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ORIGINAL PAPER

Strongly Spanning Trailable Graphs with Small Circumference and Hamilton-Connected Claw-Free Graphs

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Abstract

A graph G is *strongly spanning trailable* if for any $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G)$ (possibly $e_1 = e_2$), $G(e_1, e_2)$, which is obtained from G by replacing e_1 by a path $u_1v_{e_1}v_1$ and by replacing e_2 by a path $u_2v_{e_2}v_2$, has a spanning (v_{e_1}, v_{e_2})-trail. A graph G is *Hamilton-connected* if there is a spanning path between any two vertices of $V(G)$. In this paper, we first show that every 2-connected 3-edge-connected graph with circumference at most 8 is strongly spanning trailable with an exception of order 8. As applications, we prove that every 3-connected $\{K_{1,3}, N_{1,2,4}\}$-free graph is Hamilton-connected and every 3-connected $\{K_{1,3}, P_{10}\}$-free graph is Hamilton-connected with a well-defined exception. The last two results extend the results in Hu and Zhang (Graphs Comb 32: 685–705, 2016) and Bian et al. (Graphs Comb 30: 1099–1122, 2014) respectively.

Keywords Strongly spanning trailable · Hamilton-connected · Supereulerian · Collapsible · Reduction

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1 Introduction

For the notation or terminology not defined here, see [2]. A graph is called *trivial* if it has only one vertex, *non-trivial* otherwise. An *empty graph* is one in which no two vertices are adjacent. For a connected graph G, we use $\kappa(G)$, $\kappa'(G)$, $c(G)$ and $g(G)$ to denote the *connectivity*, *edge connectivity*, *circumference* and *girth* of G, respectively. Throughout this paper, we use P_n, C_n to denote a path or a cycle of order n. The graph $N_{i,j,k}$ is a triangle with disjoint paths of length i, j, k each attaching to distinct vertices of the triangle; H_i denotes the graph formed from two triangles, which are connected by a single path of length i. The graph $N_{i,j,k}$ is defined but we are defining $B_{i,j} = N_{i,j,0}$ and $Z_i = N_{i,0,0}$ here.

A graph G is **Hamilton-connected** if there is a spanning path between any pair vertices of $V(G)$. For a collection $\mathcal{H}$ of graphs, graph G is said to be $\mathcal{H}$**-free** if G does not contain H as an induced subgraph for all $H \in \mathcal{H}$ (see [11]). Any Hamilton-connected graph is 3-connected. Then it is natural to consider which forbidden pairs of graphs $\{R, S\}$ imply that a 3-connected $\{R, S\}$-free graph G is Hamilton-connected. Faudree and Gould in [10] showed that one of them must be $K_{1,3}$. We now list the known graphs S which, together with the $K_{1,3}$, imply that a 3-connected $\{K_{1,3}, S\}$-free graph is Hamilton-connected.

Theorem 1 *Let G be a 3-connected $\{K_{1,3}, S\}$-free graph satisfying one of the following:*

(1) (Shepherd [24]) $S \cong N_{1,1,1}$,
(2) (Faudree and Gould [10]) $S \cong Z_2$,
(3) (Chen and Gould [8]) $S \in \{B_{1,2}, Z_3, P_6\}$,
(4) (Faudree et al. [9]) $S \in \{N_{1,1,3}, N_{1,2,2}, P_8\}$,
(5) (Bian et al. [1]) $S \cong P_9$,
(6) (Hu and Zhang [12]) $S \cong N_{1,2,3}$,
(7) (Broersma et al. [3]) $S \cong H_1$.

Then G is Hamilton-connected.

Theorem 1 shows that the progress in forbidden pair guaranteeing a 3-connected graph to be Hamilton-connected is very slowly, although it is also popular. Motivated by the above results, we intend to extend Theorem 1(1)–(6).

The **line graph** of a given graph G, denoted by $L(G)$, is a graph with vertex set $E(G)$ such that two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are incident to a common vertex in G. Following [2], the Wagner graph, denoted by W_8, is obtained from the cycle C_8 by adding all four pairs of vertices of maximum distance in C_8 as four chords in C_8, and is depicted in Fig. 1. Now we define a set of graphs $\mathcal{G} = \{L(W) : W$ is obtained from W_8 by adding at least one pendant edge at each vertex of $W_8\}$.

Theorem 2 *Let G be a 3-connected graph. Then each of the following holds.*

(1) *If G is $\{K_{1,3}, P_{10}\}$-free, then G is Hamilton-connected or G is a spanning subgraph of a member in $\mathcal{G}$.*

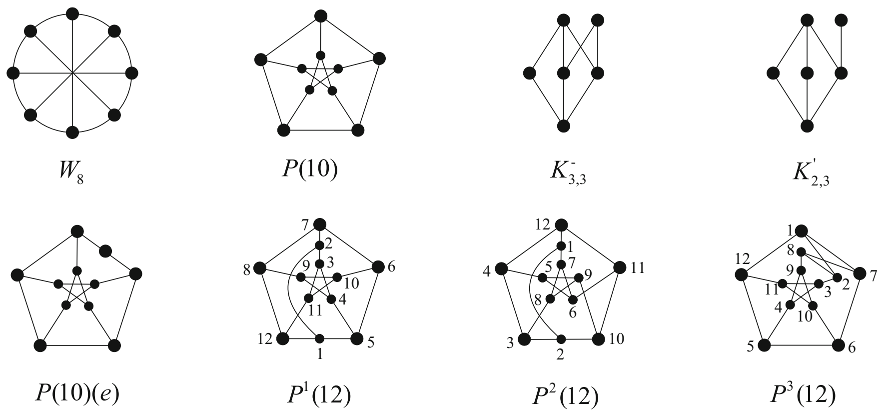

Fig. 1 Eight special graphs

(2) *If G is $\{K_{1,3}, N_{1,2,4}\}$-free, then G is Hamilton-connected.*

In fact, Faudree et al. [9] showed that if i, j, k are positive integers such that every 3-connected $\{K_{1,3}, N_{i,j,k}\}$-free graph is Hamilton-connected, then $i + j + k \leq 7$. Hence Theorem 2(2) is sharp.

We use (u, v)-trail, $P(u, v)$ to denote a trail and a path with u, v as end-vertices, respectively. A graph is called **supereulerian** if it contains a spanning Eulerian subgraph. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ denote two edges of G. If $e_1 \neq e_2$, then the graph $G(e_1, e_2)$ is obtained from G by replacing e_1 by a path $u_1v_{e_1}v_1$ and by replacing e_2 by a path $u_2v_{e_2}v_2$, where v_{e_1}, v_{e_2} are two new vertices not in $V(G)$. If $e_1 = e_2$, then the graph $G(e_1, e_2)$ is also denoted by $G(e)$ and is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_{e_1}v_1$. A graph G is **strongly spanning trailable** if for any $e_1, e_2 \in E(G)$, $G(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail. As $e_1 = e_2$ is possible, strongly spanning trailable graphs are supereulerian.

It is known [14, 21] that the line graph of a strongly spanning trailable graph is Hamilton-connected. In order to prove Theorem 2, we need the following associate result, which is itself interesting and shall have potential useful applications.

Theorem 3 *Every 2-connected 3-edge-connected graph G with $c(G) \leq 8$ other than W_8 is strongly spanning trailable.*

The proofs of Theorems 3 and 2 are placed in Sects. 3 and 4, respectively. In the rest of this section, we prepare some terminology and notation to be used in this article. For the notation or terminology not defined here, see [2]. The *degree* of a vertex u in a graph G, denoted by $d_G(u)$, is the number of edges of G incident with u, each loop counting as two edges. Call u a *k-vertex* if $d_G(u) = k$. Define $D_i(G) = \{u \in V(G) : d_G(u) = i\}$ and $D_{\geq i}(G) = \{u \in V(G) : d_G(u) \geq i\}$. We denote by $\Delta(G)$ and $\delta(G)$ the *maximum degree* and *minimum degree* of the vertices of G. For subsets $S \subseteq V(G)$ and $E \subseteq E(G)$, we denote by $G - S$ and $G - E$ the subgraphs of G induced by $V(G) \backslash S$ and $E(G) \backslash E$, respectively, define $N_G(S)$ to be the set of vertices in $V(G)\backslash S$ that are adjacent to a vertex in S and $N_G[S] = N_G(S) \cup S$. Define $E(u, S) = \{us : s \in S\}$. When $S = \{s\}$, $E = \{e\}$, we use $G - s$, $N_G(s)$, $N_G[s]$ and $G - e$ for $G - \{s\}$, $N_G(\{s\})$, $N_G[\{s\}]$ and $G - \{e\}$, respectively. We use $H \subseteq G$, $H \cong G$ to denote the fact that H is a subgraph of G, H and G are isomorphic. For any two sets S_1, S_2, define $S_1 \triangle S_2 = (S_1 \cup S_2)\backslash(S_1 \cap S_2)$.

2 Reductions and Reduced Graphs

In this section, we prepare some definitions and additional results and prove two theorems.

For a graph G and $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the edges in X. If $X = \{e\}$, then we use G/e for $G/\{e\}$. When H is a subgraph of G, then we use G/H for $G/E(H)$. If H is connected, then the vertex in G/H onto which H is contracted is denoted by v_H, and H is the **preimage** of v_H in G.

For a graph G, let $O(G)$ denote the set of odd degree vertices in G. In [5], Catlin defined collapsible graphs. A graph G is *collapsible* if for any even subset R of $V(G)$, G has a spanning connected subgraph Γ_R with $O(\Gamma) = R$. The **reduction** of G is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph.

Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Catlin (Theorem 2 of [6]) shows that a connected graph G is collapsible if $F(G) = 0$. Let $K_{m,n}$ be the complete bipartite graph with partition sets of size m and n. Fig. 1 depicts some of the related graphs in this paper, including the Petersen graph $P(10)$.

We summarize some results on Catlin's reduction method and other related facts below.

