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Edge-disjoint spanning trees and forests of graphs

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ABSTRACT

The spanning tree packing number of a graph G is the maximum number of edge-disjoint spanning trees in G, and the arboricity of G is the minimum number of edge-disjoint forests needed to partition the edge set of G. In this paper, we give bounds on the spanning tree packing number and the arboricity of graphs in terms of effective resistances. As applications, we show that equiarboreal graphs are uniformly dense, and determine the maximum number of edge-disjoint spanning c-forests of equiarboreal graphs, including edge-transitive graphs, 1-walk-regular graphs, distance-regular graphs and so on. The arboricity of a regular graph can be derived from our effective resistance bounds directly.

1. Introduction

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. A spanning forest of G with exactly c components is called a *spanning c-forest* of G. Let $\sigma_c(G)$ denote the maximum number of edge-disjoint spanning c-forests in G. Clearly, $\sigma_1(G)$ is the maximum number of edge-disjoint spanning trees in G, which is also called the *spanning tree packing number* [19] of G. The spanning tree packing number of maximal planar graphs, complete graphs, complete bipartite graphs, quasi-random graphs and some cartesian products can be found in [14,19].

Let $\omega(G)$ denote the number of components of a graph G. In [17] and [21], Nash-Williams and Tutte proved the following result.

Theorem 1.1. *For any connected graph G and positive integer k, $\sigma_1(G) \geq k$ if and only if $|X| \geq k(\omega(G-X)-1)$ for each $X \subseteq E(G)$.*

The *c-order edge toughness* [5,6] of a graph G is defined as

$$\tau_c(G) = \min\left\{\frac{|X|}{\omega(G-X)-c} : X \subseteq E(G), \omega(G-X) > c\right\}.$$

By Theorem 1.1, we have $\sigma_1(G) = \lfloor \tau_1(G) \rfloor$. Theorem 1.1 can be extended as follows.

Theorem 1.2 ([5,6]). *The maximum number of edge-disjoint spanning c-forests of a connected graph G is $\sigma_c(G) = \lfloor \tau_c(G) \rfloor$.*

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The *arboricity* $a(G)$ of a graph G is the minimum number of edge-disjoint forests needed to partition the edge set of G. The *fractional arboricity* of G is defined as

$$\gamma(G) = \max\left\{\frac{|E(H)|}{|V(H)|-1}\right\},$$

where the maximum is taken over all nontrivial subgraphs of G. In [18], Nash-Williams gave the following fundamental theorem on $a(G)$.

Theorem 1.3. *The arboricity of a connected graph G is $a(G) = \lceil\gamma(G)\rceil$.*

A graph G is *uniformly dense* [5] if $\gamma(G) = \frac{|E(G)|}{|V(G)|-1}$, which is also called strongly balanced [20]. It is known [5] that G is uniformly dense if and only if $\tau_1(G) = \gamma(G)$. Uniformly dense graphs are useful in random graphs [20] and web networks [10].

For two vertices i, j in a connected graph G, the *resistance* between i and j, denoted by $r_{ij}(G)$, is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The resistance is a distance function on graphs [12], which is a useful tool to study random walk parameters of graphs [2,3,13,15]. There are fruitful methods for computing resistance in graphs, including algebraic formulas via generalized inverses of the Laplacian matrix [12,26], sum rules via the local structure [24], recursion formula via deleting edges [23] and so on.

A connected graph is called *equiarboreal* [8,25] if the number of spanning trees containing a given edge is independent of the choice of the edge. In [8], Godsil proved that any graph which is a colour class in an association scheme is equiarboreal. Many important graph classes in algebraic graph theory are equiarboreal graphs, including edge-transitive graphs, distance-regular graphs and 1-walk-regular graphs (see [8,25]).

The aim of this paper is to introduce an effective resistance method for studying the spanning tree packing number, edge toughness and (fractional) arboricity of graphs. The paper is organized as follows. In Section 2, we give some auxiliary lemmas. In Section 3, we use effective resistance to give bounds on the spanning tree packing number, edge toughness and the (fractional) arboricity of graphs. We use these bounds to show that equiarboreal graphs are uniformly dense, and determine the maximum number of edge-disjoint spanning c-forests of equiarboreal graphs. The arboricity of a regular graph can be derived from our resistance bounds directly. In Section 4, we list the maximum number of edge-disjoint spanning c-forests in several classes of equiarboreal graphs, including edge-transitive graphs, 1-walk-regular graphs, k-subdivision graphs and some double graphs.

