a $(y, x)$-walk; and is connected if $G(D)$ is connected. For the digraphs $H$ and $D$, by $HD \subset D$ we mean that $H$ is a subdigraph of $D$. Following [3], for a digraph $D$ with $X, Y \subset V(D)$, define 

$$(X, Y)_D = \{xy \in A(D) : x \in X, y \in Y\}.$$ 

when $Y = V(D) - X$, we define 

$$\partial^+_D (X) = (X, Y)_D \text{ and } \partial^-_D (X) = (Y, X)_D.$$ 

For a vertex $v$ in $D$, $d^+_D (v) = |\partial^+_D \{v\}|$ and $d^-_D (v) = |\partial^-_D \{v\}|$ are the out-degree and the in-degree of $v$ in $D$, respectively. We use the following notation:

$N^+_D (v) = \{u \in V(D) - v : vu \in A(D)\}$ and $N^-_D (v) = \{w \in V(D) - v : vw \in A(D)\}$.

The sets $N^+_D (v)$, $N^-_D (v)$ and $N_D (v) = N^+_D (v) + N^-_D (v)$ are called the out-neighbourhood, in-neighbourhood and neighbourhood of $v$. We called the vertices in $N^+_D (v)$, $N^-_D (v)$ and $N_D (v)$ the out-neighbours, in-neighbours and neighbours of $v$.

Let $D$ be a digraph. We define $D$ to be a circulation if for any $v \in V(D)$ we have $d^+_D (v) = d^-_D (v)$; and a strong digraph $D$ is eulerian if for any $v \in V(D)$, $d^+_D (v) = d^-_D (v)$. $D$ is eulerian if $D$ is a connected circulation. Thus, by definition, an eulerian digraph is also a strong digraph. It is known [3] that a digraph $D$ is a
circulation if and only if $D$ is an arc-disjoint union of cycles. A subdigraph $F$ of $D$ is a cycle factor of $D$ if $F$ is spanning circulation of $D$. Define $f(D) = \min\{k : D$ has a cycle factor with $k$ components\}. The following is well-known or immediately from the definition.

**Theorem 1.1.** (Euler, see Theorem 1.7.2 of [2] and Veblen [3]) Let $D$ be a digraph. The following are equivalent.

(i) $D$ is eulerian.

(ii) $D$ is a spanning closed trail.

(iii) $D$ is a disjoint union of cycles and $D$ is connected.

The supereulerian problem was introduced by Boesch, Suffel, and Tindell in [4], seeking to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank in [5] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been lots of research on this topic. For more literature on supereulerian graphs, see Catlin’s informative survey [6], as well as the later updates in [7] and [8]. The supereulerian problem in digraphs is considered by Gutin [9]-[10]. A digraph $D$ is supereulerian if $D$ contains a spanning eulerian subdigraph, or equivalently, a connected cycle factor. Thus, supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

The supereulerian digraph problem is to characterize the strong digraphs that contain a spanning closed trail.

Other than the researches on hamiltonian digraphs, a number of studies on supereulerian digraphs have been conducted recently. In particular, Hong et al in [11]-[12] and Bang-Jensen and Maddaloni [13] presented some best possible sufficient degree conditions for supereulerian digraphs. Several researches on various conditions of supereulerian digraphs can be found in [13]-[23], among others.

Following [24], some digraph products are defined as follows.

**Definition 1.2.** Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs,

$$V_1 = \{u_1, u_2, \ldots, u_n\}, \quad V_2 = \{v_1, v_2, \ldots, v_n\}$$

Then the Cartesian product, the Direct product and the Strong product of $D_1$ and $D_2$ are defined as follows,

(i) **The Cartesian product** denoted by $D_1 \square D_2$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 \square D_2) = \{(u_i, v_j) : u_i \in A_1 \text{ and } v_j \in A_2 \}$$

(ii) **The Direct product** denoted by $D_1 \times D_2$ is the digraph with vertex set $V_1 \times V_2$ and

$$A(D_1 \times D_2) = \{(u_i, v_j) : u_i \in A_1 \text{ and } v_j \in A_2 \}$$

(iii) **The Strong product** denoted by $D_1 \boxtimes D_2$ is the digraph with vertex set
and
\[ A(D_1 \Box D_2) = \{ (u_i, v_j) : u_i \in A_i \text{ or } u_i, u_j \in A_i \text{ and } v_j = v_i \text{ or both } u_i, u_j \in A_i \text{ and } v_j, v_i \in A_j \} \]

It is often of interest to investigate natural conditions on the factors of a product to assure hamiltonicity of the product, as seen in Problem 6 of [25]. Researchers have investigated conditions on factors of digraph products to warrant the product to be supereulerian. Alsatami, Liu, and Zhang in [17] introduced eulerian vertex cover of a digraph $D$ to study the supereulerian digraph problem.

