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Polynomially determine if a graph is $(s, 3)$-supereulerian

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ABSTRACT

For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Pulleyblank in 1979 showed that determining whether a graph is $(0, 0)$-supereulerian, even when restricted to planar graphs, is NP-complete. We investigate the value of the smallest integer $j(s, t)$ such that every $j(s, t)$-edge-connected graph is (s, t)-supereulerian, and show that

$$j(s,t) = \begin{cases} \max\{4, t+2\} & \text{if } 0 \leq s \leq 1, \text{ or } (s,t) \in \{(2,0), (2,1), (3,0)\}, \\ 5 & \text{if } (s,t) \in \{(2,2), (3,1)\}, \\ s+t+\frac{1-(-1)^s}{2} & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases}$$

As applications, we obtain a characterization of (s, t)-supereulerian graphs when $t \geq 3$ in terms of edge-connectivities, and show that when $t \geq 3$, there exists a polynomial time algorithm to determine if a graph is (s, t)-supereulerian.

1. The problem

We consider finite graphs without loops but permitting multiple edges, and follow [2] for undefined terms and notation. For a vertex subset or an edge subset X of a graph G, let $G[X]$ denote the subgraph induced by X. When X is an edge subset of G, we denote $G - X = G[E(G) - X]$. As in [2], we use $\delta(G)$ and $\kappa'(G)$ to denote the minimum degree and the edge-connectivity of a graph G, respectively. Following [2], a set of vertices no two of which are adjacent is referred as a **stable set**. For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains every edge in X and avoids all the edges in Y. In particular, a $(0, 0)$-supereulerian graph G, commonly known as a **supereulerian graph**, is a graph that contains a spanning closed trail. The supereulerian graph problem was first introduced by Boesch, Suffel and Tindell [1], which seeks to characterize supereulerian graphs. Pulleyblank [17] proved that determining whether a graph is supereulerian, even when restricted to planar graphs, is NP-complete. There have been intensive studies on supereulerian graphs, as seen in Catlin's survey [4] and its updates in [7,12]. The (s, t)-supereulerian problem, determining whether a given graph is (s, t)-supereulerian for given values of s and t, is an attempt to generalize the supereulerian problem. A number of research results on the (s, t)-supereulerian problem and similar topics have been obtained, as seen in [8,9,11,13–15,18,19], among others. As it is known that determining whether a graph is $(0, 0)$-supereulerian is NP-complete, the complexity of determining if a graph G is

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(s,t)-supereulerian for other values of s and t becomes of interests. This motivates the current research. A main result of this paper is a polynomial time verifiable characterization of (s,t)-supereulerian graphs when $t \geq 3$.

The notion of (s,t)-supereulerian was formally introduced in [13,14], as a generalization of supereulerian graphs. Lei et al. considered sufficient conditions for a graph to be (s,t)-supereulerian using local connectivity. For a vertex $u \in V(G)$ in a graph G, let $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ denote the set of the neighbors of u in G. The vertex v is locally k-edge-connected if $\kappa'(G[N_G(v)]) \geq k$, and a connected graph G is **locally k-edge-connected** if every vertex v of G is locally k-edge-connected. In [14], Lei et al. showed that (Theorems 1.3 and 1.4 of [14]) if k, s and t are nonnegative integers such that $k \geq s+t+1$, then every locally k-edge-connected graph is (s,t)-supereulerian; and if $k = s+t$, then locally k-edge-connected graphs are (s,t)-supereulerian with well characterized exceptional situations.

Studies involving generic $(s,0)$-supereulerian graphs were considered much earlier. A best possible edge connectivity sufficient condition for $(s,0)$-supereulerian graphs was considered in Theorem 3.3 of [11]. This was later extended by Chen, Chen and Luo in [9] for (s,t)-supereulerianicity of graphs when the parameters s and t are in certain ranges. They proved that (Theorem 4.1 of [9]) for an integer $r \geq 1$, and for edge subsets X and Y of a graph G satisfying $X \cap Y = \emptyset$, $|Y| \leq \lfloor \frac{r+1}{2} \rfloor$, and $|X| + |Y| \leq r$, $G - Y$ has an eulerian subgraph H containing X if and only if $\kappa'(G) \geq r+1$. The problem whether all these conditions posed above are necessary is naturally coming up. Motivated by these prior results, in the current research we aim to find, for any given nonnegative integers s and t, the smallest positive integer N such that every graph G with edge-connectivity at least N must be (s,t)-supereulerian. We have the following necessary conditions.