2. Preliminaries

The following is an inequality which will be used in this paper.

Lemma 2.1 (*[16]*)**.** *If $q_1, \ldots, q_m$ are positive numbers, then*

$$\frac{p_1 + \cdots + p_m}{q_1 + \cdots + q_m} \le \max_{1\le i\le m} \frac{p_i}{q_i},$$

for any nonzero real numbers $p_1, \ldots, p_m$.

Lemma 2.2 (*[6]*)**.** *Let G be a connected graph with n vertices and m edges. For integer c satisfying $1 \le c \le n-1$, we have $\tau_c(G) = \frac{m}{n-c}$ if and only if*

$$\gamma(G) \le \frac{m}{n-c}.$$

A connected graph is called 2-*connected* if it has no cut vertices. A *block* in graph G is a maximal 2-connected induced subgraph or a bridge of G. The edge set of G can always be partitioned into the blocks of G.

Lemma 2.3. *Suppose that there are t blocks $H_1, \ldots, H_t$ in graph H. Then*

$$|V(H)| - 1 = \sum_{i=1}^{t}(|V(H_i)| - 1).$$

Proof. Let G be the bipartite graph with bipartition $V_1 \cup V_2$, where $V_1 = \{H_1, \ldots, H_t\}$, V_2 is the set of cut vertices of H, and $u \in V_2$ and H_i are adjacent in G if and only if u is contained in H_i in H. By computation, we have

$$\sum_{i=1}^{t}|V(H_i)| = |V(H)| - |V_2| + \sum_{u\in V_2} d_u(G) = |V(H)| - |V_2| + |E(G)|,$$

where $d_u(G)$ is the degree of vertex u in G. It is known [3] that G is a tree. Then $|E(G)| = |V_2| + t - 1$. Hence

$$\sum_{i=1}^{t} |V(H_i)| = |V(H)| - 1 + t,$$

$$\sum_{i=1}^{t} (|V(H_i)| - 1) = |V(H)| - 1. \quad \square$$

Lemma 2.4. *Let G be a connected graph with at least one cycle. Then G has a 2-connected induced subgraph H such that*

$$\gamma(G) = \frac{|E(H)|}{|V(H)| - 1}.$$

Proof. From the definition of $\gamma(G)$, there exists an induced subgraph H such that $\gamma(G) = \frac{|E(H)|}{|V(H)|-1}$. If H is not 2-connected, then we can assume that there are t blocks $H_1, \ldots, H_t$ in H. By Lemma 2.3, we have

$$\gamma(G) = \frac{|E(H)|}{|V(H)| - 1} = \frac{\sum_{i=1}^{t} |E(H_i)|}{\sum_{i=1}^{t} (|V(H_i)| - 1)}.$$

By Lemma 2.1, there is a block H_i satisfying

$$\gamma(G) = \frac{|E(H)|}{|V(H)| - 1} \leq \frac{|E(H_i)|}{|V(H_i)| - 1}.$$

It follows from the definition of $\gamma(G)$ that $\frac{|E(H_i)|}{|V(H_i)|-1} = \gamma(G)$. Since G has at least one cycle, we have

$$\frac{|E(H_i)|}{|V(H_i)| - 1} = \gamma(G) \geq \frac{|E(G)|}{|V(G)| - 1} > 1.$$

Hence the block H_i is not a bridge, and it is a 2-connected induced subgraph of G such that $\gamma(G) = \frac{|E(H_i)|}{|V(H_i)|-1}$. $\square$

For an edge $e = uv$ of a connected graph G, the resistance $r_{uv}(G)$ between two end-vertices of e is called the *edge resistance* of e in G, denoted by $r_e(G)$. The second part of the following lemma is known as the Foster's Theorem.

Lemma 2.5 (*[25]*)**.** *Let G be a connected graph with t spanning trees, and let t_e be the number of spanning trees containing an edge e of G. Then*

$$r_e(G) = \frac{t_e}{t}, \quad \sum_{e \in E(G)} r_e(G) = n - 1.$$

The following is a lower bound of edge resistances.

Lemma 2.6 (*[22]*)**.** *Let G be a connected graph, and let d_u denote the degree of a vertex u in G. For an edge $e = uv$ of G, we have*

$$r_e(G) \geq \frac{d_u + d_v - 2}{d_u d_v - 1}.$$

Equiarboreal graphs can be characterized by resistances as follows.