**Definition 1.3.** Let $D$ be a digraph, $C_1, C_2, \cdots, C_k$ be eulerian subdigraphs of $D$ and set $F = \{ C_1, C_2, \cdots, C_k \}$ where $k > 0$ is an integer.

(i) $F$ is called a cycle vertex cover of $D$ if each $C_i$ in $F$ is a cycle, and both (i-1) and (i-2) hold:

(i-1) $V(D) = \bigcup_{i \in F} V(C_i)$.

(i-2) $F = \bigcup_{i \in F} C_i$ is weakly connected.

(ii) For any $u, v \in V(D)$, $F$ is called an eulerian chain joining $u$ and $v$, if each of the following holds.

(ii-1) $u \in V(C_i)$ and $v \in V(C_i)$.

(ii-2) $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for any $i$ with $1 \leq i \leq k - 1$.

A subdigraph $F$ of a digraph $D$ is a circulation if $d^+_i (v) = d^-_i (v) > 0$ holds for every $v \in V(F)$, and a spanning circulation of $D$ is a cycle factor of $D$.

Let $e = [v_i, v_j] \in A(D)$ denote an arc of $D$ which is either $v_i, v_j$ or $v_j, v_i$. Define $D/e$ to be the digraph obtained from $D - e$ by identifying $v_i$ and $v_j$ into a new vertex $v_s$ and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is a symmetric arc subset, then define the contraction $D/W$ to be the digraph obtained from $D$ by contracting each arc $e \in W$, and deleting any resulting loops. Thus even $D$ does not have parallel arcs, a contraction $D/W$ is loopless but may have parallel arcs, with $A(D/W) \subseteq A(D)\setminus W$. If $H$ is a subdigraph of $D$, then we often use $D/H$ for $(D, A(H))$. If $L$ is a connected symmetric component of $H$ and $v_L$ is the vertex in $D/H$ onto which $L$ is contracted, then $L$ is the contraction preimage of $v_L$. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a trivial vertex if the preimage of $v$ is a single vertex (also denoted by $v$) in $D$. Hence, we often view trivial vertices in a contraction $D/W$ as vertices in $D$.

**Definition 1.4.** Let $F$ be a circulation of a digraph $D$ and $D/F$ denote the digraph formed from $D$ by contracting arcs in $A(F)$. For any circulation $F$ of $D$, define

(i) $f_{D/F} = \min \{ |C| : C \text{ is a cycle vertex cover of } D/F \}$ and,

(ii) $f(D) = \min \{ f_{D/F} : F \text{ is a circulation of } D \}$.

By definition, if $D$ is a circulation, then every component of $D$ is eulerian. By Theorem 1.1, we observe the following.

Every circulation is an arc-disjoint union of cycles. (2)
There have been some former results concerning the Cartesian products of digraphs to be eulerian and to be supereulerian.

**Theorem 1.5.** Let $D_1$ and $D_2$ be nontrivial strong digraphs.

(i) (Xu [26]) If $D_1$ and $D_2$ are eulerian digraphs. Then the Cartesian product $D_1 \square D_2$ is eulerian.

(ii) (Alsatami, Liu, and Zhang [17]) If such that $D_1$ is supereulerian and $D_2$ has a cycle vertex cover $C'$ with $|C'| \leq |V(D_1)|$, then the Cartesian product $D_1 \square D_2$ is supereulerian.

The current research is motivated by Problem 6 of [25] and Theorem 1.5. We prove the following.

**Theorem 1.6.** Let $D_1$ and $D_2$ be strong digraphs. If $f(D_1) \leq |V(D_1)|$ and if for some cycle factor $F$ of $D_1$, $D_1/F$ is hamiltonian, then the strong product $D_1 \boxtimes D_2$ is supereulerian.

In the next section, we develop some lemmas which will be used in our arguments. The proof of the main result will be given in the last section.

2. Lemmas

Let $k \geq 0$ be an integer. We use $Z_k = \{1, 2, \ldots, k\}$ to denote the cyclic group of order $k$ and with the additive binary operation $+_k$ and with $k$ being the additive identity in $Z_k$. Let $H$ and $H'$ denote two digraphs. Define $H \cup H'$ to be the digraph with $V(H \cup H') = V(H) \cup V(H')$ and $A(H \cup H') = A(H) \cup A(H')$.