Theorem 4 *Let G be a connected graph, $H \subseteq G$ be a collapsible subgraph and G' be the reduction of G, respectively. Then each of the following holds.*

(1) (Catlin [5]) *G is collapsible if and only if G/H is collapsible. And G is collapsible if and only if G' is K_1.*
(2) (Catlin [5]) *G is reduced if and only if G has no non-trivial collapsible subgraphs.*
(3) (Catlin [5]) $g(G') \geq 4$ *and* $\delta(G') \leq 3$.
(4) (Catlin [6], *see also Theorem* 3.4 *of* [19]) $F(G') = 2|V(G')| - 2 - |E(G')|$.
(5) (Catlin et al. [7]) *If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some $t \geq 1$; if $F(G) \leq 2$ and $\kappa'(G) \geq 3$, then G is collapsible. Consequently, $K_{3,3}^-$ is collapsible.*
(6) (Lai et al. [15]) *If $\delta(G) \geq 3$ and $|V(G)| \leq 13$, then $G' \in \{K_1, K_2, K_{1,2}, K_{1,3}, P(10), P^1(12), P^2(12), P^3(12)\}$.*

For two disjoint subsets V_1, V_2 and a 4-cycle $C = x_1x_2x_3x_4x_1$ of graph G, define $G/\pi(V_1, V_2)$ to be the graph obtained from $G - E(G[V_1 \cup V_2])$ by identifying V_1 to form a vertex v_1, by identifying V_2 to form a vertex v_2, and by adding a new edge $e_\pi = v_1v_2$, and define $G/\pi(C) = G/\pi(\{x_1, x_3\}, \{x_2, x_4\})$.

Theorem 5 (*Catlin* [6]) *For the graphs G and* $G/\pi(C)$ *defined above, if* $G/\pi(C)$ *is collapsible, then G is collapsible.*

In [20], the authors gave a method to verify whether a subgraph of G is collapsible. They construct a C-subpartition (X_1, X_2) of G starting with a 4-cycle $x_1x_2x_3x_4x_1 \subseteq G$.

1. $X_1 := \{x_1, x_3\}$, $X_2 := \{x_2, x_4\}$, $\{i, j\} = \{1, 2\}$
2. **While** $u \in N_G(X_1 \cup X_2) \neq \emptyset$, $N_G(X_1) \cap N_G(X_2) = \emptyset$ and $N_G(u) \cap N_G[X_1 \cup X_2] \neq \emptyset$ **do**

$$\{X_i := X_i \cup \{u\}, X_j := X_j, if |E(u, X_i)| \geq 2; X_i := X_i \cup (N_G(X_i) \cap N_G[u]),$$
$$X_j := X_j, elseif N_G(X_i) \cap N_G[u] \neq \emptyset; X_i := X_i \cup (N_G(X_j) \cap N_G(u)),$$
$$X_j := X_j \cup \{u\}, else.\}$$

The following result would play an important role in the proofs in Sects. 2 and 3.

Lemma 1 (*Liu et al.* [20]) *Let G be a graph with* $g(G) = 4$ *and* (X_1, X_2) *be a C-subpartition of G. Then*

(1) $G[X_1 \cup X_2 \cup X_{12}]$ *is collapsible for any non-empty set* $X_{12} \subseteq N_G(X_1) \cap N_G(X_2)$,
(2) *if* $G/\pi(X_1, X_2)$ *is collapsible, then G is collapsible.*

An edge cut X is *essential* if $G - X$ has at least two non-trivial components. A graph G is *essentially k-edge-connected* if G does not have an essential edge cut X with $|X| < k$.

Theorem 6 (*Lai et al.* [16]) *Let G be a graph. If* $\kappa'(G) \geq 3$ *and* $c(G) \leq 8$, *then G is supereulerian.*

The following theorem extends Theorem 6.

Theorem 7 *Let G be an essentially* 3*-edge-connected graph such that* $\kappa'(G) \geq 2$, $c(G) \leq 8$ *and* $|D_2(G)| \leq 1$. *Then G is collapsible.*

Proof By contradiction, assume that G is a counter-example with $|V(G)|$ minimized. Then G is reduced; for otherwise, the reduction G' of G is a non-trivial counter-example with smaller order than G, a contradiction. By Theorem 4(2), G has no non-trivial collapsible subgraphs.

Besides, $\kappa(G) \geq 2$; for otherwise, each block of G is collapsible by the minimality of G if G has a cut-vertex, a contradiction.

We then claim that $g(G) = 4$. If not, then by Theorem 4(3), $g(G) \geq 5$. Take a longest path $P_0 = x_1x_2 \cdots x_l$ of G with $d_G(x_1) \geq d_G(x_l)$. Since $|D_2(G)| \leq 1$, $d_G(x_1) \geq 3$, and so x_1 has at least three neighbors in P_0. As $g(G) \geq 5$ and

$c(G) \le 8$, $\{x_1x_5, x_1x_8\} \subseteq E(G)$. Using the alternative longest path $x_4x_3x_2x_1x_5x_6 \cdots x_l$, we get $x_4x_8 \in E(G)$ by the same argument if $d_G(x_4) \ge 3$, yielding a $C_4 = x_1x_5x_4x_8x_1$. This means that $D_2(G) = \{x_4\}$. Using the alternative longest path $x_7x_6x_5x_4x_3x_2x_1x_8 \cdots x_l$, we get $x_7x_3 \in E(G)$. Since $g(G) \ge 5$ and $c(G) \le 8$, $E(x_6, V(P_0) \backslash \{x_5, x_7\}) = \emptyset$, and so x_6 has a neighbor x_6' outside P_0 such that $E(x_6', V(P_0) \backslash \{x_6\}) = \emptyset$. Therefore, there is a longer path $x_6''x_6'x_6x_7x_3x_4x_5x_1x_8 \cdots x_l$ of order $l+1$ for any $x_6'' \in N_G(x_6') \backslash V(P_0)$ than P_0, a contradiction.

So G has a 4-cycle $C_4 = x_1x_2y_1y_2x_1 \subseteq G$. As every cycle in $G/\pi(C_4)$ corresponds to a cycle in G, we have $c(G/\pi(C_4)) \le c(G) \le 8$. As $|D_2(G)| \le 1$, $|D_2(G/\pi(C_4))| \le 1$. If $\kappa'(G/\pi(C_4)) \ge 3$, then the minimality of G implies that $G/\pi(C_4)$ is collapsible. Thus by Theorem 5, G is collapsible, a contradiction. Therefore, we must have $\kappa'(G/\pi(C_4)) \le 2$. We consider the following two cases to finish our proof.

Case 1. $\kappa'(G/\pi(C_4)) = 1$.

Then e_π must be the cut-edge of $G/\pi(C_4)$, and so $G - E(C_4)$ has two components G_1, G_2 such that $x_1, y_1 \in V(G_1)$, $x_2, y_2 \in V(G_2)$ and $V(G_1) \subseteq D_{\ge 3}(G)$. As G is essentially 3-edge-connected, $V(C_4) \subseteq D_{\ge 3}(G)$. Therefore, we can choose longest paths $P(x_i, y_i)$ between x_i and y_i in G_i for $i \in \{1, 2\}$. Since $g(G) = 4$, $|E(P(x_i, y_i))| \ge 2$.

We first claim that $|E(P(x_1, y_1))| \ge 3$. Since otherwise, assume that $P(x_1, y_1) = x_1wy_1$. Then w has a neighbor w' outside $\{x_1, x_2\}$ such that $G_1 - w$ has a path between w' and $\{x_1, y_1\}$ since G is 2-connected, which would produce a longer (x_1, y_1)-path, a contradiction.

If $|E(P(x_1, y_1))| = 3$, assume that $P(x_1, y_1) = x_1w_1w_2y_1$, then w_1 has a neighbor w_1' outside $\{x_1, w_2\}$ such that $G_1 - w_1$ has no path between w_1' and $\{w_2, x_1\}$ and no path of order at least 2 between w_1' and y_1 by the choice of $P(x_1, y_1)$. Hence $w_1'y_1 \in E(G)$ since G is 2-connected. By symmetry, w_2 has a neighbor w_2' such that $w_2'x_1 \in E(G)$, and so $x_1w_2'w_2w_1w_1'y_1$ is a longer path than $P(x_1, y_1)$, a contradiction.

This implies that $|E(P(x_1, y_1))| = 4$ and $|E(P(x_2, y_2))| = 2$ since $c(G) \le 8$. Assume that $P(x_1, y_1) = x_1w_1w_2w_3y_1$, $P(x_2, y_2) = x_2wy_2$. Since $g(G) = 4$ and by the choice of $P(x_1, y_1)$, w_2 has a neighbor w_2' outside $V(P(x_1, y_1))$ such that $G - w_2$ has no path between w_2' and $\{w_1, w_3\}$ and no path of order at least 2 between w_2' and $\{x_1, y_1\}$. Then $\{w_2'x_1, w_2'y_1\} \not\subseteq E(G)$, since otherwise, $K_{3,3}^- \subseteq G[\{x_1, x_2, y_1, y_2, w, w_2'\}]$, a contradiction. Then w_2' has a neighbor w_2'' outside $V(P(x_2, y_2)) \cup \{w_2'\}$ such that $G - \{w_1w_2, w_2w_3\}$ has no path between $w_2w_2'w_2''$ and C by the choice of $P(x_2, y_2)$, i.e., $\{w_1w_2, w_2w_3\}$ is an essential 2-edge-cut of G, a contradiction.

Case 2. $\kappa'(G/\pi(C_4)) = 2$

If $G/\pi(C_4)$ is essentially 3-edge-connected, then $G/\pi(C_4)$ has a 2-vertex $u_0 \in V(e_\pi)$, and so $V(C) \cap D_2(G) \neq \emptyset$. Then $D_2(G/\pi(C_4)) = 1$, and so $G/\pi(C_4)$ is collapsible by the minimality of G, and hence G is collapsible by Theorem 5, a contradiction. This implies that $G/\pi(C_4)$ has an essential 2-edge-cut $\{e_\pi, z_1z_2\}$ such that $G - V(C_4)$ has a cut-edge z_1z_2 such that $(G - V(C_4)) - z_1z_2$ has two components G_1, G_2 with $z_1 \in V(G_1)$, $z_2 \in V(G_2)$ and $V(G_1) \cup \{x_1, y_1\} \subseteq D_{\geq 3}(G)$. Choose longest paths $P(x_i, z_i)$ (say) between $\{x_i, y_i\}$ and z_i in $G[V(G_i) \cup \{x_i, y_i\}]$ for $i \in \{1, 2\}$.