Lemma 2.7 (*[25]*)**.** *Let G be a connected graph with n vertices and m edges. Then G is equiarboreal if and only if $r_e(G) = \frac{n-1}{m}$ for each edge e of G.*

A *cycle cover* of G is a set $\mathfrak{C}$ of cycles of G such that each edge of G belongs to at least one cycle of $\mathfrak{C}$.

Lemma 2.8 (*[1]*)**.** *Every bridgeless graph has a cycle cover $\mathfrak{C}$ such that every edge appears in exactly 4 cycles of $\mathfrak{C}$.*

3. Main results

For a connected graph G with edge set $\{1, \ldots, m\}$, it has m edge resistances $r_1 \geq \cdots \geq r_m$. We use these edge resistances to give the following lower bounds for spanning tree packing number $\sigma_1(G)$ and edge toughness $\tau_1(G)$.

Theorem 3.1. *Let G be a connected graph with edge resistances $r_1 \geq \cdots \geq r_m$ $(m > 1)$. Then*

$$\sigma_1(G) \geq \min\left\{k : \sum_{i=1}^{k} r_i > 1\right\} - 1,$$
$$\lceil \tau_1(G) \rceil \geq \min\left\{k : \sum_{i=1}^{k} r_i \geq 1\right\},$$
$$\tau_1(G) \geq r_1^{-1}.$$

Proof. If $\sigma_1(G) < k$ for some positive integer k, then by Theorem 1.1, there exists $X \subseteq E(G)$ such that $|X| < k(\omega(G-X)-1)$. Since each spanning tree of G contains at least $\omega(G-X)-1$ edges in X, we have

$$\sum_{i\in X} t_i \geq (\omega(G-X)-1)t,$$

where t_i denotes the number of spanning trees containing the edge i, t is the number of spanning trees in G. By Lemma 2.5, we have

$$\sum_{i=1}^{|X|} r_i \geq \sum_{i\in X} r_i \geq \omega(G-X)-1.$$

By $|X| < k(\omega(G-X)-1)$, we get

$$(\omega(G-X)-1)\sum_{i=1}^{k} r_i > \sum_{i=1}^{|X|} r_i \geq \omega(G-X)-1,$$
$$\sum_{i=1}^{k} r_i > 1.$$

Hence $\sum_{i=1}^{k} r_i > 1$ if $\sigma_1(G) < k$ for some positive integer k, that is

$$\sigma_1(G) \geq \min\left\{k : \sum_{i=1}^{k} r_i > 1\right\} - 1.$$

Suppose that $Y \subseteq E(G)$ satisfies $\tau_1(G) = \frac{|Y|}{\omega(G-Y)-1}$. Since each spanning tree of G contains at least $\omega(G-Y)-1$ edges in Y, we have

$$(\omega(G-Y)-1)t \leq \sum_{i\in Y} t_i.$$

By Lemma 2.5, we have

$$\omega(G-Y)-1 \leq \sum_{i\in Y} r_i \leq \sum_{i=1}^{|Y|} r_i.$$

By $|Y| = (\omega(G-Y)-1)\tau_1(G)$, we have

$$\omega(G-Y)-1 \leq \sum_{i=1}^{|Y|} r_i \leq (\omega(G-Y)-1)\sum_{i=1}^{\lceil \tau_1(G) \rceil} r_i,$$
$$\omega(G-Y)-1 \leq \sum_{i=1}^{|Y|} r_i \leq (\omega(G-Y)-1)\tau_1(G) r_1.$$

Hence

$$\lceil \tau_1(G) \rceil \geq \min\left\{k : \sum_{i=1}^{k} r_i \geq 1\right\},$$
$$\tau_1(G) \geq r_1^{-1}. \quad \square$$

We next use edge resistances to give the following upper bounds for the fractional arboricity $\gamma(G)$. These bounds can be also regarded as resistance bounds for the arboricity $a(G)$ because $a(G) = \lceil \gamma(G) \rceil$.

Theorem 3.2. *Let G be a connected graph, and let d_u denote the degree of vertex u in G. Then*

$$\gamma(G) \le \frac{1}{\min_{e\in E(G)} r_e(G)} \le \max_{uv\in E(G)} \frac{d_u d_v - 1}{d_u + d_v - 2}.$$

If G is not a tree, then

$$\gamma(G) \le \max_{C\in\mathcal{C}(G)} \frac{|E(C)|}{\sum_{e\in E(C)} r_e(G)},$$

where $\mathcal{C}(G)$ is the set of cycles in G.