Let $T = v_1v_2 \cdots v_k$ denote a trail. We use $T[v_1, v_k]$ to emphasize that $T$ is oriented from $v_1$ to $v_k$. For any $1 \leq i < j \leq k$, we use $T[v_i, v_j] = v_iv_{i+1} \cdots v_{j-1}v_j$ to denote the sub-trail of $T$. Likewise, if $Q = u_1u_2 \cdots u_k$ is a closed trail, then for any $i, j$ with $1 \leq i < j \leq k$, $Q[u_i, u_j]$ denotes the sub-trail $u_1u_2 \cdots u_{i-1}u_i u_{j-1}u_j$. If $T' = w_1w_2 \cdots w_k$ is a trail with $v_k = w_1$ and $V(T) \cap V(T') = \{v_k\}$, then we use $TT'$ or $T[v_k, v_k]T'[v_k, w_k]$ to denote the trail $v_1v_2 \cdots v_kw_k \cdots v_k$. If $V(T') = \emptyset$ and there is a path $z_1z_2 \cdots z_k$ with $z_2, \ldots, z_k \in V(T) \cup V(T')$ and with $z_1 = v_k$ and $z_k = w_1$, then we use $Tz_2 \cdots z_kT'$ to denote the trail $v_1v_2 \cdots v_kw_k \cdots v_k$. In particular, if $T$ is a $(v, w)$-trail of a digraph $D$ and $uv, wz \in A(D) - A(T)$, then we use $uvTwz$ to denote the $(u, z)$-trail $D[A(T) \cup \{uv, wz\}]$. The subdigraphs $uvT$ and $Twz$ are similarly defined.

**Lemma 2.1.** Let $J_1, J_2, \ldots, J_k$ be vertex disjoint strong subdigraphs of a digraph $D$, and $J = \bigcup_{i=1}^k J_i$ is the disjoint union of these subdigraphs. Let $v_1, v_2, \ldots, v_k$ be vertices in $V(D/J)$ such that for each $i \in [k]$, $J_i$ is the preimage of $v_i$. Suppose that $C' = v_1, v_2, \ldots, v_k$ be a cycle of $D/J$. Each of the following holds.

(i) $D$ has a cycle $C$ with $A(C') \subseteq A(C)$ such that for each $i \in [k]$, $V(C) \cap V(J_i) \neq \emptyset$. (Such a cycle $C$ is called a lift of the cycle $C'$.)

(ii) If for each $i \in Z_k$, $e_i = v_i'v_{i+1}' \in A(C')$ is an arc in $D$ with $v_i' \in V(J_i)$ and $v_{i+1}' \in V(J_{i+1})$, then $C[v_i', v_{i+1}']$ is a path in $J_i$.

**Proof.** As (i) implies (ii), it suffices to prove (i). Let $C' = v_1, v_2, \ldots, v_k$ be a
cycle of $D/J$, and for each $i \in \mathbb{Z}_+$. By definition, the arc $\mathbf{e}_i = v'_i v''_i \in A(C')$ is an arc in $D$, and so we may assume that there exist vertices $v'_i, v''_i \in V(J_i)$ such that $\mathbf{e}_i = v''_i v'_i \in A(D)$. If $J_i$ is trivial, then we have $v'_i = v''_i$. Since $J_i$ is strong, $J_i$ contains a $(v'_i, v''_i)$-path $P_i$. Thus

$$C := P_i v'_i v''_i v'_i v''_i \cdots v'_i v''_i v'_i P_i v'_i v''_i$$

is a cycle of $D$ with $C[v'_i, v''_i]$ being a path in $J_i$, for each $i \in \mathbb{Z}_+$.

Following [2], we define a digraph to be cyclically connected if for every pair $x, y$ of distinct vertices of $D$ there is a sequence of cycles $C_1, C_2, \ldots, C_k$ such that $x$ is in $C_1$, $y$ is in $C_k$, and $C_i$ and $C_{i+1}$ have at least one common vertex for every $i \in [k-1]$. The following results are useful. Lemma 2.2 (ii) follows immediately from definition of strong digraphs.

**Lemma 2.2.** Let $D$ be a digraph.

(i) (Exercise 1.17 of [2]) A digraph $D$ is strong if and only if it is cyclically connected.

(ii) If $H_1$ and $H_2$ are strong subdigraphs of $D$ with $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also strong.

**Proposition 2.3.** (Alsatami, Liu and Zhang, Proposition 2.1 of [17]) Let $D$ be a weakly connected digraph.

Then the following are equivalent.

(i) $D$ has a cycle vertex cover.

(ii) $D$ is strong.

(iii) $D$ is cyclically connected.

(iv) For any vertices $u, v \in V(D)$, there exists an eulerian chain joining $u$ and $v$.

**Lemma 2.4.** Let $D_1$ and $D_2$ be digraphs. Each of the following holds.

(i) If $D_1$ and $D_2$ are cycles, then $D_1 \times D_2$ is a circulation.