Proposition 1.1. *Let s,t be nonnegative integers. If a graph G is (s,t)-supereulerian, then*

$$\kappa'(G) \geq \begin{cases} s+t+\frac{1-(-1)^s}{2} & \text{if } s \geq 1 \text{ and } s+t \geq 3, \\ t+2 & \text{otherwise.} \end{cases}$$

Proof. Let G be a (s,t)-supereulerian graph and $W \subseteq E(G)$ be a minimum edge-cut of G. Take a subset $Y \subseteq W$ with $|Y| = \min\{t, |W|\}$. Since G is (s,t)-supereulerian, $G - Y$ contains a spanning eulerian subgraph, and so $\kappa'(G-Y) \geq 2$. Since W is an edge-cut of G, $W - Y$ is also an edge-cut of $G - Y$. Hence $|W - Y| \geq \kappa'(G-Y) \geq 2$, and so $|Y| = t$. Thus $\kappa'(G) = |W| = |Y| + |W - Y| \geq t+2$.

Assume further that $s \geq 1$ and $s+t \geq 3$. Then $s + \frac{1-(-1)^s}{2} \geq 2$, and so $s+t+\frac{1-(-1)^s}{2} \geq t+2$. Suppose that $|W| < s+t+\frac{1-(-1)^s}{2}$. As $s \geq 1$, there exists a subset $X \subseteq W$ satisfying

$$1 \leq |X| \leq s, |W - X| \leq t, \text{ and } |X| \equiv 1 \pmod 2.$$

Set $Y = W - X$. Since G is (s,t)-supereulerian, $G - Y$ has a spanning eulerian subgraph H with $X \subseteq E(H)$. Since W is an edge-cut of G and $X = W - Y$, X is an edge-cut of $G - Y$. Since $X \subseteq E(H)$ and H is spanning subgraph of $G - Y$, X is also an edge-cut of H. As H is eulerian, every edge-cut of H must have an even size, contrary to the fact that $|X|$ is odd. This contradiction shows that we must have $\kappa'(G) = |W| \geq s+t+\frac{1-(-1)^s}{2}$. ■

For given nonnegative integers s,t, let $j(s,t)$ denote the smallest integer such that every graph G with $\kappa'(G) \geq j(s,t)$ is (s,t)-supereulerian. One of our goals is to determine the value of $j(s,t)$.

Theorem 1.2. *Let s,t be nonnegative integers. Then*

$$j(s,t) = \begin{cases} \max\{4, t+2\} & \text{if } 0 \leq s \leq 1, \text{ or } (s,t) \in \{(2,0),(2,1),(3,0)\}, \\ 5 & \text{if } (s,t) \in \{(2,2),(3,1)\}, \\ s+t+\frac{1-(-1)^s}{2} & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases} \tag{1}$$

While Theorem 1.2 presents an extremal edge connectivity sufficient condition for (s,t)-supereulerian graphs, it is natural to investigate when this sufficient condition is also necessary. As an application of Theorem 1.2, we obtain a characterization of (s,t)-supereulerian graphs when $t \geq 3$, and its corollary on the complexity of the (s,t)-supereulerian problem.

Theorem 1.3. *Let s,t be integers with $s \geq 0$ and $t \geq 3$.*
(i) Then a graph G is (s,t)-supereulerian if and only if $\kappa'(G) \geq j(s,t)$.
(ii) If $t \geq 3$, then whether a graph is (s,t)-supereulerian can be polynomially determined.

In the next section, we present the needed tools in our arguments. The main results will be justified in Section 3.

2. Mechanisms

We write $H \subseteq G$ to mean that H is a subgraph of G. If X, Y are vertex subsets of $V(G)$, then define $E_G[X,Y] = \{xy \in E(G) : x \in X, y \in Y\}$ and $\partial_G(X) = E_G[X, V(G) - X]$. If $X = \{v\}$, then we often use $\partial_G(v)$ for $\partial_G(X)$. If $X \subseteq E(G)$ is an edge subset, then the **contraction** G/X is obtained by identifying the two ends of each edge in X and then deleting all the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G and v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H.

For an integer $i \geq 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $O(G) = \cup_{j\geq 0} D_{2j+1}(G)$ be the set of all odd degree vertices of G. A graph G is **collapsible** if for any subset R of $V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H with $O(H) = R$. Collapsible graphs are introduced by Catlin in [3] in a wording that is equivalent to the definition above (see also Proposition 1 of [12]). Catlin ([4]) indicated that collapsible graphs and its associate reduction method are very useful in the study of eulerian subgraphs. As when $R = \emptyset$, a spanning connected subgraph H with $O(H) = R$ is a spanning eulerian subgraph of G, and so collapsible graphs are supereulerian graphs. Let $H_1, H_2, \ldots, H_c$ denote the list of all maximal collapsible subgraphs. The graph $G' = G/(\cup_{i=1}^{c} H_i)$ is the **reduction** of G. A graph equaling its own reduction is a **reduced** graph. Theorem 2.1 below presents useful properties related to collapsible graphs, which will be deployed in our arguments.