Proof. Suppose that H_0 is an induced subgraph of G such that

$$\frac{|E(H_0)|}{|V(H_0)|-1} = \gamma(G) > 0.$$

If H_0 is disconnected, then by Lemma 2.1, H_0 has a component F such that $\frac{|E(F)|}{|V(F)|-1} > \frac{|E(H_0)|}{|V(H_0)|-1} = \gamma(G)$, a contradiction. So H_0 is connected. By Lemmas 2.5 and 2.1, we have

$$\gamma(G) = \frac{|E(H_0)|}{|V(H_0)|-1} = \frac{|E(H_0)|}{\sum_{e\in E(H_0)} r_e(H_0)} \le \frac{1}{\min_{e\in E(H_0)} r_e(H_0)}.$$

By the Raleigh's Principle of electrical networks (see Corollary 4.3 in [15]), we have $r_e(H_0) \ge r_e(G)$ for each $e \in E(H_0)$. By Lemma 2.6, we get

$$\gamma(G) \le \frac{1}{\min_{e\in E(H_0)} r_e(H_0)} \le \frac{1}{\min_{e\in E(G)} r_e(G)} \le \max_{uv\in E(G)} \frac{d_u d_v - 1}{d_u + d_v - 2}.$$

If G is not a tree, then by Lemmas 2.4 and 2.5, G has a 2-connected induced subgraph H such that

$$\gamma(G) = \frac{|E(H)|}{|V(H)|-1} = \frac{|E(H)|}{\sum_{e\in E(H)} r_e(H)}.$$

By Lemma 2.8, H has a cycle cover $\mathfrak{C} = \{C_1, \ldots, C_t\}$ such that each edge of H is contained in exactly 4 cycles of $\mathfrak{C}$. Hence

$$\gamma(G) = \frac{4|E(H)|}{4\sum_{e\in E(H)} r_e(H)} = \frac{\sum_{i=1}^{t} |E(C_i)|}{\sum_{i=1}^{t}\sum_{e\in E(C_i)} r_e(H)}.$$

Suppose that

$$\frac{|E(C_1)|}{\sum_{e\in E(C_1)} r_e(H)} \ge \frac{|E(C_i)|}{\sum_{e\in E(C_i)} r_e(H)},\quad i = 1, \ldots, t.$$

Then by Lemma 2.1, we get

$$\begin{aligned}\gamma(G) &= \frac{\sum_{i=1}^{t} |E(C_i)|}{\sum_{i=1}^{t}\sum_{e\in E(C_i)} r_e(H)} \le \frac{|E(C_1)|}{\sum_{e\in E(C_1)} r_e(H)} \le \frac{|E(C_1)|}{\sum_{e\in E(C_1)} r_e(G)} \\ &\le \max_{C\in\mathcal{C}(G)} \frac{|E(C)|}{\sum_{e\in E(C)} r_e(G)}. \quad \square\end{aligned}$$

It is known that the arboricity of a d-regular graph is $\lfloor \frac{d}{2} \rfloor + 1$ (see Corollary 1.4 in [11]). We can derive this result from Theorem 3.2.

Corollary 3.3. *For any d-regular graph G, we have*

$$a(G) = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

Proof. If G is disconnected, then $a(G)$ is equal to the maximum arboricity of all components of G. So we only need to show $a(G) = \lfloor \frac{d}{2} \rfloor + 1$ when G is connected. By Theorem 3.2, we have

$$\frac{nd}{2(n-1)} = \frac{|E(G)|}{|V(G)|-1} \le \gamma(G) \le \frac{d^2-1}{2d-2} = \frac{d+1}{2},$$

where n is the number of vertices of G. Hence $a(G) = \lceil \gamma(G) \rceil = \lfloor \frac{d}{2} \rfloor + 1$. $\square$

By using Theorem 3.2, we show that equiarboreal graphs are uniformly dense, and determine the maximum number of edge-disjoint spanning c-forests of equiarboreal graphs as follows.

Theorem 3.4. *Let G be a equiarboreal graph with n vertices and m edges. Then G is uniformly dense and*

$$\left\lfloor \frac{m}{n-c} \right\rfloor = \sigma_c(G) \leq \tau_c(G) = \frac{m}{n-c}, \ c = 1, \ldots, n-1.$$

If $\frac{m}{n-c}$ is an integer, then the edge set of G can be partitioned into $\frac{m}{n-c}$ edge-disjoint spanning c-forests.