Note that $\{z_1x_1, z_1y_1, z_2x_2, z_2y_2\} \not\subseteq E(G)$ since $K_{3,3}^- \not\subseteq G[\{x_1, y_1, z_1, x_2, y_2, z_2\}]$. Then $\max\{|E(P(x_1, z_1))|, |E(P(x_2, z_2))|\} \geq 2$. By symmetry, assume that $P(x_2, z_2) = x_2w_1 \cdots w_tz_2$ for some $t \geq 1$. Since $c(G) \leq 8$, $t \leq 2$. Suppose first that $t = 1$. Then $N_G(w_1) \subseteq \{x_2, y_2, z_2\}$, since otherwise, w_1 has a neighbor w_1' outside $\{x_2, y_2, z_2\}$ such that $G - w_1$ has no path between w_1' and $\{x_2, y_2, z_2\}$ by the choice of $P(x_2, z_2)$, i.e., w_1 is a cut-vertex of G, a contradiction. Besides, $N_{G_2}(z_2) \subseteq \{x_2, y_2, w_1\}$. (Otherwise, since G is 2-connected and by the choice of $P(x_2, z_2)$, z_2 has a neighbor z_2' outside $\{x_2, y_2, w_1\}$ such that $z_2'w_1 \notin E(G)$ and $E(z_2', \{x_2, y_2\}) \neq \emptyset$. By the symmetry of w_1 and z_2', $N_G(z_2') \subseteq \{z_2, x_2, y_2\}$. Since $c(G) \leq 8$, $|E(P(x_1, z_1))| = 1$, i.e., $\{z_1x_1, z_1y_1\} \subseteq E(G)$. Hence $K_{3,3}^- \subseteq G[\{x_1, y_1, z_1, x_2, y_2, z_2, z_2'\}]$, a contradiction.) Then $|E(P(x_1, z_1))| \geq 2$ and $\{w_1, z_2\} \cap D_2(G) \neq \emptyset$ since $\{y_2w_1, y_2z_2\} \not\subseteq E(G)$. By the symmetry of $P(x_1, z_1)$ and $P(x_2, z_2)$, $|E(P(x_1, z_1))| \geq 3$ since $|D_2(G)| \leq 1$, and so $G[V(P(x_1, z_1) \cup P(x_2, z_2) \cup C_4)]$ has a cycle of order at least 9, a contradiction. Suppose now that $t = 2$. Since $c(G) \leq 8$, $|E(P(x_1, z_1))| = 1$ and $\{z_1x_1, z_1y_1\} \subseteq E(G)$. Then $d_G(w_1) = 2$. (Otherwise, assume that w_1 has a neighbor w_1'. By the choice of $P(x_2, z_2)$ and since G is 2-connected, $w_1'z_2 \in E(G)$. Note that $\{w_2, w_1'\} \not\subseteq D_2(G)$. By symmetry, either w_2 has a neighbor w_2' outside $\{x_2, y_2, z_2, w_1'\}$ such that $G - w_2$ has no path between w_2' and $\{x_2, y_2, z_2, w_1, w_1'\}$ by the choice of $P(x_2, z_2)$ or $E(w_2', \{x_2, y_2, z_2, w_1'\}) \neq \emptyset$ and $G[\{x_1, y_1, z_1, x_2, y_2, z_2, w_1, w_2, w_1', w_2'\}]$ is collapsible, a contradiction.) Hence w_2 has a neighbor w_2' outside $\{x_2, y_2, z_2\}$ such that $G - w_2$ has no path between w_2' and $\{x_2, y_2.z_2, w_1\}$ by the choice of $P(x_2, z_2)$ and $|D_2(G)| \leq 1$, a contradiction. □

Theorem 8 (*Ma et al.* [22]) *Let G be a* 3*-edge-connected graph. Then each of the following holds.*

(1) *If* $c(G) \leq 11$, *then G is supereulerian or G is contractible to* $P(10)$.
(2) *If G is reduced,* $g(G) = 4$ *and* $c(G) \leq 11$, *then there is a* 4*-cycle C such that* $\kappa'(G/\pi(C)) \geq 3$.
(3) *If G is reduced,* $|V(G)| \geq 14$ *and* $g(G) \geq 5$, *then* $c(G) \geq 12$.

The following theorem extends Theorem 8(1) and will play an important role in the proof of Theorem 2.

Theorem 9 *Let G be a* 2*-connected* 3*-edge-connected graph with* $c(G) \leq 11$ *and* G' *be the reduction of G. Then either G is collapsible or* $G' \cong P(10)$.

Proof By contradiction, assume that G is a counter-example with $|V(G)|$ minimized. Then G is reduced. Otherwise, G has a collapsible subgraph H. Then G/H is 2-edge-connected, 3- edge-connected with $c(G/H) \leq 11$ and v_H is the contraction image of H. If $\kappa(G/H) \geq 2$, then either G/H is collapsible, and then G is collapsible or the reduction G' of G/H is isomorphic to $P(10)$, a contradiction. If $\kappa(G/H) = 1$, then the reduction G' of G/H has at least two blocks $B_1 \cong B_2 \cong P(10)$ sharing one cut-vertex v_H. Since $\kappa(G) \geq 2$, $|N_G(V(B_1)\backslash\{v_H\}) \cap V(H)| \geq 2$ and $|N_G(V(B_2)\backslash\{v_H\}) \cap V(H)| \geq 2$. Hence G has a cycle of order at least 18, contradicting $c(G) \leq 12$.

Furthermore, $g(G) \geq 5$. If not, then G has a 4-cycle $C_0 = x_1y_1x_2y_2x_1$ such that $\kappa'(G/\pi(C_0)) \geq 3$ by Theorem 8(2). Let G'_1 be the reduction of $G/\pi(C_0)$ and $e_\pi = xy$. Then $|V(G'_1)| \leq |V(G/\pi(C_0))| < |V(G)|$, $c(G'_1) \leq c(G/\pi(C_0)) \leq 11$. The minimality of $|V(G)|$ implies that each block of G'_1 is isomorphic to $P(10)$. If $\kappa(G/\pi(C_0)) \geq 2$, then either $G/\pi(C_0) \cong G'_1 \cong P(10)$ and $G \cong P^3(12)$ (see Fig. 1), and hence $c(G) = 12$, or G has a subgraph H such that $V(C_4) \cap V(H) = \{x_1, x_2\}$ (or $\{y_1, y_2\}$), $H/\{x_1, x_2\}$ (or $H/\{y_1, y_2\}$) is collapsible and $(G/\pi(C_0))/H \cong P(10)$, and hence $c(G) \geq c(P^3(12)) \geq 12$, a contradiction. Then $G/\pi(C_0)$ has two blocks B_1, B_2 such that $e_\pi \in E(B_1)$ and $V(B_1) \cap V(B_2) = \{x\}$ (or $\{y\}$). This implies that G has a subgraph H such that $C_0 \subseteq H$ and the reduction of $H/\pi(C_0)(= B_1)$ is isomorphic to $P(10)$. Then $c(G) \geq c(H) \geq 12$.

As $c(G) \leq 11$ and $g(G) \geq 5$, by Theorem 8(3), $|V(G)| \leq 13$. By Theorem 4(6), $G' \in \{P^1(12), P^2(12)\}$. Therefore, G' has a 12-cycle (see Fig. 1), contradicting $c(G) \leq 11$. □

3 Proof of Theorem 3

Before presenting the proof, we need to prepare some results. The graphs $K'_{2,3}, P(10)(e)$ are depicted in Fig. 1.

Theorem 10 *It holds the following.*

(1) (Li et al. [18]) *Every connected graph G with $|V(G)| \leq 12$, $|D_1(G)| = 0$, $|D_2(G)| \leq 1$ either is supereulerian with 12 vertices or the reduction of G is in $\{K_1, K_2, P_3, K_{2,3}, K'_{2,3}, P(10), P(10)(e)\}$.*

(2) (Wang [25]) *Every 3-edge-connected graph G with $|V(G)| \leq 8$ other than W_8 is strongly spanning trailable.*

(3) (Li et al. [18]) *Let G be a 3-edge-connected graph with blocks $B_1, \ldots, B_k$. Then G is strongly spanning trailable if and only if B_i is strongly spanning trailable for every $i = 1, \ldots, k$.*

Let $\mathcal{W}_0$ be the set of graphs obtained from W_8 by subdividing one edge of W_8 and then adding at least one edge between the new vertex and exactly one of its neighbor.

Corollary 1 *Every 3-edge-connected graph G with $|V(G)| \leq 9$ other than a member of $\{W_8\} \cup \mathcal{W}_0$ is strongly spanning trailable.*

Proof Let G be a counter-example. Then $|V(G)| = 9$ by Theorem 10(2) and for some pair of edges e_1, e_2, $G(e_1, e_2)$ does not have a spanning (v_{e_1}, v_{e_2})-trail. Let H be the graph obtained from $G(e_1, e_2)$ by adding a new vertex z and two edges zv_{e_1}, zv_{e_2}. Then H is 2-edge-connected, essentially 3-edge-connected and nonsupereulerian with 12 vertices if $e_1 \neq e_2$ or 11 vertices if $e_1 = e_2$. Besides, the reduction H' of H is

2-edge-connected, essentially 3-edge-connected and nonsupereulerian with $|D_2(H')| \leq 1$. By Theorem 10(1), $H' \in \{P(10), P(10)(e)\}$. If $H' \cong P(10)$, then H has a collapsible subgraph H_1 containing z. Since z is not in a triangle, $|V(H_1)| \geq 4$, and then $|V(H)| \geq 13$, a contradiction. Hence $H' \cong P(10)(e)$. If $H' = H$, then $H = W_8$, a contradiction. If $H' \neq H$, then H has a collapsible subgraph H_1 with $|V(H_1)| = 2$ since $|V(H)| = 12$, and then $H \in \mathcal{W}_0$, a contradiction. □

Let G be a graph and $S \subseteq V(G)$ be a subset with |S| even. A subgraph $L_S \subseteq G$ is an S-**join** if $O(L_S) = S$. Thus a graph G is collapsible if for every even vertex subset S, G has a spanning connected S-join.