(ii) If $H_1$ and $H_2$ are arc-disjoint subdigraphs of $D_i$, then $H_1 \times D_2$ and $H_2 \times D_1$ are arc-disjoint subdigraphs of $D_1 \times D_2$.

(iii) If each of $D_1$ and $D_2$ has a cycle factor, then $D_1 \times D_2$ has a cycle factor.

**Proof.** For (i), let $V_1$ and $V_2$ be the vertex sets of $D_1$ and $D_2$, respectively. It suffices to prove that for each $(u, v_j) \in V_1 \times V_2$, $d^{u \times u}_{D_1 \times D_2}((u, v_j)) = d^{d_1 \times d_2}_{D_1 \times D_2}((u, v_j)) = d^{d_2 \times d_2}_{D_1 \times D_2}((u, v_j))$.

Let $(u, v_j) \in V_1 \times V_2$. Since $D_1$ and $D_2$ are cycles, we have $|N^+_{D_1}(u)| = |N^{-}_{D_1}(u)|$ and $|N^+_{D_2}(v_j)| = |N^{-}_{D_2}(v_j)|$. By Definition 1.2, we have the following, which implies (i).

$$d^{d_1 \times d_2}_{D_1 \times D_2}((u, v_j)) = |N^+_{D_1 \times D_2}((u, v_j))|$$

$$= |\{(u, v_j) \in V_1 \times V_2 : (u, v_j) (u, v_j) \in A(D_1 \times D_2)\}|$$

$$= |\{(u, v_j) \in V_1 \times V_2 : u u \in A(D_1) \text{ and } v_j v_j \in A(D_2)\}|$$

$$= \sum_{u \in N_{D_1}^{-}(u)} \sum_{v_j \in N_{D_2}^{-}(v_j)} |\{(u, v_j) \in V_1 \times V_2\}|$$

$$= |N^+_{D_1}(u)||N^{-}_{D_2}(v_j)| + |N^{-}_{D_1}(u)||N^+_{D_2}(v_j)|$$
\[
= \sum_{u_1 \in N_{D_1}(v_1)} \sum_{v_1 \in N_{D_2}(v_1)} \left| \{(u_1, v_1) \in V_1 \times V_2 : u_1 u_i \in A(D_1) \text{ and } v_j v_1 \in A(D_2)\} \right|
\]
\[
= \left| \{(u_1, v_1) \in V_1 \times V_2 : u_1 u_i \in A(D_1) \text{ and } v_j v_1 \in A(D_2)\} \right|
\]
\[
= \left| N^{-1}_{D_1 \times D_2} \left( (u_1, v_1) \right) \right|
\]
\[
= \left| \{(u_1, v_1) \in V_1 \times V_2 : (u_3, v_1)(u_i, v_1) \in A(D_1 \times D_2)\} \right|
\]
\[
= d^-_{D_1 \times D_2} \left( (u_1, v_1) \right)
\]

To prove (ii), let \( H_1 \) and \( H_2 \) be an arc-disjoint subdigraph of \( D_i \). If there exists an arc
\[
(u_1, v_1)(u_3, v_1) \in A(H_1 \times D_2) \cap A(H_2 \times D_2),
\]
then by Definition 1.2, we must have \( u_1 u_i \in H_1 \cap H_2 \). Hence if \( H_1 \) and \( H_2 \) are arc-disjoint subdigraphs of \( D_i \), then \( H_1 \times D_2 \) and \( H_2 \times D_2 \) are arc disjoint subdigraphs of \( D_1 \times D_2 \).

To prove (iii), let \( F_1 \) and \( F_2 \) be the spanning circulations of \( D_1 \) and \( D_2 \), respectively. By Definition 1.2, \( F_1 \times F_2 \) is spanning subdigraph of \( D_1 \times D_2 \). By (i), \( F_1 \times F_2 \) is a circulation, and so \( F_1 \times F_2 \) is the spanning circulation of \( D_1 \times D_2 \).

\textbf{Lemma 2.5.} Let \( D_i \), \( D_j \) be digraphs and \( F \) be a subdigraph of \( D_i \). Then
\[
A(F \square D_j) \cap A(F \times D_j) = \emptyset.
\]
\textbf{Proof.} Suppose that there exists an arc
\[
(u_1, v_1)(u_3, v_1) \in A(F \square D_j) \cap A(F \times D_j).
\]
By Definition 1.2 (i), as
\[
(u_1, v_1)(u_3, v_1) \in A(F \square D_j), \text{ we have either } u_1 = u_i \text{ and } v_j v_1 \in A(D_2) \text{ or } u_1 u_i \in A(F) \text{ and } v_j = v_1.
\]
By Definition 1.2 (ii), if \( u_1 = u_i \) or if \( v_j = v_1 \), then
\[
(u_1, v_1)(u_3, v_1) \in A(F \times D_j).
\]
It follows that
\[
A(F \square D_j) \cap A(F \times D_j) = \emptyset.
\]