Theorem 2.1. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*
(i) (Catlin, Theorem 3 of [3]) G is collapsible (or supereulerian, respectively) if and only if G/H is collapsible (or supereulerian, respectively). In particular, G is collapsible if and only if the reduction of G is K_1.
(ii) (Catlin, Theorem 5 of [3]) A graph is reduced if and only if it does not have a nontrivial collapsible subgraph.

For a graph G, let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in G, and $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning trees. Thus $\tau(G) \geq 2$ if and only if $F(G) = 0$.

Theorem 2.2. *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Theorem 7 of [3]) If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$. In particular, every graph G with $\tau(G) \geq 2$ is collapsible.
(ii) (Catlin et al., Theorem 1.3 of [6]) If $F(G) \leq 2$, then either G is collapsible or its reduction is a member in $\{K_2, K_{2,t} : t \geq 1\}$.

Utilizing the well-known spanning tree packing theorem of Nash-Williams [16] and Tutte [20], the following is obtained.

Theorem 2.3 *(Theorems 1.1 and 1.3 of [5]). Let G be a graph, $\epsilon \in \{0, 1\}$ and $\ell \geq 1$ be integers. The following are equivalent.*
(i) G is $(2\ell + \epsilon)$-edge-connected.
(ii) For any $X \subseteq E(G)$ with $|X| \leq \ell + \epsilon$, $\tau(G - X) \geq \ell$.

Theorem 2.3 has a seemingly more general corollary, as stated below.

Corollary 2.4. *Let G be a connected graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0, 1\}$, $\ell \geq 2$ and $2 \leq k \leq \ell$. The following are equivalent.*
(i) $\kappa'(G) \geqslant 2\ell + \epsilon$.
(ii) For any $X \subseteq E(G)$ with $|X| \leq 2\ell - k + \epsilon$, $\tau(G - X) \geq k$.

Proof. To show (i) implies (ii), we pick a subset $X \subseteq E(G)$ with $|X| \leq 2\ell - k + \epsilon$. Choose $X_1 \subseteq X$ with $|X_1| = \min\{\ell + \epsilon, |X|\}$. By (i) and by Theorem 2.3, $\tau(G - X_1) \geq \ell$. Let $X_2 = X - X_1$. Then $|X_2| \leq |X| - |X_1| \leq \ell - k$. Thus among the ℓ edge-disjoint spanning trees of $G - X_1$, at least k of them are edge-disjoint from X_2, and so $\tau(G - X) \geq k$. Conversely, we observe that Corollary 2.4(ii) implies Theorem 2.3(ii). Hence by Theorem 2.3, $\kappa'(G) \geq 2\ell + \epsilon$. ■

One application of Corollary 2.4 is to extend Theorem 1.5 of [10] to the form expressed in Theorem 2.5 below.

Theorem 2.5 *(Gu et al., [10]). Let $m \geq 4$ be an integer, let G be an m-edge-connected graph and let $X \subset E(G)$ be an edge subset with $|X| \leq m - 1$. Then $G - X$ is collapsible if and only if $\kappa'(G - X) \geq 2$.*

Proof. Suppose that $\kappa'(G) \geq m$ and $X \subseteq E(G)$ is an edge subset with $|X| \leq m - 1$. As collapsible graphs must be 2-edge-connected, it suffices to assume that $\kappa'(G - X) \geq 2$ to show $G - X$ is collapsible. Let $X_1 \subseteq X$ be such that $|X_1| \leq m - 2$ and $|X - X_1| \leq 1$. By Corollary 2.4 with $k = 2$, $\tau(G - X_1) \geq 2$. As $|X - X_1| \leq 1$, we have $F(G - X) \leq 1$. By Theorem 2.2(i) and as $\kappa'(G - X) \geq 2$, $G - X$ is collapsible. ■

Before stating our corollary of Theorem 2.5, we need an additional tool. A **elementary subdivision** of a graph G at an edge $e = uv$ is an operation to obtain a new graph $G(e)$ from $G - e$ by adding a new vertex v_e and two new edges uv_e and $v_e v$. For a subset $X \subseteq E(G)$, we define $G(X)$ to be the graph obtained from G by elementarily subdividing every edge of X. By definitions, for a subset $X \subseteq E(G)$,

$$G \text{ has a spanning closed trail containing } X \text{ if and only if } G(X) \text{ is supereulerian.} \tag{2}$$

Corollary 2.6. *Let G be a connected graph with $\kappa'(G) \geq 4$, and let $X, Y \subseteq E(G)$ be disjoint edge subsets with $|Y| \leq 1$.*
(i) If $|X| = 2$, then $G - Y$ has a spanning closed trail that contains X.
(ii) If $|X| = 3$, then G has a spanning closed trail that contains X.
(iii) If $|X| = 3$ and $\kappa'(G) \geq 5$, then $G - Y$ has a spanning closed trail that contains X.