Lemma 2 *Let $G \cong K_{2,t}$ for integer $t \geq 2$ and $S \subseteq V(G)$ be an even subset such that $S \cap D_2(G) \neq \emptyset$. Then for any $\{u_1, u_2\} \subseteq V(G)$, exactly one of the following holds,*

(1) *$t = 2$, $S = \{u_1, u_2\}$ and $u_1u_2 \notin E(G)$,*
(2) *G has a spanning S-join L such that either L is connected (if $D_2(G) \nsubseteq S$) or L has exactly two components L_1, L_2 such that $u_1 \in V(L_1), u_2 \in V(L_2)$ (if $D_2(G) \subseteq S$).*

Proof Let w_1, w_2 be two nonadjacent vertices of degree t in G and $v_1, \ldots, v_t$ be the other vertices of G. Let $V_1 = \{v_1, \ldots, v_t\} \cap S$ and $V_2 = \{v_1, \ldots, v_t\} \backslash S$. Let $\{i, j\} = \{1, 2\}$.

Suppose that $t = 2$. Then, without loss of generality, either $u_1 = v_1, u_2 = v_2$ or $u_1 = v_1, u_2 = w_1$. If $S = \{w_1, w_2, w_3, w_4\}$, then set $L_1 = v_1w_2, L_2 = v_2w_1$. If $S = \{w_1, w_2\}$, then set $L_1 = v_1, L_2 = w_1v_2w_2$. If $S = \{v_1, v_2\}$, then either $u_1 = v_1, u_2 = v_2$ and (i) holds, or $u_1 = v_1, u_2 = w_1$ and set $L_1 = w_1, L_2 = v_1w_2v_2$. We then assume $S = \{v_1, w_1\}$, then set $L = v_iw_iv_jw_j$. Therefore, we then assume that $t \geq 3$. Then $V_1 \neq \emptyset$.

Case 1. $V_2 = \emptyset$.

It suffices to construct a spanning S-join L of G that has exactly two components L_1, L_2 such that $\{u_1, u_2\} \cap V(L_1) = \{u_1\}$. If t is odd, then $\{w_1, w_2\} \cap S = \{w_i\}$ and V_1 has a partition (V_1^1, V_1^2) such that $|V_1^1|$ is odd, $|V_1^2|$ is even, $(V_1^1 \cup \{w_i\}) \cap \{u_1, u_2\} = \{u_1\}$, and hence set $L_1 = G[E(w_i, V_1^1)]$, $L_2 = G[E(w_j, V_1^2)]$.

If t is even, then either $\{w_1, w_2\} \subseteq S$ or $\{w_1, w_2\} \cap S = \emptyset$. If $\{w_1, w_2\} \subseteq S$, then V_1 has a partition (V_1^3, V_1^4) such that $|V_1^3|$, $|V_1^4|$ are odd and $(V_1^3 \cup \{w_1\}) \cap \{u_1, u_2\} = \{u_1\}$, and hence set $L_1 = G[E(w_1, V_1^3)]$, $L_2 = G[E(w_2, V_1^4)]$. If $\{w_1, w_2\} \cap S = \emptyset$, then V_1 has a partition (V_1^5, V_1^6) such that $|V_1^5|$, $|V_1^6|$ are even and $(V_1^5 \cup \{w_1\}) \cap \{u_1, u_2\} = \{u_1\}$, and set $L_1 = G[E(w_1, V_1^5)]$, $L_2 = G[E(w_2, V_1^6)]$.

Case 2. $V_2 \neq \emptyset$.

Then V_1 has a partition (V_1^7, V_1^8) such that $|V_1^8|$ is odd. It suffices to construct a spanning connected S-join L of G.

Suppose first that t is odd. If $\{w_1, w_2\} \subseteq S$, then $|V_1|$ is even, $|V_2|$ is odd, and set $L = G - E(w_2, V_1)$. If $\{w_1, w_2\} \cap S = \{w_i\}$, then $|V_1|$ is odd, $|V_2|$ is even, and set $L = G - E(w_j, V_1)$. If $\{w_1, w_2\} \cap S = \emptyset$, then $|V_1|$ is even, $|V_1^7|$, $|V_2|$ are odd, and set

$L = G - (E(w_1, V_1^8) \cup E(w_2, V_1^7))$.

Suppose then t is even. If $\{w_1, w_2\} \subseteq S$, then $|V_1|, |V_2|$ are even, $|V_1^7|$ is odd, and set $L = G - (E(w_1, V_1^8) \cup E(w_2, V_1^7))$. If $\{w_1, w_2\} \cap S = \{w_i\}$, then $|V_1|$, $|V_2|$ are odd, $|V_1^7|$ is even, and set $L = G - (E(w_i, V_1^7) \cup E(w_j, V_1^8))$. If $\{w_1, w_2\} \cap S = \emptyset$, then $|V_1|$, $|V_2|$ are even, and set $L = G - E(w_2, V_1)$. □

Lemma 3 *Let G be a graph and H be a subgraph of G such that H has* 2 *edge-disjoint spanning trees. If either H is essentially* 3*-edge-connected, or G is* 3*-edge-connected, then*

(1) *if G is strongly spanning trailable, then G/H is strongly spanning trailable,*
(2) *if G/H is strongly spanning trailable, then either G is strongly spanning trailable, or G has only one pair edges e, e' such that $H = G[\{e, e'\}] \cong C_2$ and $G(e, e')$ has no spanning $(v_e, v_{e'})$-*trail.

Proof

(1) Suppose that G is strongly spanning trailable and let e_1, e_2 be two edges in G/H. As $e_1, e_2 \in E(G) - E(H)$, $G(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail T. Since $G/H(e_1, e_2) = G(e_1, e_2)/H$, $T/E(H) \cap E(T)$ is a spanning (v_{e_1}, v_{e_2})-trail of G/H. Hence by definition, G/H is strongly spanning trailable.
(2) Assume that G/H is strongly spanning trailable, and let v_H denote the vertex in G/H onto which H is contracted. For any $e_1, e_2 \in E(G)$, we shall show that $G(e_1, e_2)$ always has a spanning (v_{e_1}, v_{e_2})-trail. If $\{e_1, e_2\} \cap E(H) = \emptyset$, then $e_1, e_2 \in E(G/H)$. As G/H is strongly spanning trailable, G/H has a spanning (v_{e_1}, v_{e_2})-trail T_1 containing the vertex v_H. Let $X_1 = V(H) \cap O(G[E(T_1)]$. Then since v_H has even degree in T_1, $|X_1|$ is even. Then H has a spanning connected X_1-join L_1. It follows by definition that $G[E(T_1) \cup E(L_1)]$ is a spanning (v_{e_1}, v_{e_2})-trail in G.

Suppose next that $|\{e_1, e_2\} \cap E(H)| = 1$, and by symmetry we may assume that $e_1 \in E(H)$ and $e_2 \notin E(H)$. Since H has 2-edge-disjoint spanning trees, $H(e_1)$ is collapsible. Let $e_1' \neq e_2$ be an edge in G/H incident with v_H. Then $e_1', e_2 \in E(G/H)$. Since G/H is strongly spanning trailable, $G/H(e_1', e_2)$ has a spanning $(v_{e_1'}, v_{e_2})$-trail T_2'. Since e_1' is incident with v_H, T_2' can be adjusted to a spanning (v_H, v_{e_2})-trail T_2 in $G/H(e_2)$, where

$$T_2 = \begin{cases} T_2' - v_{e_1'} v_H & \text{if } v_{e_1'} v_H \in E(T_2') \\ T_2' - v_{e_1'} + e_1' & \text{if } v_{e_1'} v_H \notin E(T_2'). \end{cases}$$

Let $X_2 = V(H) \cap O(G[E(T_2)]$. Then since v_H has odd degree in T_2, $|X_2|$ is odd, and so $X_2' = X_2 \triangle \{v_{e_1}\}$ is an even subset of $V(H(e_1))$. Since $H(e_1)$ is collapsible, $H(e_1)$ has a spanning connected X_2'-join. It follows by definition that $G[E(T_2) \cup E(L_2)]$ is a spanning (v_{e_1}, v_{e_2})-trail in G.

Therefore, we assume that $\{e_1, e_2\} \subseteq E(H)$. If $H(e_1, e_2)$ is collapsible, then since G/H is strongly spanning trailable, G/H has a spanning closed trail T_3. Let $X_3 = V(H) \cap O(G[E(T_3)])$. Since v_H has even degree in T_3, $|X_3|$ is even, and so

$X_3' = X_3 \cup \{v_{e_1}, v_{e_2}\}$ is also an even subset. Since $H(e_1, e_2)$ is collapsible, $H(e_1, e_2)$ has a spanning connected X_3'-join L_3. It follows by definition that $G[E(T_3) \cup E(L_3)]$ is a spanning (v_{e_1}, v_{e_2})-trail in G.