Proof. By Lemma 2.7, every edge of G has edge resistance $r_e(G) = \frac{n-1}{m}$. By Theorem 3.2, we have

$$\gamma(G) \leq \frac{1}{\min_{e\in E(G)} r_e(G)} = \frac{m}{n-1}.$$

Since $\gamma(G) \geq \frac{m}{n-1}$, we get

$$\gamma(G) = \frac{m}{n-1}.$$

Hence G is uniformly dense.

Since $\gamma(G) = \frac{m}{n-1} \leq \frac{m}{n-c}$ for any integer c satisfying $1 \leq c \leq n-1$, by Lemma 2.2 and Theorem 1.2, we have

$$\tau_c(G) = \frac{m}{n-c}, \ c = 1, \ldots, n-1,$$

$$\sigma_c(G) = \lfloor \tau_c(G) \rfloor = \left\lfloor \frac{m}{n-c} \right\rfloor, \ c = 1, \ldots, n-1.$$

Notice that each spanning c-forest in G contains $n-c$ edges of G. If $\frac{m}{n-c}$ is an integer, then $\sigma_c(G) = \frac{m}{n-c}$, that is, the edge set of G can be partitioned into $\frac{m}{n-c}$ edge-disjoint spanning c-forests. □

Remark 3.5. By Lemma 2.7 and Theorem 3.4, we know that the bounds in Theorems 3.1 and 3.2 are both attained for equiarboreal graphs.

4. Examples

In this section, we list the maximum number of spanning c-forests for several classes of equiarboreal graphs.

4.1. Edge-transitive graphs

A graph is *edge-transitive* if its automorphism group acts transitively on its edge set. We can obtain the following result from Theorem 3.4.

Proposition 4.1. *Let G be an edge-transitive connected graph with n vertices and m edges. Then G is uniformly dense and*

$$\left\lfloor \frac{m}{n-c} \right\rfloor = \sigma_c(G) \leq \tau_c(G) = \frac{m}{n-c}, \ c = 1, \ldots, n-1.$$

If $\frac{m}{n-c}$ is an integer, then the edge set of G can be partitioned into $\frac{m}{n-c}$ edge-disjoint spanning c-forests.

The complete graph K_n is edge-transitive. By Proposition 4.1, we can obtain the following result.

Example 4.2. Let K_n be the complete graph with n vertices. Then K_n is uniformly dense and

$$\left\lfloor \frac{n(n-1)}{2(n-c)} \right\rfloor = \sigma_c(K_n) \leq \tau_c(K_n) = \frac{n(n-1)}{2(n-c)}, \ c = 1, \ldots, n-1.$$

If $\frac{n(n-1)}{2(n-c)}$ is an integer, then the edge set of K_n can be partitioned into $\frac{n(n-1)}{2(n-c)}$ edge-disjoint spanning c-forests.

The complete bipartite graph K_{n_1,n_2} is another classical example of edge-transitive graphs. By Proposition 4.1, we can obtain the following result.

Example 4.3. Let K_{n_1,n_2} be the complete bipartite graph with n_1+n_2 vertices and n_1n_2 edges. Then K_{n_1,n_2} is uniformly dense and

$$\left\lfloor \frac{n_1n_2}{n_1+n_2-c} \right\rfloor = \sigma_c(K_{n_1,n_2}) \leq \tau_c(K_{n_1,n_2}) = \frac{n_1n_2}{n_1+n_2-c}, \ c = 1, \ldots, n_1+n_2-1.$$

If $\frac{n_1n_2}{n_1+n_2-c}$ is an integer, then the edge set of K_{n_1,n_2} can be partitioned into $\frac{n_1n_2}{n_1+n_2-c}$ edge-disjoint spanning c-forests.

More examples for edge-transitive graphs can be found in [9]. The maximum number of edge-disjoint spanning c-forests in these edge-transitive graphs can be obtained from Proposition 4.1.

4.2. 1-walk-regular graphs

Let G be a connected graph with adjacency matrix A and diameter D. It is known that $(A^k)_{uv}$ is equal to the number of walks of length k from vertex u to vertex v. For a given integer $m \leqslant D$, G is called *m-walk-regular* if the number $(A^k)_{uv}$ only depends on the distance h between u and v, provided that $h \leqslant m$ (see [7]). Actually, G is D-walk-regular if and only if it is distance-regular. An m-walk-regular graph is also called an m-homogeneous graph in [8].

Clearly, an m-walk-regular graph is also $(m-1)$-walk-regular, and 0-walk-regular graphs are exactly walk regular graphs. Godsil [8] proved that 1-walk-regular graphs are equiarboreal. Hence the following result follows from Theorem 3.4.