Proof. As $\kappa'(G) \geq 4$, we have $\kappa'(G - Y) \geq 3$. By Theorem 2.3, $\tau(G - Y) \geq 2$ and so $F((G - Y)(X)) \leq 2$. As $\kappa'(G - Y) \geq 3$, the only edge-cuts of size 2 in $(G - Y)(X)$ are those of the form $\partial_{(G-Y)(X)}(v_e)$, for some $e \in X$. It follows by Theorem 2.2(ii) that either $(G - Y)(X)$ is collapsible or the reduction of $(G - Y)(X)$ is a $K_{2,\ell}$, for some integer $\ell \geq 2$. In the latter, as $|X| = 2$ and $\kappa'(G - Y) \geq 3$, we must have $\ell = 2$, which implies that $G - Y$ has an edge-cut of size 2, contrary to the fact that $\kappa'(G - Y) \geq 3$. Hence $(G - Y)(X)$ must be collapsible, and so $(G - Y)(X)$ is supereulerian. This proves (i) by (2).

Now assume that $|X| = 3$. If $\kappa'(G - X) \geq 2$, then by Theorem 2.5, $G - X$ is collapsible. Let $R = O(G[X])$. Then $R \subseteq V(G - X)$ and $|R| \equiv 0 \pmod 2$. As $G - X$ is collapsible, $G - X$ has a spanning connected subgraph H with $O(H) = R$. It follows that $G[E(H) \cup X]$ is a spanning eulerian subgraph containing X. Hence we may assume that $\kappa'(G - X) = 1$, and so G has an edge-cut W with $|W| = 4$ and $X \subset W$. Let $W = \{e_1, e_2, e_3, e_4\}$ with $X = W - \{e_4\}$. By Theorem 2.3, $\tau(G - \{e_3, e_4\}) \geq 2$, and so $F((G - \{e_3, e_4\})(e_1, e_2)) \leq 2$. By definition and as $F((G - \{e_3, e_4\})(\{e_1, e_2\})) \leq 2$, it follows that $F(G(W)) \leq 2$. By Theorem 2.1(ii), either $G(W)$ is collapsible, or for some integer $\ell \geq 2$ the reduction of $G(W)$ is a $K_{2,\ell}$. As $\kappa'(G) \geq 4$, the only edge-cut of size 2 in $G(W)$ are those of the form $\partial_{G(W)}(v_{e_i})$, with $1 \leq i \leq 4$. Thus again by $\kappa'(G) \geq 4$, if the reduction of $G(W)$ is a $K_{2,\ell}$, then $\ell = 4$. Hence in any case, the reduction of $G(W)$ is always supereulerian. By Theorem 2.1(i), $G(W)$ is supereulerian, and so by (2), G has a spanning eulerian subgraph containing X. This proves (ii).

To prove (iii), we assume that $\kappa'(G) \geq 5$ and $t = 1$. Then we have $\kappa'(G - Y) \geq 4$, and by Corollary 2.6(ii), $G - Y$ has a spanning closed trail that contains X. ■

3. Proofs of the main results

We first show that Theorem 1.3 follows from Theorem 1.2. When $t \geq 3$, Theorem 1.3(i) indicates that determining if a graph G is (s, t)-supereulerian amounts to determining the edge-connectivity of G. It is well-known (for example, Section 7.3 of [2]) that the edge-connectivity can be determined by using an integral maximum flow algorithm, which is known to be a polynomial algorithm. Hence Theorem 1.3(ii) follows from Theorem 1.3(i).

We assume the validity of Theorems 1.2 to prove Theorems 1.3(i). Suppose that $t \geq 3$. By the definition of $j(s, t)$, any graph G with $\kappa'(G) \geq j(s, t)$ is (s, t)-supereulerian. Conversely, we assume that G is (s, t)-supereulerian. If $0 \leq s \leq 1$, then by Proposition 1.1, $\kappa'(G) \geq t + 2 \geq 4$, and so by Theorem 1.2, $\kappa'(G) \geq j(s, t)$. Assume that $s \geq 2$. Since $t \geq 3$, we have $s + t \geq 5$, and so by Proposition 1.1 and Theorem 1.2, we have

$$\kappa'(G) \geq s + t + \frac{1 - (-1)^s}{2} = j(s, t).$$

This proves Theorems 1.3(i). Thus to prove Theorems 1.2 and 1.3, it suffices to justify Theorem 1.2.