\textbf{Theorem 2.6.} (Hammack, Theorem 10.3.2 of [24]) Let \( m \) and \( n \) be integers with \( m \geq n \geq 2 \) and let \( C_m \) and \( C_n \) denote the cycles of order \( m \) and \( n \), respectively. Let \( \gcd(m, n) \) and \( \text{lcm}(m, n) \) be the greatest common divisor and the least common multiplier of \( m \) and \( n \), respectively. Then the direct product \( C_m \times C_n \) is a vertex disjoint union of \( \gcd(m, n) \) cycles, each of which has length \( \text{lcm}(m, n) \).

We can show a bit more structural properties in the direct product revealed by Theorem 2.6, which are stated in Lemma 2.7.

\textbf{Lemma 2.7.} Let \( D_i \) and \( D_j \) be digraphs with vertex set notation in (1).

(i) Suppose that \( D_1 \) and \( D_2 \) are cycles and \( v \in V(D_1) \) is an arbitrarily given vertex. Then for any cycle \( C \) in \( D_1 \times D_2 \), there exists a vertex \( u \in V(D_1) \) such that the vertex \( (u, v) \in V(C) \).

(ii) Suppose that \( D_1 \) and \( D_2 \) are circulations and \( v \in V(D_2) \) is an arbitrarily given vertex. Then \( D_1 \times D_2 \) is also a circulation. Moreover, for any eulerian subdigraph \( F \) in \( D_1 \times D_2 \), there exists a vertex \( u \in V(D_1) \) such that the vertex \( (u, v) \in V(F) \).
Proof. Suppose $D_1 = u_1u_2\ldots u_{n_1}$ and $D_2 = v_1v_2\ldots v_{n_2}$ are cycles, and by symmetry, assume that $v = v_1$. Let $C$ be a cycle in $D_1 \times D_2$. Thus $C$ contains a vertex $(u_i, v_j)$. It follows by Definition 1.2 that

$$C = (u_1, v_1) (u_{j_1+1}, v_{j_1}) (u_{j_1+1}, v_{j_1+1}) \ldots (u_{j_1+n_2}, v_{j_1+n_2})$$

where the subscripts of vertices in $D_1$ are taken in $\mathbb{Z}_{n_1}$ and those of vertices in $D_2$ are taken in $\mathbb{Z}_{n_2}$. It follows that $u = u_{j_1+n_2-j_1+1}$. This proves (i). Suppose that $D_1$ and $D_2$ are circulations. By (2), each of $D_1$ and $D_2$ is an arc-disjoint union of cycles. By Lemma 2.4, $D_1 \times D_2$ is also a circulation. Let $F$ be an eulerian subdigraph in $D_1 \times D_2$. By (2), $F$ is also an arc-disjoint union of cycles $C_1, C_2, \ldots$. Applying Lemma 2.7 (i) to each cycle $C_i$, we conclude that (ii) holds as well.

3. Proofs of Theorem 1.6

Assume that $D_1$ and $D_2$ are two strong digraphs, and for some cycle factor $F$ of $D_1$, $D_1/F$ is hamiltonian with $f(D_1) \leq |V(D_1)|$. We start with some notation for the copies of factors in the Cartesian product.

**Definition 3.1.** Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two strong digraphs with $V_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$. For $i \in \{1, 2\}$, let $H_i$ be a subdigraph of $D_i$.

(i) For each $u \in V_1$, let $D_2^u$ be the subdigraph of $D_1 \boxtimes D_2$ induced by $V(D_2^u) = \{(u, v_1) : 1 \leq i \leq n_2\}$. The subdigraph $D_2^u$ is called the $u$-copy of $D_2$ in $D_1 \boxtimes D_2$.

(ii) For each $v \in V_2$, let $D_1^v$ be the subdigraph of $D_1 \boxtimes D_2$ induced by $V(D_1^v) = \{(u_1, v) : 1 \leq i \leq n_1\}$. The subdigraph $D_1^v$ is called the $v$-copy of $D_1$ in $D_1 \boxtimes D_2$.

(iii) More generally, for each $u \in V_1$ (or $v \in V_2$, respectively), let $H_2^u$ (or $H_1^v$, respectively) be the subdigraph of $D_2^u$ (or $D_1^v$, respectively) induced by $A(H_2^u) = \{(u, v) : (u, v) \in A(H_2^u)\}$ (or $A(H_1^v) = \{(u, v) : (u, v) \in A(H_1^v)\}$, respectively). The subdigraph $H_2^u$ is called the $u$-copy of $H_2$ in $D_1 \boxtimes D_2$ and the subdigraph $H_1^v$ is called the $u$-copy of $H_1$ in $D_1 \boxtimes D_2$.