Thus we may assume that $H(e_1, e_2)$ is not collapsible. If $F(H(e_1, e_2)) \leq 1$, then $H(e_1, e_2)$ is collapsible. Hence $F(H(e_1, e_2)) = 2$. Let H' be the reduction of $H(e_1, e_2)$. Thus there exists a subgraph J of $H(e_1, e_2)$ such that each component of J is collapsible and such that $H(e_1, e_2)/J = H'$. By Theorem 4(5), $H' = K_{2,t}$ for some $t \geq 2$. If $|\{v_{e_1}, v_{e_2}\} \cap V(H')| \leq 1$, then $F(H') \leq F(H) + 1 \leq 1$, contrary to the fact $H' = K_{2,t}$. Hence v_{e_1}, v_{e_2} must be two distinct vertices in $D_2(H')$, and each of $\{v_{e_1}, v_{e_2}\}$ is not incident with any edges in $E(G)$. As G/H is strongly spanning trailable, G/H has a spanning closed trail T_4. Let $X_4 = V(H) \cap O(G[E(T_4)])$. Since v_H has even degree in T_4, $|X_4|$ is even, and so $X_4' = X_4 \cup \{v_{e_1}, v_{e_2}\}$ is also an even subset. Define $X'' = \{v \in V(H') :$ the preimage of v in $H(e_1, e_2)$ contains an odd number of vertices in $X_4'\}$. Then $|X''|$ is even with $v_{e_1}, v_{e_2} \in X''$. If $t \geq 3$, then by Lemma 2, H' has a spanning X''-join L such that either L is connected (if $D_2(H') \nsubseteq X''$), or L has exactly two components L_1 and L_2 with the preimage of L_i in $H(e_1, e_2)$ containing u_i for $i \in \{1, 2\}$ (if $D_2(H') \subseteq X''$). Note that if $D_2(H') \subseteq X''$, then there exist vertices $u_1, u_2 \in V(H(e_1, e_2))$ such that u_1, u_2 are in the same component of $G[E(T_4)]$ and such that u_1 and u_2 are contained in different vertices of H'. It happens that $G/J[E(T_4) \cup E(L)]$ is a spanning (v_{e_1}, v_{e_2})-trail of G/J. Since each component of J is collapsible, $G/J[E(T_4) \cup E(L)]$ can be lifted to a spanning (v_{e_1}, v_{e_2})-trail of G by replacing each vertex $v \in V(H')$ by a spanning connected subgraph of its preimage in $H(e_1, e_2)$. We then assume that $t = 2$ and $H' = u_1 v_{e_1} u_2 v_{e_2} u_1$. Then $\{e, e'\} = \{e_1, e_2\} = \{u_1 u_2, u_1 u_2\}$ and $H = G[\{e, e'\}] \cong C_2$. □

Let $P(10) + e$ be a graph obtained from the Petersen graph $P(10)$ by adding an additional edge e between two adjacent vertices x, y. In fact, e, xy are multiple edges. Then $c(P(10) + e) = 9$. By Corollary 1, $(P(10) + e)/\{e, xy\}$ is strongly spanning trailable. On the other hand, $(P(10) + e)(e, xy)$ has no spanning (v_e, v_{xy})-trail. This implies that the condition $c(G) \leq 8$ in Lemma 4 is sharp.

Lemma 4 *Let G be a 3-edge-connected graph with $c(G) \leq 8$. If G has a subgraph H such that H has 2 edge-disjoint spanning trees, then G/H is strongly spanning trailable if and only if G is strongly spanning trailable.*

Proof By Lemma 3(2), assume that G/H is strongly spanning trailable, it suffices to prove that for one pair edges e_1, e_2 of G such that $H = G[\{e_1, e_2\}] \cong C_2$, $G(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail. Let G be a counter-example with $|V(G)|$ minimized. By Theorem 10(3), G is 2-connected. Furthermore, $G - \{e_1, e_2\}$ is reduced. If not, assume that $G - \{e_1, e_2\}$ has a nontrivial collapsible subgraph H_1. As $e_1, e_2 \notin E(H_1)$ and by the definition of contractions, $G/H_1(e_1, e_2) = G(e_1, e_2)/H_1$. By the choice of G and as $|V(G/H_1)| < |V(G)|$, G/H_1 is strongly spanning trailable, and so $G(e_1, e_2)/H_1 = G/H_1(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail. Since H_1 is collapsible, it follows that $G(e_1, e_2)$ also has a spanning (v_{e_1}, v_{e_2})-trail, a contradiction.

Assume that $\{e_1, e_2\} = \{x_1 x_2, x_2 x_1\}$. If $G - e_1$ has an essential 2-edge-cut $\{x_1 x_2, uv\}$ for some $uv \in E(G)$, then $G - \{x_1, x_2\} - uv$ has two components F_1, F_2

such that $u \in V(F_1)$, $v \in V(F_2)$ and $E(x_1, F_2) = E(x_2, F_1) = \emptyset$. Since G is 3-edge-connected, $|N_G(x_1) \cap V(F_1)| \geq 2$ and $|N_G(x_2) \cap V(F_2)| \geq 2$. Choose longest paths $P_1(u_1, u)$ between $N_G(x_1) \cap V(F_1)$ and u in F_1 and $P_2(v_1, v)$ between $N_G(x_2) \cap V(F_2)$ and v in F_2. Then $|E(P_1(u_1, u))| \geq 1$. Assume that $P_1(u_1, u) = u_1 \cdots u_s u$. If $s \leq 2$, then u_1 has a neighbor u_1' outside $V(P_1(u_1, u))$. By the choice of $P_1(u_1, u)$, either $G - u_1$ has no path between u_1' and $\{x_1, u\}$ (if $s = 1$) or $G - \{u_1, u\}$ has no path between u_1' and $\{x_1, u\}$ and $G - u_1$ has no path of order at least 2 between u_1' and u (if $s = 2$). Then $s \geq 2$ and if $s = 2$, then $u_1'u \in E(G)$ and u_1' has a neighbor u_1'' such that $G - u_1'$ has no path between u_1'' and $\{x_1, u_1, u_2, u\}$, i.e., u_1' is a cut-vertex, a contradiction. Therefore $s \geq 3$, i.e., $|E(P_1(u_1, u))| \geq 3$. By symmetry, $|E(P_2(v_1, v))| \geq 3$. Then $x_1 u_1 P_1(u_1, u) u v P_2(v_1, v) v_1 x_2 x_1$ is a cycle of order at least 10, a contradiction.

Hence $G - e_1$ is essentially 3-edge-connected. Note that $c(G - e_1) \leq c(G) \leq 8$ and $|V_{\leq 2}(G - e_1)| = |V_2(G - e_1)| \leq 1$. Then $G - e_1$ is collapsible by Theorem 7. Let G_1 be the graph obtained from $G(e_1, e_2)$ by adding an additional vertex v and adding edges vv_{e_1}, vv_{e_2}. Note that there is a C-subpartition $(\{x_1, v\}, \{x_2, v_{e_1}, v_{e_2}\})$ such that $G_1/\pi(\{x_1, v\}, \{x_2, v_{e_1}, v_{e_2}\}) \cong G - e_1$. Then G_1 is collapsible and also is supereulerian by Lemma 1(2). Then G_1 has a closed spanning trail T_0 such that $T_0 - v$ is a spanning (v_{e_1}, v_{e_2})-trail of $G(e_1, e_2)$. □

Proof of Theorem 3 Let G be a counterexample with $|V(G)|$ minimized. By Corollary 1, $|V(G)| \geq 10$. If G has a 2-cycle C_0, then the minimality implies that G/C_0 is strongly spanning trailable. Since $F(C_0) = 0$ and by Lemma 4, G is strongly spanning trailable. Then $g(G) \geq 3$. Note that G has edges e_1, e_2 (or possibly $e_1 = e_2$) such that $G(e_1, e_2)$ has no spanning (v_{e_1}, v_{e_2})-trail. □

Claim *1. $G - \{e_1, e_2\}$ is reduced.*

Proof By contradiction, assume that $G - \{e_1, e_2\}$ has a nontrivial collapsible subgraph H_1. Then as $e_1, e_2 \notin E(H_1)$ and by the definition of contractions, $G/H_1(e_1, e_2) = G(e_1, e_2)/H_1$. By the choice of G and as $|V(G/H_1)| < |V(G)|$, G/H_1 is strongly spanning trailable, and so $G(e_1, e_2)/H_1 = G/H_1(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail. Since H_1 is collapsible, it follows that $G(e_1, e_2)$ also has a spanning (v_{e_1}, v_{e_2})-trail, a contradiction. □

Claim *2.For any connected subgraph H containing e_1, e_2, $|E(H)| \leq 2|V(H)| - 3$.*

Proof By Claim 1, $H_1 = H - \{e_1, e_2\}$ is reduced. By Theorem 4(4), $F(H_1) = 2|V(H)| - (|E(H)| - 2) - 2$. By Lemma 3(2), $F(H) \geq 1$. Then $F(H_1) \geq F(H) + 2 \geq 3$ and $|E(H)| \leq 2|V(H)| - 3$. □

Since G is 2-connected, G has a cycle $C = x_1 x_2 \cdots x_l x_1$ containing e_1, e_2 with l maximized. Then $3 \leq l \leq 8$. Since $\kappa(G) \geq 2$ and $V(G) - V(C) \neq \emptyset$, there exists a maximum path $P_0 = u_1 u_2 \cdots u_t$ in $G - V(C)$ such that $N_G(u_1) \cap V(C) \neq \emptyset, N_G(u_t) \cap V(C) \neq \emptyset$ and $|N_G(\{u_1, u_2\}) \cap V(C)| \geq 2$. Let $V_0 = V(C) \cup V(P_0)$.

Claim *3.*

(1) If $t \leq 2$, then $N_G(P(u_1, u_t)) \subseteq V(C)$,

(2) if $t=3$, then $N_G(\{u_1,u_3\}) \subseteq V(C) \cup \{u_2\}$ and either $N_G(u_2) \subseteq V(C) \cup \{u_1,u_3\}$ or $N_G(u_2') \subseteq V(C)$ for any $u_2' \in N_G(u_2) \backslash \{u_1,u_3\}$.

Proof

(1) It is true for $t=1$. We then assume that $t=2$. Without loss of generality, assume that u_2 has a neighbor u_2' outside V_0. By the choice of P_0, $N_G(u_2') \cap V_0 \subseteq \{u_2,x_1\}$ if $|N_G(u_1) \cap V(C)|=1$ or $N_G(u_2') \cap V_0 = \{u_2\}$ if $|N_G(u_1) \cap V(C)| \geq 2$. Then $|N_G(u_2') \cap V_0| \leq 2$, and so u_2' has a neighbor u_2'' outside V_0. By the choice of P_0, $G-\{u_2,u_2'\}$ has no path between u_2'' and $V_0 \backslash \{u_2\}$, and so $G-u_2'$ has a path between u_2 and u_2'', and hence $G-u_2$ has no path between $\{u_2',u_2''\}$ and $V_0 \backslash \{u_2\}$, which means that u_2 is a cut-vertex of G, a contradiction.