Throughout the rest of this paper, let s and t be nonnegative integers. We start with some examples.

Example 3.1. Let G_1, G_2 be disjoint graphs satisfying $\kappa'(G_1) \geq 3$ and $\kappa'(G_2) \geq 3$, and let $v_1 \in D_3(G_1)$ with $N_{G_1}(v_1) = \{x_1, x_2, x_3\}$ and $v_2 \in D_3(G_2)$ with $N_{G_2}(v_2) = \{y_1, y_2, y_3\}$. Define a new graph $G_1 \circ G_2$ from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding three new edges x_1y_1, x_2y_2, x_3y_3.
(i) If G_1 is not supereulerian, then $G_1 \circ G_2$ is not supereulerian. Moreover, if both $\kappa'(G_1) \geq 3$ and $\kappa'(G_2) \geq 3$, then $\kappa'(G_1 \circ G_2) \geq 3$.
(ii) Let G_1 be a 3-edge-connected nonsupereulerian graph, (for example, we can choose G_1 to be the Petersen graph), and let G_2 be any 3-edge-connected graph with a vertex of degree 3. Then $G_1 \circ G_2$ is also a 3-edge-connected nonsupereulerian graph.
(iii) For any integers $s \geq 0$ and $t \geq 0$, $j(s, t) \geq 4$.

Catlin [3] observed that any contraction of a supereulerian graph is supereulerian (for example, Lemma 3 of [3] with $S = O(G)$). As $(G_1 \circ G_2)/G_2 = G_1$ is not supereulerian, it follows that $G_1 \circ G_2$ is not supereulerian. The conclusion on the edge-connectivity of $G_1 \circ G_2$ follows from the fact that any minimum edge-cut of $G_1 \circ G_2$ corresponds to an edge-cut of G_1 or G_2, and so $\kappa'(G_1 \circ G_2) \geq 3$. Hence Example 3.1(i) can be observed. Example 3.1(ii), an immediate consequence of Example 3.1(i), suggests that there exist infinitely many 3-edge-connected nonsupereulerian graphs, and so for any values of s and t, we must have $j(s, t) \geq 4$.

Example 3.2. Let $n \geq 3$ be an integer, $\mathbb{Z}_n$ be the cyclic group of order n, and $\{J_i : i \in \mathbb{Z}_n\}$ be a collection of mutually disjoint 4-edge-connected graphs. Obtain a graph $C(J_1, \ldots, J_n)$ from the disjoint union of $J_1, J_2, \ldots, J_n$ by adding these new edges $E' = \{x_ix_{i+1}, y_iy_{i+1} : x_i, y_i \in V(J_i), x_{i+1}, y_{i+1} \in V(J_{i+1})$ and $i \in \mathbb{Z}_n\}$. Each of the following holds.
(i) $\kappa'(C(J_1, \ldots, J_n)) = 4$.
(ii) $C(J_1, \ldots, J_n)$ is not $(2, 2)$-supereulerian.
(iii) $j(2, 2) \geq 5$.

We have one observation of Example 3.2. Example 3.2(i) follows from the fact that each J_i is 4-edge-connected, and the construction of $C(J_1, \ldots, J_n)$. Let $G = C(J_1, \ldots, J_n)$ and choose $X = \{x_1x_2, y_1y_2\}$ and $Y = \{x_2x_3, y_3y_4\}$, where the subscripts are taken in $\mathbb{Z}_n$. Then in $G - Y$, each of $\{x_1x_2, y_1y_2, y_2y_3\}$ and $\{x_1x_2, y_1y_2, x_3x_4\}$ is an edge-cut of $G - Y$. If G has a spanning closed trail H that contains $\{x_1x_2, y_1y_2\}$, then as $E(H)$ intersecting any edge-cut of $G - Y$ must be an even size set, we conclude that $y_2y_3, x_3x_4 \notin E(H)$, and so H cannot be spanning and connected, a contradiction. This justifies Example 3.2(ii), which, by the definition of $j(s,t)$, implies Example 3.2(iii).

We shall determine the value of $j(s,t)$ according to the different ranges from which of s and t take their values. By Example 3.1(iii), we always have $j(s,t) \geq 4$ in the rest of the discussions.

Case 1. Either $0 \leq s \leq 1$ or $(s,t) \in \{(2,0), (2,1), (3,0)\}$.

Let G be a graph with $\kappa'(G) = j(s,t)$. By Proposition 1.1, $j(s,t) = \kappa'(G) \geq t+2$. By Example 3.1(iii), $j(s,t) \geq 4$. Hence, $j(s,t) \geq \max\{4, t+2\}$.