If two digraphs $D$ and $H$ are isomorphic, then we write $D \cong H$. The following is an immediate observation from Definition 3.1 for the Cartesian product $D_1 \boxtimes D_2$ of two digraphs $D_1$ and $D_2$.

For any $v \in V(D_2)$, $D_1 \cong D_2^v$, and for any $u \in V(D_1)$, $D_2 \cong D_2^u$. (3)

Let $F$ be a cycle factor of $D_1$ such that $D_1/F$ has a Hamilton cycle. Since $F$ is a cycle factor of $D_1$, each component of $F$ is an eulerian subdigraph of $D_1$. Let

$$F_1, F_2, \ldots, F_k$$

be the components of $F$, and $J = D_1/F$. (4)

Then $V(J) = \{w_1, w_2, \ldots, w_k\}$, where for each $i \in [k]$, $w_i$ is the contraction image in $J$ of the eulerian subdigraph $F_i$ in $D_1$. Since $J$ is hamiltonian, we may by symmetry assume that $C_1 = w_1w_2\ldots w_kw_1$ is a hamilton cycle of $J$. It follows by Lemma 2.1 that

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$D_i$ has a cycle $C$ with $A(C') \subseteq A(C)$. \hfill (5)

Now we consider $D_i$. Let $f(D_i) = m \leq |V(D_i)|$ and $F'$ be a circulation of $D_i$ such that $D_i/F'$ has a cycle vertex cover $C' = \{C_1', C_2', \ldots, C_m'\}$. Let $F_1', F_2', \ldots, F_m'$ be the components of $F'$, $w_{i+1}', \ldots, w_i'$ be the vertices in $V(D_i) - V(F')$. We define, for each $i$ with $k' + 1 \leq i \leq t$, $F_i'$ to be the digraph with $V(F_i') = \{w'_i\}$ and $A(F_i') = \emptyset$. With these definitions, we have

$$V(D_i/F') = \{w_i', w_{i+1}', \ldots, w_t', w_{t+1}', \ldots, w_i'\} \hfill (6)$$

By Lemma 2.1, for each $j \in [m]$, $C_j'$ in $C'$ can be lifted to a cycle $C_j$ in $D_i$. To construct a spanning eulerian subdigraph of $D_i \boxtimes D_2$, we start by justifying the following claims.

Claim 1. Each of the following holds.

(i) For any $i \in [k]$, and $j \in [t]$, $F_i \times F_j'$ is a circulation.

(ii) For any $i \in [k]$, and $j \in [r]$, $F_i \boxtimes F_j'$ is an eulerian digraph.

(iii) For any $i \in [k]$, and for each $j \in [r]$, if $v \in V(F_j')$, then $F_i^v \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \boxtimes F_j'$.

Proof. For each $i \in [k]$, $F_i$ is an eulerian subdigraph of $D_i$, so $F_i$ is a disjoint union of cycles. Similarly, for each $j \in [k']$, $F_j'$ is an eulerian subdigraph of $D_2$, so $F_j'$ is a disjoint union of cycles. By Lemma 2.7, $F_i \times F_j'$ is a circulation.

By assumption, for each $i \in [k]$, $F_i$ is an eulerian subdigraph of $D_i$. If $j \in [k']$, then as $F_j'$ is an eulerian subdigraph of $D_2$, it follows by Theorem 1.5(i) that $F_i \boxtimes F_j'$ is an eulerian digraph.

Now assume that $k' + 1 \leq i \leq t$. Then $V(F_j') = \{w'_i\}$, and so by (3), $F_i \boxtimes F_j' = F_i^v \cong F_i$ is eulerian. This proves (ii).