(2) Without loss of generality, assume that u_3 has a neighbor u_3' outside V_0. By the choice of P_0, either $N_G(u_3') \cap V_0 \subseteq \{u_3,x_1\}$ or $N_G(u_3') \cap V_0 \subseteq \{u_1,u_3\}$. Then u_3' has a neighbor u_3'' outside V_0 such that $N_G(u_3'') \cap V_0 \subseteq \{x_1\}$. Then u_3'' has a neighbor u_3''' outside $V_0 \cup \{u_3,u_3',u_3''\}$ such that $G-\{u_3',u_3''\}$ has no path between u_3''' and $V_0 \backslash \{u_3\}$. Since G is 2-connected, $G-u_3''$ has a path between u_3''' and $\{u_3,u_3'\}$. By the choice of P_0, $G-u_3$ has no path between $\{u_3',u_3'',u_3'''\}$ and $V_0 \backslash \{u_3\}$, i.e., u_3 is a cut-vertex of G, a contradiction.

If u_2' has a neighbor u_2'' outside V_0, then by the choice of P_0, $G-\{u_2,u_2'\}$ has no path between u_2'' and $V_0 \backslash \{u_2\}$. Note that $G-u_2'$ has a path between u_2'' and u_2 of order at least 3. Then $G-u_2$ has no path between $\{u_2',u_2''\}$ and $V_0 \backslash \{u_2\}$, and so u_2 is a cut-vertex of G, a contradiction. □

If $l=3$, by symmetry, then $\{e_1,e_2\}=\{x_1x_2,x_2x_3\}$. By the choice of C, $(G-x_2)-x_1x_3$ has no path between x_1 and x_3. Then since G is 3-edge-connected, G has paths P_1, P_2 with end-vertices x_1,x_2, and x_2,x_3, respectively, such that $V(P_1) \cap V(P_2)=\{x_2\}$ and $E(x_3,P_1)=E(x_1,P_2)=\emptyset$. By Claims 2 and 3(1), $|V(P_1)| \geq 3$, $|V(P_2)| \geq 3$, and so $x_1P_1x_2P_2x_3x_1$ is a cycle of order at least 9, a contradiction. Then $4 \leq l \leq 8$. Without loss of generality, assume that $u_1x_1 \in E(G)$. Since $c(G) \leq 8$, $t \leq 5$. We shall distinguish the following three cases.

Case 1. $t \in \{4,5\}$.

Since $c(G) \leq 8$, $l \leq 6$. We then claim that $|N_G(P_0) \cap V(C)|=2$. Otherwise, assume that $\{u_0x_i, u_tx_j\} \subseteq E(G)$ for some $u_0 \in V(P_0)$ and $1<i<j \leq l$. If $E(x_jx_{j+1}\cdots x_lx_1) \cap \{e_1,e_2\}=\emptyset$, then $|V(x_jx_{j+1}\cdots x_lx_1)| \geq 6$, since otherwise, $|V(V(x_1u_1P_0u_tx_j))| \geq 6 > |V(x_jx_{j+1}\cdots x_lx_1)|$, and then $x_1u_1P_0u_tx_jx_{j-1}\cdots x_1$ is a cycle containing e_1,e_2 of order bigger than C, contradicting the choice of C. Thus $x_jx_{j+1}\cdots x_1u_1u_2\cdots u_tx_j$ is a cycle of order at least 10, a contradiction. Hence $E(x_jx_{j+1}\cdots x_lx_1) \cap \{e_1,e_2\} \neq \emptyset$. Then either $E(x_1x_2\cdots x_i) \cap \{e_1,e_2\}=\emptyset$ or $E(x_ix_{i+1}\cdots x_j) \cap \{e_1,e_2\}=\emptyset$. By the choice of C, either $|V(P(x_1x_2\cdots x_i))| > |V(u_1P_0u_0)|+2$ or $|V(P(x_ix_{i+1}\cdots x_j))| > |V(u_0P_0u_t)|+2$. Hence $j \geq 5$ for $u_0 \notin \{u_1,u_t\}$ or $j \geq 4$ for $u_0 \in \{u_1,u_t\}$. Hence $u_0 \in \{u_1,u_4\}$ and $t=4$, since otherwise, $x_1x_2\cdots x_ju_tu_{t_1}\cdots u_1x_1$ is a cycle of order at least 9, a contradiction. Without loss of generality, assume that $\{u_1x_3, u_4x_4\} \subseteq E(G)$. Then $\{e_1, e_2\} = \{x_1x_4, x_3x_4\}$. By Claim 1, $u_1u_3 \notin E(G)$, and so u_3 has a neighbor u'_3 outside $\{u_2, u_4\}$. By the choices of C and P_0, $G - \{u_1, u_3, x_4\}$ has no path between u'_3 and $\{x_1, x_2, x_3, u_2, u_4\}$ and $G - u_3$ has no path of order at least two between u'_3 and $\{u_1, x_4\}$. Then $\{u'_3u_1, u'_3x_4\} \subseteq E(G)$. By the choice of P_0 and since $K^-_{3,3} \nsubseteq G[\{x_4, u_1, u_2, u_3, u_4, u'_3\}]$, $N_G(u_2) \cap V_0 = \{u_1, u_3\}$, and so u_2 has a neighbor u'_2 outside $V_0 \cup \{u'_3\}$ such that $G - u_2$ has no path between u'_2 and $V_0 \cup \{u'_3\}$, and hence u_2 is a cut-vertex of G, a contradiction.

Suppose that $l = 4$. If $u_tx_2 \in E(G)$, then $t = 4$ since $c(G) \leq 8$. Then at least one of $\{x_3, x_4\}$ has neighbor outside V_0, since otherwise, $|E(G[V(C)])| \geq 6$, contradicting Claim 2. By symmetry, assume that $x_3x'_3 \in E(G)$ for some $x'_3 \notin V_0$. Since $c(G) \leq 8$ and by the choice of P_0, $N_G(x'_3) \cap V_0 \subseteq \{x_1, x_3\}$, and so x'_3 has a neighbor x''_3 outside $V_0 \cup \{x'_3\}$ such that $G - \{x_3, x'_3\}$ has no path between x''_3 and V_0, and hence $G - x_3$ has no path between $\{x'_3, x''_3\}$ and V_0, i.e., x_3 would be a cut-vertex of G, a contradiction. Hence $u_tx_2, u_tx_4 \notin E(G)$ and $u_tx_3 \in E(G)$. Then x_2, x_4 have no neighbor outside V_0.(Otherwise, assume that $x'_2x_2 \in E(G)$ for some $x'_2 \notin V_0$. Since $c(G) \leq 8$ and by the choice of C, either $N_G(x'_2) \cap V_0 \subseteq \{x_1, x_2\}$ or $N_G(x'_2) \cap V_0 \subseteq \{x_2, x_3\}$, and so x'_2 has a neighbor x''_2 outside V_0 such that $G - \{x_2, x'_2\}$ has no path between x''_2 and V_0, and hence $G - x_2$ has no path between $\{x'_2, x''_2\}$ and $V_0 \backslash \{x_2\}$, i.e., x_2 is a cut-vertex of G, a contradiction.) Then $x_2x_4 \in E(G)$ by Claim 2. By symmetry, $\{e_1, e_2\} = \{x_1x_2, x_2x_3\}$, and so $x_1x_2x_3u_tu_{t-1} \cdots u_1x_1$ is a longer cycle containing e_1, e_2, or $\{e_1, e_2\} = \{x_1x_2, x_3x_4\}$, and so $x_1x_2x_4x_3u_tu_{t-1} \cdots u_1x_1$ is a longer cycle containing e_1, e_2, or $\{e_1, e_2\} = \{x_1x_2, x_1x_4\}$, and so $G - \{e_1, e_2\}$ has a collapsible subgraph $x_2x_3x_4x_2$, contradicting Claim 1. Suppose that $l = 5$. Then since $c(G) \leq 8$, $t = 4$ and $E(u_4, \{x_2, x_5\}) = \emptyset$. By symmetry, assume that $u_4x_3 \in E(G)$. By the same argument above, x_2, x_4, x_5 have no neighbor outside V_0, i.e., $N_G(x_i) \subseteq V(C)$ for $i \in \{2, 4, 5\}$. Since $c(G) \leq 8$, $E(x_2, \{x_4, x_5\}) = \emptyset$. Then $|E(G[V(C)])| \geq 8$, contradicting Claim 2. Suppose that $l = 6$. Then $t = 4$ and $u_4x_4 \in E(G)$. By the same argument above, x_2, x_3, x_5, x_6 have no neighbor outside V_0, i.e., $N_G(x_i) \subseteq V(C)$ for $i \in \{2, 3, 5, 6\}$. Since $c(G) \leq 8$, $E(G[\{x_2, x_3, x_5, x_6\}]) = \{x_2x_3, x_5x_6\}$. Then $|E(G[V(C)])| \geq 10$, contradicting Claim 2.

Case 2. $t \in \{2, 3\}$.

Suppose that $t = 2$. By Claims 1 and 3(1), there are four distinct vertices $x_1, x_p \in N_G(u_1) \cap V(C)$ and $x_m, x_n \in N_G(u_1) \cap V(C)$ $(m < n)$. Note that those four vertices divide C into four paths whose set is defined by $\mathcal{P}_0$ and at least two of them do not contain e_1, e_2. Then $p \notin [m, n]$, since otherwise, at least two paths in $\mathcal{P}_0$ has order at least 4 by the choice of C, and so there is a cycle containing u_1u_2 with order at least 10, a contradiction. By symmetry, assume that $p \in [1, m]$. Since $c(G) \leq 8$ and by the choice of C, $\{p, m, n\} = \{3, 4, 6\}$, $C = x_1x_2x_3x_4x_5x_6x_1$ and $\{e_1, e_2\} = \{x_3x_4, x_1x_6\}$, and so $G - \{x_1, x_3\}$ has no path between x_2 and $\{u_1, u_2, x_4, x_5, x_6\}$, which means that $d_G(x_2) = 2$, a contradiction.