Suppose that $(s,t) \in \{(2,0), (2,1), (3,0)\}$. By Corollary 2.6, we always have $j(s,t) \leq 4$. Hence in this case, $j(s,t) = 4 = \max\{4, t+2\}$. Now assume that $0 \leq s \leq 1$. To establish $j(s,t) \leq \max\{4, t+2\}$, we shall assume that $m = \max\{4, t+2\}$ and G is a graph with $\kappa'(G) \geq m$ to show that G is (s,t)-supereulerian.

Let $Y \subseteq E(G)$ be an arbitrarily edge subset with $|Y| \leq t$ and let $X \subseteq E(G-Y)$ with $|X| = s$. If $t \leq 1$, then $m = 4$, and so by Corollary 2.6(i), G is (s,t)-supereulerian. Hence we assume that $m = t+2 \geq 4$. As $|Y| \leq t = m-2$, it follows by Corollary 2.4 with $k = 2$ that $\tau(G-Y) \geq 2$, and so as $|X| \leq 1$, we conclude that both $F((G-Y)(X)) \leq 1$ and $\kappa'(G-Y)(X)) \geq 2$. By Theorem 2.2(i) that $(G-Y)(X)$ is collapsible, and so supereulerian. Hence $G-Y$ has a spanning closed trail containing all edges in X. Therefore in this case, we always have $j(s,t) = \max\{4, t+2\}$.

Case 2. $(s,t) \in \{(2,2), (3,1)\}$.

By Example 3.2(iii), $j(2,2) \geq 5$; by Proposition 1.1, $j(3,1) \geq 5$. It remains to show that $j(2,2) \leq 5$ and $j(3,1) \leq 5$. Let G be a graph with $\kappa'(G) \geq 5$, and let X, Y be disjoint edge subsets of G with $|X| \leq s$ and $|Y| \leq t$. If $s = 3$ and $t = 1$, then by Corollary 2.6(iii), $(G-Y)(X)$ is supereulerian, and so $j(3,1) \leq 5$.

Hence we may assume that $s = t = 2$. Denote $X = \{e_1, e_2\}$. We shall show that $(G-Y)(X)$ has a spanning eulerian subgraph. By Corollary 2.4, $\tau(G-Y) \geq 2$ and $\kappa'(G-Y) \geq 3$. As $|X| = 2$, we have $F((G-Y)(X)) \leq 2$, and any 2-edge-cut of $(G-Y)(X)$ must be either $\partial_{(G-Y)(X)}(v_{e_1})$ or $\partial_{(G-Y)(X)}(v_{e_1})$. It follows by Theorem 2.1(ii) that either $(G-Y)(X)$ is collapsible, or the reduction of $(G-Y)(X)$ is a $K_{2,2}$. In either case, by Theorem 2.1(i), $(G-Y)(X)$ is supereulerian. Hence we have $j(2,2) \leq 5$. This completes the proof for this case.

Case 3. $s \geq 2$ and $s+t \geq 5$.

Let $m = s+t+\frac{1-(-1)^s}{2}$. Then $m \geq 5$. In this case, we are to show $j(s,t) = m$. By Proposition 1.1, $j(s,t) \geq \kappa'(G) \geq m$. To compete the proof, we need to show $j(s,t) \leq m$.

We argue by contradiction and assume that there exists a graph G with $\kappa'(G) \geq m$, but G is not (s,t)-supereulerian. Hence there exist edge subsets $X, Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|X| \leq s$ and $|Y| \leq t$, such that

$$G - Y \text{ does not have an eulerian subgraph containing all edges in } X. \tag{3}$$

By adding edges to X, we may assume that $|X| = s$. Let $X = \{e_1, e_2, \ldots, e_s\}$. Then $|(X - \{e_1, e_2\}) \cup Y| \leq s+t-2 \leq m-2$. By Corollary 2.4 with $k = 2$, $\tau(G - ((X - \{e_1, e_2\}) \cup Y)) \geq 2$. Let $H = (G - ((X - \{e_1, e_2\}) \cup Y))(e_1, e_2)$. Then both $F(H) \leq 2$ and $\kappa'(H) \geq 2$. Let H' denote the reduction of H. By Theorem 2.2(ii), either H is collapsible, or H' is a $K_{2,\ell}$ for some integer $\ell \geq 2$.