For each $i \in [k]$, each $j \in [r]$ and a fixed vertex $v \in V(F_j')$, let $J' = F_i^v \cup (F_i \times F_j')$. By (i), $F_i \times F_j'$ is a circulation. By (3), $F_i^v \cong F_i$ is an eulerian digraph. By Lemma 2.5, $A(F_i^v) \cap A(F_i \times F_j') = \emptyset$. It follows that for any vertex $z \in V(J)$, $d_j^+ (z) = d_j^-(z) + d_j^+ (z) + d_j^- (z) = d_j^-(z)$ and so $J'$ is a circulation. Without loss of generality, we denote $V(F) = \{u_1, u_2, \ldots, u_k\}$ and $V(F_j') = \{v_j, v_2, \ldots, v_j\}$ with $v = v_j$. To prove that $J'$ is connected, let $z_0 = (u_1, v_j) \in V(J')$ and let $J_1$ be the connected component of $J'$ that contains $z_0$. If $J'$ is not connected, then by symmetry, we may assume that there exists a vertex $(u_2, v_j) \in V(J.1)$, $J_1$ is a eulerian subdigraph of $F_i \times F_j'$ with $(u_2, v_2) \in V(F)$. By Lemma 2.7 (ii), there exists a vertex $u' \in V(D_i)$ such that $(u', v_j) \in V(F)$. Thus by Definition 3.1(ii), $V(F) \cap V(F_j') \neq \emptyset$. By (3) and (4), $F_j' \cong F_j$ is connected, and so both $(u', v_j)$ and $(u', v_j)$ must be in the same component of $J'$. This implies that $(u', v_j) \in V(J_1)$. Since $(u_2, v_j)$ and $(u', v_j)$ are in the same component of $J'$, it follows that $(u_2, v_2) \in V(J_1)$ also, contrary to the assumption that $(u_2, v_2) \in V(J') \cap V(J_1)$. Hence $J'$
must be connected, and so $F_i' \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $D_i \square F_j'$.

**Claim 2.** Let $C'$ be a Hamilton cycle of $J$ and $C$ be a lift of $C'$ in $D_1$ as warranted by (5). For each $v \in V(D_1)$, let $C''$ denote the $v$-copy of $C$ in $D_1 \square D_2$. For each $j \in [i]$, if $v, v' \in V(F_j')$ are two distinct vertices, then

$$H_{v,v';j} := \bigcup_{i=1}^{k} (F_i' \cup (F_i \times F_j')) \cup C''$$

is a spanning eulerian subdigraph $D_i \square F_j'$.

**Proof.** By Lemma 2.1, for any $v \in V(D_2)$, $C''$ has the property that for any $i \in [k]$, $V(C'') \cap V(F_i') \neq \emptyset$. By Claim 1 (iii), for any $i \in [k]$ and for any $j \in [i]$, $F_i'' \cup (F_i \times F_j')$ is a spanning eulerian subdigraph $F_i \square F_j'$ and so $F_i'' \cup (F_i \times F_j')$ is a strong subdigraph of $D_i \square F_j'$. Since for any $i \in [k]$, $V(C'') \cap V(F_i') \neq \emptyset$, we may assume that for some vertex $u \in V(F_j')$, $(u,v) \in V(C'') \cap V(F_i')$. As $v \in V(F_i')$, we have $(u,v) \in V(C'') \cap V(F_i') \cup (F_i \times F_j')$ and so $F_i'' \cup (F_i \times F_j') \cup C''$ is connected. Since $v \neq v'$, $A(C'') \cup A(F_i'' \cup (F_i \times F_j')) = \emptyset$, we conclude from the facts that $C''$ and $F_i'' \times F_j'$ are circulations (see Claim 1 (i)) that $F_i'' \cup (F_i \times F_j') \cup C''$ is eulerian. As $i \in [k]$ is arbitrary, we conclude that

$$H_{v,v';j} := \bigcup_{i=1}^{k} (F_i'' \cup (F_i \times F_j')) \cup C''$$

is an eulerian subdigraph with vertex set $V(H_{v,v';j}) = \bigcup_{i=1}^{k} (F_i \times F_j') = V(D_i \square F_j')$. This proves Claim 2.

**Claim 3** Let $u \in V(D_1)$ be an arbitrary vertex, $F'$ be a circulation of $D_2$ such that $D_2 // F'$ has a cycle vertex cover $C' = \{C_1', C_2', \ldots, C_m'\}$ with $m = f(D_1) \leq |V(D_1)|$. Each of the following holds.

(i) $F''$ is a circulation of $D_2'$.

(ii) For any $j \in [m]$, $C''_j$ is a cycle of $D_2' // F''$ and $\{C''_1, C''_2, \ldots, C''_m\}$ is a cycle vertex cover of $D_2' // F''$.

(iii) Let $u \in V(D_1)$ be a vertex, $h \in [m]$ be arbitrarily given. For any vertex $v \in V(C_h')$, let $v(j), v'(j)$ be two distinct vertices in $V(F_j')$, and $C_h'$ be a lift of $C_h'$ in $D_2$. Then

$$H_h'' = \bigcup_{v(j), v'(j) \in [C_h'] \cup V(F_j')} \bigcup_{v(j), v'(j) \in [C_h'] \cup V(F_j')}$$

is an eulerian digraph with $V(H_h'') = \bigcup_{v(j), v'(j) \in [C_h'] \cup V(F_j')}$.