Proposition 4.4. *Let G be a 1-walk-regular graph with n vertices and degree $d < n-1$. Then G is uniformly dense and*

$$\left\lfloor \frac{nd}{2(n-c)} \right\rfloor = \sigma_c(G) \le \tau_c(G) = \frac{nd}{2(n-c)}, \quad c = 1, \dots, n-1.$$

If $\frac{nd}{2(n-c)}$ is an integer, then the edge set of G can be partitioned into $\frac{nd}{2(n-c)}$ edge-disjoint spanning c-forests.

Let Q be a set of size q. The Hamming graph [4] $H(d,q)$ is the graph with vertex set Q^d, where two vertices are adjacent when they agree in $d-1$ coordinates. In particular, $H(d,2)$ is the d-cube. Since $H(d,q)$ is distance-regular (see [4]), it is also 1-walk-regular. The following result follows from Proposition 4.4.

Example 4.5. The Hamming graph $H(d,q)$ is a distance-regular graph with q^d vertices and degree $(q-1)d$. Then $H(d,q)$ is uniformly dense and

$$\left\lfloor \frac{q^d(q-1)d}{2(q^d-c)} \right\rfloor = \sigma_c(H(d,q)) \le \tau_c(H(d,q)) = \frac{q^d(q-1)d}{2(q^d-c)}, \quad c = 1, \dots, q^d-1.$$

If $\frac{q^d(q-1)d}{2(q^d-c)}$ is an integer, then the edge set of $H(d,q)$ can be partitioned into $\frac{q^d(q-1)d}{2(q^d-c)}$ edge-disjoint spanning c-forests.

More examples for distance-regular graphs can be found in [4]. Examples for 1-walk-regular graphs which are not distance regular can be found in [7]. The maximum number of edge-disjoint spanning c-forests in these graphs can be obtained from Proposition 4.4.

4.3. Subdivision graphs

For a graph G, let $S_k(G)$ denote the *k-subdivision* graph obtained from G by replacing each edge $ij \in E(G)$ by a path $p_{ij} = ie_1 \cdots e_k j$ of length $k+1$ connecting i and j. It is known [25] that $S_k(G)$ is equiarboreal if and only if G is equiarboreal. So we can get the following result from Theorem 3.4.

Proposition 4.6. *Let G be a equiarboreal graph with n vertices and $m > n-1$ edges. Then $S_k(G)$ is uniformly dense and*

$$\left\lfloor \frac{(k+1)m}{km+n-c} \right\rfloor = \sigma_c(S_k(G)) \le \tau_c(S_k(G)) = \frac{(k+1)m}{km+n-c}, \quad c = 1, \dots, km+n-1.$$

If $\frac{(k+1)m}{km+n-c}$ is an integer, then the edge set of $S_k(G)$ can be partitioned into $\frac{(k+1)m}{km+n-c}$ edge-disjoint spanning c-forests.

4.4. Double graphs

Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$. Take another copy H of G with vertex set $V(H) = \{u_1, \dots, u_n\}$ such that $u_iu_j \in E(H)$ if and only if $v_iv_j \in E(G)$. The *double graph* of G, denoted by $D(G)$, is the graph with vertex set $V(D(G)) = V(G) \cup V(H)$ and edge set $E(D(G)) = E(G) \cup E(H) \cup \{v_iu_j : v_iv_j \in E(G)\}$. Naturally, iterated double graphs are defined as $D^0(G) = G$, $D^k(G) = D(D^{k-1}(G))$ $(k = 1, 2, \dots)$.

If G is a regular equiarboreal graph or edge-transitive, then $D^k(G)$ is equiarboreal for any positive integer k (see [25]). So we can get the following result from Theorem 3.4.

Proposition 4.7. *Let G be a connected graph with n vertices and m edges. If G is a regular equiarboreal graph or edge-transitive, then $D^k(G)$ is uniformly dense and*

$$\left\lfloor \frac{4^k m}{2^k n - c} \right\rfloor = \sigma_c(D^k(G)) \le \tau_c(D^k(G)) = \frac{4^k m}{2^k n - c}, \quad c = 1, \dots, 2^k n - 1.$$

If $\frac{4^k m}{2^k n-c}$ is an integer, then the edge set of $D^k(G)$ can be partitioned into $\frac{4^k m}{2^k n-c}$ edge-disjoint spanning c-forests.

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