Assume first that H is collapsible. By their definitions, H is a subgraph of $(G-Y)(X)$ and $(G-Y)(X)/H$ is a graph containing the vertices v_H (the contraction image of H), and $v_{e_3}, v_{e_4}, \ldots, v_{e_s}$, satisfying $|E_{(G-Y)(X)/H}[\{v_H\}, \{v_{e_i}\}]| = 2$, for any i with $3 \leq i \leq s$. It follows that $\tau((G-Y)(X)/H) \geq 2$, and so $(G-Y)(X)/H$ is collapsible. As H is collapsible, by Theorem 2.1(i), $(G-Y)(X)$ is also collapsible, and so supereulerian. Thus $G-Y$ has a spanning closed trail that contains every edge in X, contrary to (3). Hence we assume that

$$H' \text{ is isomorphic to a } K_{2,\ell} \text{ for some integer } \ell \geq 2. \tag{4}$$

We have the following claim.

Claim 1. *Let $\Gamma = G - ((X - \{e_1, e_2\}) \cup Y)$, $H = \Gamma(e_1, e_2)$ and H' be the reduction of H satisfying (4). Each of the following holds.*
(i) $\tau(\Gamma) \geq 2$.
(ii) Each of v_{e_1} and v_{e_2} is not lying in a nontrivial collapsible subgraph of H. Consequently, v_{e_1} and v_{e_2} are nonadjacent vertices in $D_2(H')$.
(iii) If $W' \subseteq E(H')$ is an edge-cut of H' with $|W'| = 2$ such that $W' \neq \partial_{H'}(v_{e_i})$ for $i \in \{1, 2\}$, then there exists a minimum edge-cut

W of G such that $(X - \{e_1, e_2\}) \cup Y \cup W' = W$.
(iv) There exist two vertices u_1, u_2 *in* $D_\ell(H')$, *each of which is adjacent to both* v_{e_1} *and* v_{e_2} *in* H', *and let* H_{u_1} *and* H_{u_2} *be maximal collapsible subgraphs of H that are contracted to* u_1 *and* u_2, *respectively.*
(v) $\ell = 2$.
(vi) $X \cup Y$ *is an edge-cut of G with* $|X| = s$ *and* $|Y| = t$, *such that* $G - (X \cup Y)$ *has two components* H_{u_1} *and* H_{u_2}, *and* $s \equiv 0$ *(mod 2).*

To justify this claim, we first observe that (i) follows from Corollary 2.4 with $k = 2$ and the assumption that $m \geq 5$. To prove (ii), we argue by contradiction and assume that H has a maximal nontrivial collapsible subgraph N that contains v_{e_1} or v_{e_2} or both. For $i \in \{1, 2\}$, since $\kappa'(N) \geq 2$ and $v_{e_i} \in D_2(H)$, if $v_{e_i} \in V(N)$, then $\partial_H(v_{e_i}) \subseteq E(N)$.

Assume that N contains v_{e_1} but not v_{e_2}. Pick an edge $e_1' \in \partial_H(v_{e_1})$. Then $N' = N/\{e_1'\}$ can be viewed as a subgraph of Γ. By Claim 1(i), we also have $\tau(\Gamma/N') \geq 2$. It follows that $F((\Gamma/N')(e_2)) \leq 1$ and $\kappa'((\Gamma/N')(e_2)) \geq 2$. By Theorem 2.2(i), $H/N = \Gamma(e_1, e_2)/N = (\Gamma/N')(e_2)$ is collapsible. By Theorem 2.1(i), H is collapsible, and so supereulerian. This implies that $G - Y$ has a spanning closed trail that contains X, contrary to (3). The proof for the cases when H has a maximal nontrivial collapsible subgraph that contains v_{e_2} or both v_{e_1} and v_{e_2} are similar and omitted. This proves Claim 1(ii).

To show (iii), let $W' \subseteq E(H')$ be an edge-cut of H' with $|W'| = 2$ such that $W' \neq \partial_{H'}(v_{e_i})$ for $i \in \{1, 2\}$. Then as W' is an edge-cut of H', there must be a proper nonempty subset $Z' \subset V(H')$ such that $W' = \partial_{H'}(Z')$. Let $Z \subset V(G)$ be a vertex subset consisting of all vertices of the preimages of the vertices in Z'. Then Z is a proper nonempty subset of $V(G)$. Let $W = \partial_G(Z)$. Since W' is an edge-cut of H' different from $\partial_{H'}(v_{e_i})$ for $i \in \{1, 2\}$, we have $|W' \cap \partial_{H'}(v_{e_i})| \leq 1$. To simplify our notation, we take the following convention: if for some $i \in \{1, 2\}$, $W' \cap \partial_{H'}(v_{e_i}) = e_i'$, then we view this e_i' in W' as e_i in W. With this notation, $W' \subset W$. Since H' is the reduction of $H = (G - ((X - \{e_1, e_2\}) \cup Y)(e_1, e_2)$, and since W' is an edge-cut of H' different from $\partial_{H'}(v_{e_i})$ for $i \in \{1, 2\}$, it follows that $W = W' \cup (W \cap (\{e_1, e_2\} \cup Y))$. Hence

$$\begin{aligned}|W| \geq \kappa'(G) \geq m = s + t + \frac{1-(-1)^s}{2} &= |(X - \{e_1, e_2\})| + 2 + |Y| + \frac{1-(-1)^s}{2} \\ &\geq |(X - \{e_1, e_2\})| + |Y| + |W'| \geq |W' \cup (W \cap (\{e_1, e_2\} \cup Y))| = |W|.\end{aligned}$$