**Proof.** Each of (i) and (ii) follows from (3) and the definition of $C'$. It remains to prove (iii). By Lemma 2.1, $C_h'$ can be lifted to a cycle $C_h$ in $D_2$. For any $w_j' \in V(C_h')$, pick two distinct vertices $v, v' \in V(F_j')$. By Claim 2, $H_{v,v';j}$ defined in Claim 2 is a spanning eulerian subdigraph $D_i \square F_j'$. By Lemma 2.5, $C_h'' = D_1[u] \square C_h$ is arc-disjoint from each $H_{v,v';j}$, and so by the facts that $C_h''$ is a directed cycle and $H_{v,v';j}$ is eulerian, it follows that $H_h''$ is a circulation. By Definition 3.1 (iii) and by Lemma 2.5, $w_j' \in V(C_h')$ if and only if $V(C_h') \cap V(F_j') \neq \emptyset$. This is equivalent to saying that a vertex $w_j' \in V(C_h')$ if
and only if for some vertex $v^* \in V(F'_j)$ with $(u,v^*) \in V(C'_u)$, since $C'_u$ is a cycle, and since, for each $w'_j \in V(C'_u)$, there exists some vertex $v^* \in V(F'_j)$ with $(u,v^*) \in V(C'_u)$, we obtain that $V(H_{u,v^*}) \cap V(C'_u)$ contains a vertex $(u,v^*)$, it follows that $H_{u,v^*}$ must be connected. Hence $H_{u,v^*}$ is a connected circulation, and so it must be eulerian. To complete the justification of Claim 3 (iii), we note that by definition,

$$V(C'_u) \subseteq \bigcup_{w'_j \in V(C'_u)} V(D_i \boxtimes F'_j).$$

This, together with Claim 2, implies

$$V(H_{u,v^*}) = \bigcup_{w'_j \in V(C'_u)} \left( V(H_{u,v^*}) \cup V(C'_u) \right) = \bigcup_{w'_j \in V(C'_u)} V(D_i \boxtimes F'_j) = \bigcup_{j=1}^n V(D_i \boxtimes F'_j).$$

This completes the proof of Claim 3.

Recall that $V(D_i) = \{u_1, u_2, \ldots, u_n\}$ with $n_1 \geq m = f(D_2)$. We will complete the proof of Theorem 1.6 by proving that

$$H = \bigcup_{k=1}^m H_{u_k}$$

is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$. By Claim 3 (iii), we conclude that

$$V(H) = \bigcup_{j=1}^n V(D_i \boxtimes F'_j) = V(D_i \boxtimes D_2).$$

As $u_1, u_2, \ldots, u_n$ are mutually distinct, and as $F'_1, F'_2, \ldots, F'_n$ are mutually vertex disjoint, we conclude that the $H_{u_k}$’s are mutually arc-disjoint. By Claim 3 (iii), each $H_{u_k}$ is eulerian, and so $H$ is a circulation. It remains to show that $H$ is connected. By Claim 3 (iii), $H$ has a component $H'$ that contains $H_{u_k}$. If $H = H'$, then done. Assume that $V(H) - V(H') \neq \emptyset$.

Since $H'$ is a component, if some $H_{u_k}$ contains a vertex in $H'$, then $H'$ contains $H_{u_k}$ as subdigraph. Thus every $H_{u_k}$ is either contained in $H'$ or totally disjoint from $H'$. Let $W = \{w'_j \in V(D_2 \setminus F'): H_{u_k}^{w'_j}$ is contained in $H'\}$. Then as $H \neq H'$, $V(D_2 \setminus F') - W \neq \emptyset$. Since $C'$ is a cycle vertex cover of $D_2 \setminus F'$, it follows by Definition 1.3 (i-2) that there must be a cycle $C'_j \in C'$ such that $C'_j$ contains a vertex $w'_j \in W$ and a vertex $w''_j \in (D_2 \setminus F') - W$. Since $w'_j \in W$, $H_{u_k}^{w'_j}$ is contained in $H'$. Since $w'_j, w''_j \in V(C'_j)$, it follows that $w''_j \in W$, contrary to the fact that $w''_j \in (D_2 \setminus F') - W$. This contradiction indicates that we must have $H = H'$, and so $H$ is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$.

4. Concluding Remark

This research provides new conditions to ensure digraph products to be supereulerian, and adds novel knowledge to the literature of supereulerian digraph theory. Analogues to Problem 6 proposed in [25], it would also be of interest to seek natural conditions to assure supereulerian products of digraphs. Current results in this direction in [17] and in the current research also involve certain cycle cover properties on the factor digraphs. It would be of interest to see if there exist sufficient conditions on supereulerian digraphs products that do not
depend on cycle cover properties.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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