Suppose that $t = 3$. Assume that $u_3x_j \in E(G)$ for some $x_j \in V(C) \backslash \{x_1\}$. We claim that $G - \{u_1, u_3\}$ has no path between u_2 and $V(C) \backslash \{x_1, x_j\}$. Suppose otherwise. Then $G - \{u_1, u_3\}$ has a path $P(u_2, x_i)$ by Claim 3(2) for some $i < j$. Since

$c(G) \leq 8$ and by the choice of C, $P(u_2, x_i) = u_2x_i$ and either $i = 4, j = 5$, $C = x_1x_2\cdots x_5x_1$ and $\{e_1, e_2\} = \{x_1x_5, x_4x_5\}$ or $i = 2, j = 3$, $C = x_1x_2\cdots x_6x_1$ and $\{e_1, e_2\} = \{x_1x_2, x_2x_3\}$, and so $|E(G[V_0])| > 2|V_0| - 3$, contradicting Claim 2. We then claim that $|N_G(P_0) \cap V(C)| \geq 3$, since otherwise, $\{u_1x_1, u_1x_j, u_3x_1, u_3x_j, u_2'x_1, u_2'x_j\} \subseteq E(G)$ for some $u_2' \in N_G(u_2)\backslash V_0$ by Claims 1 and 3(2), and then $G[\{x_1, x_j, u_1, u_2, u_3, u_2'\}] - \{e_1, e_2\} \cong K_{3,3}^-$ is collapsible, contradicting Claim 1. Furthermore, $|N_G(P_0) \cap V(C)| = 3$ since $c(G) \leq 8$. By symmetry, assume that $u_1x_i \in E(G)$ for some $i < j$. Then u_2 has a neighbor u_2' such that either $\{u_2'x_1, u_2'x_i\} \subseteq E(G)$ or $\{u_2'x_1, u_2'x_j\} \subseteq E(G)$. Note that x_1, x_i, x_j divide C into three paths such that at least one of them does not contain e_1, e_2, and so it has order at least 5. By symmetry, assume that $i \geq 5$. Then $x_1x_2\cdots x_iu_1u_2u_3x_j\cdots x_1$ is a cycle of order at least 9, a contradiction.

Case 3. $t = 1$.

Then $G[V(G)\backslash V(C)]$ is an empty graph. Recall $|V(G)| \geq 10$. There is a subset $V_1 \subseteq V(G)\backslash V(C)$ such that $u_1 \in V_1$, $|V_1| = 10 - l$ and $|E(G[V_1 \cup V(C)])| \geq 3 \times (10 - l) + l$. By Claim 2, $|E(G[V_1 \cup V(C)])| \leq 17$. Then $l \geq 7$.

Subcase 3.1 $l = 7$.

Since $|E(G[V_1 \cup V(C)])| = |E(G[V(C)])| + |E(V_1, V(C))| \leq 17$ and $|E(V_1, V(C))| \geq 3 \times (10 - 7) = 9$, $|E(G[V(C)])| \leq 8$. Without loss of generality, at least one of the following holds: $\{u_1x_1, u_1x_2, u_1x_3\} \subseteq E(G)$, $\{u_1x_1, u_1x_2, u_1x_4\} \subseteq E(G)$, $\{u_1x_1, u_1x_2, u_1x_5\} \subseteq E(G)$ or $\{u_1x_1, u_1x_3, u_1x_5\} \subseteq E(G)$.

If $\{u_1x_1, u_1x_2, u_1x_3\} \subseteq E(G)$, then $\{e_1, e_2\} = \{x_1x_2, x_2x_3\}$. We claim that x_4, x_7 have no neighbor outside $V(C)$. Suppose otherwise. By symmetry, choose $x_4' \in N_G(x_4)\backslash V(C)$. Since $c(G) \leq 8$, $E(x_4', \{x_2, x_3, x_5\}) = \emptyset$. Besides, $x_4'x_7 \notin E(G)$; for otherwise, $E(x_4', \{x_1, x_6\}) = \emptyset$, and so $d_G(x_4') = 2$, a contradiction. So $\{x_4'x_1, x_4'x_6\} \subseteq E(G)$. Note that $x_5x_7 \notin E(G)$. Then either x_7 has a neighbor x_7' outside $V(C)$ or x_5 has a neighbor x_5' outside $V(C)$ such that $N_G(x_7') \subseteq \{x_7\}$ or $N_G(x_5') \subseteq \{x_5\}$ since $c(G) \leq 8$, a contradiction. Since $|E(G[V(C)])| \leq 8$, $x_4x_7 \in E(G)$ and x_5 has a neighbor x_5' outside $V(C)$ such that $N_G(x_5') \subseteq \{x_3, x_5\}$, a contradiction.

Suppose next that $\{u_1x_1, u_1x_2, u_1x_4\} \subseteq E(G)$. Since $c(G) \leq 8$, $x_1x_2 \in \{e_1, e_2\}$. Note that $N_G(x_3') \subseteq \{x_3, x_6\}$, $N_G(x_5') \subseteq \{x_1, x_5, x_7\}$ and $\{x_5'x_1, x_5'x_7\} \nsubseteq E(G)$ for any $x_3' \in N_G(x_3)\backslash V(C)$ and any $x_5' \in N_G(x_5)\backslash V(C)$. Since $|E(G[V(C)])| \leq 8$, x_3, x_5 have no neighbor outside $V(C)$ and $x_3x_5 \in E(G)$. Then x_7 has a neighbor x_7' outside $V(C)$ such that $N_G(x_7') \subseteq \{x_5, x_7\}$, a contradiction.

Suppose then that $\{u_1x_1, u_1x_2, u_1x_5\} \subseteq E(G)$. Then $x_1x_2 \in \{e_1, e_2\}$. Besides, x_4, x_6 have no neighbor outside $V(C)$. (Otherwise, by symmetry, assume that there is a vertex $x_6' \in N_G(x_6)\backslash V(C)$. Since $c(G) \leq 8$, $E(x_6', \{x_2, x_3, x_5, x_7\}) = \emptyset$ and $\{x_6'x_1, x_6'x_4\} \nsubseteq E(G)$, i.e., $d_G(x_6') = 2$, a contradiction.) Then $x_4x_6 \in E(G)$ and x_7 has a neighbor x_7' outside $V(C)$ such that $N_G(x_7') \subseteq \{x_7\}$, a contradiction.

Therefore, we assume that $\{u_1x_1, u_1x_3, u_1x_5\} \subseteq E(G)$. Then x_2, x_4 have no neighbor outside $V(C)$. (Otherwise, by symmetry, assume that x_4 has a neighbor x_4' outside $V(C)$. By symmetry, $E(x_4', \{x_3, x_5\}) = \emptyset$. Since $c(G) \leq 8$,

$E(x_4', \{x_2, x_6, x_7\}) = \emptyset$. Then $d_G(x_4') \leq 2$, a contradiction.) Then $x_4x_6 \in E(G)$ and x_6 has a neighbor x_6' outside $V(C)$ such that $N_G(x_6') \subseteq \{x_1, x_6\}$, a contradiction.

Subcase 3.2 $l = 8$.

Since $|E(G[V_1 \cup V(C)])| = |E(G[V(C)])| + |E(V_1, V(C))| \leq 17$ and $|E(V_1, V(C))| \geq 3 \times (10 - 8) = 6$, $|E(G[V(C)])| \leq 11$. Without loss of generality, at least one of the following holds: $\{u_1x_1, u_1x_3, u_1x_5\} \subseteq E(G)$ or $\{u_1x_1, u_1x_3, u_1x_6\} \subseteq E(G)$.

If $\{u_1x_1, u_1x_3, u_1x_5\} \subseteq E(G)$, then x_2, x_4 have no neighbor outside $V(C)$, since otherwise, $N_G(x_i') \subseteq \{x_i\}$ for any $x_i' \in N_G(x_i)$ and $i \in \{2, 4\}$, a contradiction. Besides, x_6, x_8 have no neighbor outside $V(C)$. (Otherwise, by symmetry, choose $x_6' \in N_G(x_6)$. Since $c(G) \leq 8$, $E(x_6', \{x_1, x_4, x_6, x_7, x_8\}) = \emptyset$ and $\{x_6'x_2, x_6'x_3\} \nsubseteq E(G)$. Then $d_G(x_6') \leq 2$, a contradiction.) Since $c(G) \leq 8$, $E(G[\{x_2, x_4, x_6, x_8\}]) \subseteq \{x_6x_8\}$. Then $x_6x_8 \in E(G)$ since $|E(G[V(C)])| \leq 11$, and hence $E(x_7, \{x_2, x_4\}) = \emptyset$ and x_7 has a neighbor x_7' outside $V(C)$ such that $N_G(x_7') \subseteq \{x_7\}$, a contradiction.

Suppose then that $\{u_1x_1, u_1x_3, u_1x_6\} \subseteq E(G)$. Then x_2 has no neighbor outside $V(C)$; for otherwise, $N_G(x_2') \subseteq \{x_2, x_6\}$ for any $x_2' \in N_G(x_2)$ since $c(G) \leq 8$, a contradiction. Besides, x_5, x_7 have no neighbor outside $V(C)$; for otherwise, without loss of generality, $N_G(x_5') \subseteq \{x_3, x_5\}$ for any $x_5' \in N_G(x_5)$ since $c(G) \leq 8$, a contradiction. What's more, x_4, x_8 have no neighbor outside $V(C)$. Suppose otherwise. By symmetry, assume that there is a vertex $x_4' \in N_G(x_4)$, then $E(x_4', \{x_2, x_5, x_7\}) = \emptyset$ and $x_4'x_8 \notin E(G)$ since $c(G) \leq 8$. Then $\{x_4'x_1, x_4'x_6\} \subseteq E(G)$. Note that any pair $\{x_2, x_5, x_7, x_8\}$ are nonadjacent in $G - x_7x_8$ since $c(G) \leq 8$. Then $|E(G[V(C) \cup \{u_1, x_4'\}])| \geq 18$, contradicting Claim 2. Since $c(G) \leq 8$ and $|E(G[V(C)])| \leq 11$, $\{x_4x_8, x_5x_7\} \subseteq E(G)$. However, $x_5x_7x_8x_4x_3x_2x_1u_1x_6x_5$ is a 9-cycle, a contradiction. *This completes the proof of Theorem 3.* □

4 Applications of Theorem 3

We now turn our attention to Theorem 3. Its proof will need some additional concepts and notations. A vertex $x \in V(G)$ is said to be eligible if $G[N_G(x)]$ is a connected noncomplete graph. We will use $V_{EL}(G)$ to denote the set of all eligible vertices of G. The **local completion** of G at a vertex x is the graph G_x^* obtained from G by adding all edges with both vertices in $N_G(x)$. One concept of a strong multigraph closure of a claw-free graph G was introduced in [13] as follows.