Hence Claim 1(iii) must hold.

Claim 1(iv) is a direct consequence of Claim 1(ii), as $v_{e_1}, v_{e_2} \in D_2(H')$ and $H' \cong K_{2,\ell}$.

We argue by contradiction to justify (v) and assume that $\ell \geq 3$. Fix an arbitrary vertex $v \in D_2(H') - \{v_{e_1}, v_{e_2}\}$. Let H_v be the contraction preimage of v in G. Then $|\partial_{H'}(v)| = 2$. By Claim 1(iii), with $W = (X - \{v_{e_1}, v_{e_2}\}) \cup Y \cup \partial_{H'}(v)$. If there exists a vertex $v' \in D_2(H') - \{v, v_{e_1}, v_{e_2}\}$ then by the same token, $W = (X - \{v_{e_1}, v_{e_2}\}) \cup Y \cup \partial_{H'}(v')$, which forces that $\partial_{H'}(v) = \partial_{H'}(v')$. On the other hand, as $D_2(H')$ is a stable set in H' and $v \neq v'$, we have $\partial_{H'}(v) = \partial_{H'}(v') = \partial_{H'}(v) \cap \partial_{H'}(v') = \emptyset$, contrary to the fact that $|\partial_{H'}(v)| = |\partial_{H'}(v')| = 2$. Therefore we must have $\ell = 3$.

Using the notation in Claim 1(iv), let $W_i = E_G[V(H_v), V(H_{u_i})]$, $1 \leq i \leq 2$. As $v \in D_2(H') - \{v_{e_1}, v_{e_2}\}$, we have $W = \partial_G(V(H_v)) = W_1 \cup W_2$. By Claim 1(iii), W is a minimum edge cut of G, and so $|W_1| + |W_2| = |W| = m$. By symmetry, we may assume that $|W_1| \leq \frac{m}{2}$. It follows that $\partial_G(V(H_{u_1})) = W_1 \cup \{e_1, e_2\}$ and so $m \leq \kappa'(G) \leq |\partial_G(V(H_{u_1}))| \leq \frac{m}{2} + 2$, leading to the contradiction of $5 \leq m \leq 4$. This contradiction indicates that we must have $\ell = 2$, verifying Claim 1(v).

As $\ell = 2$, the edge subset $X_1 = \{e_1, e_2\}$ is an edge-cut of $G - ((X - X_1) \cup Y)$, and so $G - (X \cup Y)$ has two components H_{u_1} and H_{u_2} with $E_G[V(H_{u_1}), V(H_{u_2})] = X \cup Y$. This implies that $s + t + \frac{1-(-1)^s}{2} = m \leq \kappa'(G) \leq |X| + |Y| \leq s + t$, and so $|X| = s$, $|Y| = t$ and $s \equiv 0 \pmod 2$. This completes the proof of Claim 1(vi), as well as the claim.

By Claim 1(vi), we have $X \subseteq E_G[V(H_{u_1}), V(H_{u_2})]$, and $|X| = s \equiv 0 \pmod 2$. It follows that $((G - Y)(X))/(H_{u_1} \cup H_{u_2}) \cong K_{2,s}$ is an eulerian graph. By Claim 1(iv), each of H_{u_1} and H_{u_2} is collapsible, and so by applying Theorem 2.1(i), we conclude that $(G - Y)(X)$ is supereulerian. This implies that $G - Y$ has a spanning closed trail that contains X, contrary to (3). This proves that in Case 3, we must have $j(s, t) \leq m$. This completes the proof of the theorem. ■

We conclude this paper by the following remark. Pulleyblank proved that determining $(0, 0)$-supereulerianicity is NP complete. In this paper, we have shown that, for any integers s and t with $s \geq 0$ and $t \geq 3$, it is polynomial to decide if a graph G is (s, t)-supereulerian. Therefore, it is of interests to understand the computational complexity for (s, t)-supereulerianicity for other values of s and t. These are to be investigated.

Data availability statement

The research has no associate data.

Declaration of competing interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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