For a given claw-free graph G, we construct a **strong multigraph closure** (or briefly an SM-closure) G^M of graph G by the following construction.

(1) If G is Hamilton-connected, we set $G^M = cl(G)$.
(2) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

   (a) $G_1 = G$,
   (b) $G_{i+1} = (G_i)_{x_i}^*$ for some $x_i \in V_{EL(G_i)}$, $i = 1, \ldots, k$,

(c) G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
(d) for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected, and set $G^M = G_k$.

The following results show the properties of G^M.

Theorem 11 *Let G be a claw-free graph and let G^M be the SM-closure. Then*

1. (Kužel et al. [13]) G^M *is Hamilton-connected if and only if G is Hamilton-connected.*
2. (Brousek et al. [4]) *If G is H-free, then G^M is H-free for any integers $i, j, k \geq 1$ and $H \in \{N_{i,j,k}, P_i\}$.*

Given a trail T and an edge e in a multigraph H, we say that e is **dominated** (**internally dominated**) by T if e is incident to a vertex (to an internal vertex) of T, respectively. A trail T in H is called an **internally dominating trail**, shortly IDT, if T internally dominates all the edges in H.

Theorem 12 (*Li et al.* [17]) *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Define the **core** of H, denoted by H_0, to be the graph obtained from H by deleting all the vertices of degree 1, and contracting the edge xy for each path xyz with $y \in D_2(H)$.

Theorem 13 (*Shao* [23]) *Let H be a connected, essentially* 3*-edge-connected graph. Then the core H_0 of H satisfies the following.*

(1) H_0 *is uniquely defined and* $\kappa'(H_0) \geq 3$,
(2) if H_0 *is strongly spanning trailable, then L(H) is Hamilton-connected.*

We say H has a H_1-minor if H_1 is isomorphic to the contraction image of a subgraph of H. The graph $T_{i,j,k}$ is obtained by identifying one vertex v with an end-vertex of three paths P_{i+1}, P_{j+1} and P_{k+1}, respectively.

Proof of Theorem 2 Assume that G is not Hamilton-connected. By Theorem 11, we may assume that G is *SM*-closed and H is a multigraph such that $L(H) = G$. Let H_0 be the core of H. By Theorem 13(1), $\kappa'(H_0) \geq 3$. Then we shall obtain a $T_{2,3,5}$-minor and either obtain a P_{11}-minor or $L(H) \in \mathcal{G}$. By Theorem 12, there are at least two edges $e_1 = u_1v_1, e_2 = u_2v_2$ of H such that H has no internally dominating (e_1, e_2)-trail. Without loss of generality, assume that $u_1, u_2 \in V(H_0)$. Note that the graph H can be regarded as the graph obtained from H_0 by adding an additional vertex set V_1 such that $V_1 = D_1(H)$, and by subdividing each edge of an edge subset $E_1 \subseteq E(H_0)$.

Let H_0' be the graph obtained from H_0 by contracting all collapsible subgraphs of $H_0[V(H_0) - V(\{e_1, e_2\})]$. Let H' be the graph obtained from H_0' by adding an additional vertex set V_1 such that $v_1u_1 \in E(H')$ if and only if $v_1 \in V_1$, $v_1u_1' \in E(H)$ and u_1 is a contraction image of non-trivial collapsible subgraph of $H_0[V(H_0) - V(\{e_1, e_2\})]$ containing u_1', and then subdividing each edge of an edge subset $E_1' \subseteq E(H_0')$ such that $uv \in E_1'$ if and only if u, v are contraction images of two collapsible subgraphs of $H_0[V(H_0) - V(\{e_1, e_2\})]$ containing u', v' and $u'v' \in E_1$. □

Claim *1. Each internally dominating (e_1, e_2)-trail T_0 of H' can be extended an internally dominating (e_1, e_2)-trail of H.*

Proof By the construction of H', $V(\{e_1, e_2\}) \subseteq V(H')$ and $\{e_1, e_2\} \subseteq T_0$. By the definition of collapsible, we can replace each contraction image of collapsible graph by a spanning subgraph of its preimage such that the resulting graph T_1 is a (e_1, e_2)-trail, and then subdividing each edge of $E_1 \cap E(T_1)$. Then the resulting graph is an internally dominating (e_1, e_2)-trail of H. □

Note that H' and H_0' are two minors of H. Then H', H_0' have no $T_{2,3,5}$-minor and P_{11}-minor if H has no $T_{2,3,5}$-minor and P_{11}-minor. By Claim 1, $H_0'(e_1, e_2)$ has no (v_{e_1}, v_{e_2})-trail and it suffices to replace H, H_0, E_1 by H', H_0', E_1', respectively. Besides, H_0 has at most two edge-disjoint cycles with order at most 3, which contains at least one of $\{e_1, e_2\}$, respectively.

A vertex of H_0 is called *non-trivial* if it is adjacent to at least one 1-vertex in H; *trivial* otherwise. Call an edge of H_0 *non-trivial* if its two end vertices are non-trivial. For $i \in \{1, 2\}$, $e_i \in E_0$ if and only if either $e_i \subseteq H_0$ is non-trivial or $e_i \subseteq u_iv_ix_i \subseteq H$ for $v_i \in D_2(H)$ and let $u_ix_i = e_i$. Then $E_0 \subseteq H_0$.

Claim *2. If H_0 is collapsible, then $E_0 \neq \emptyset$ and $H_0 - E_0$ is not collapsible.*

Proof

(1) If $\min\{d_H(v_1), d_H(v_2)\} = 2$, then $E_0 \neq \emptyset$. If not, then $e_1, e_2 \in E(H_0)$. Since H_0 is collapsible, H_0 has a spanning (u_1, u_2)-trail T_1. If $\{e_1, e_2\} \cap E(T_1) = \emptyset$, then subdivide some edges of $T_1 \cup \{e_1, e_2\}$ and the resulting trail is an internally dominating (e_1, e_2)-trail of H, a contradiction. Then by symmetry, assume that $e_1 \subseteq T_1 \subseteq H_0$ and u_1 is non-trivial in H_0. If v_1 is non-trivial, then $e_1 \in E_0$. Hence we assume that v_1 is trivial. Note that H_0 has a spanning (v_1, u_2)-trail T_2. By symmetry, $e_2 \subseteq T_2 \subseteq H_0$ and u_2 is non-trivial in H_0. Then v_2 is non-trivial and $e_2 \in E_0$; for otherwise, H_0 has a spanning (v_1, v_2)-trail T_3, and then the trail by subdividing some edges in T_3 is an internally dominating (e_1, e_2)-trail of H, a contradiction.

(2) Assume that $H_0 - E_0$ is collapsible. Then $H_0 - E_0$ has a spanning (u_1, u_2)-trail T_4. Let $T_4 = T_4 \cup e_i$ if $e_i \nsubseteq T_4$ for any $i \in \{1, 2\}$. Then at least one of $\{e_1, e_2\}$, by symmetry, assume $e_1 \subseteq T_4$ and u_1 is non-trivial, v_1 is trivial. Note that $H_0 - E_0$ has a spanning (v_1, u_2)-trail T_5. By symmetry, v_2 is trivial and $H_0 - E_0$ has a spanning (v_1, v_2)-trail, which can be extended to an internally dominating (e_1, e_2)-trail of H, a contradiction.

□

Choose a longest cycle $C_0 = x_1x_2\cdots x_lx_1 \subseteq H_0$. We then consider the following two cases to finish our proof.

Case 1. $l \geq 9$.

Claim *3. H has P_{11}-minor and $T_{2,3,5}$-minor.*

Proof We argue by contradiction. Then if H_0 has a cycle C_0 of order at least 10, then $V(C_0) = V(H_0)$. Since otherwise, there is a vertex $y_1 \in N_{H_0}(x_1)$ outside $V(C_0)$ such that H_0 has a P_{11}. Besides, either $N_{H_0}(y_1) = \{x_1\}$ and $d_{H_0}(y_1) = 1$ or H_0 has a $T_{2,3,5}$ as its subgraph, a contradiction.

We then claim that $l \leq 11$; for otherwise, $P_{11} \subseteq H_0$ and either $H_0[V(C_0)]$ contains a $T_{2,3,5}$ or x_1, x_5, x_9 are in three edge-disjoint cycles of order at most 3, a contradiction.

Besides, $P(10)$ is not an induced subgraph of H_0; for otherwise, either $H_0 \cong P(10)$ with at least one non-trivial vertex or cut-vertex of H_0, and hence there are $T_{2,3,5}$, P_{11} in any cases of them, a contradiction.

Then H_0 is collapsible by Theorem 9 and $E_0 \neq \emptyset$ by Claim 2. Suppose that $10 \leq l \leq 11$. Then $10 \leq |V(H_0)| \leq 11$ and H has a P_{11}-minor. If there is an edge $x_1x_1' \notin E(C_0)$, then either H has a $T_{2,3,5}$-minor or $x_2x_l \notin E(H_0)$, $x_jx_i \notin E(H_0)$ for $i,j \neq 1 \in \{1,\ldots,l\}$ and $|j-i| \geq 3$, and so x_2, x_5, x_l are in three vertex-disjoint cycles of order at most 3, a contradiction. We then assume that $x_1x_2 \in E_1$. Replace x_1x_2 by $x_1v_1x_2$ in H_0. Then either x_1, x_4, x_8 are in three vertex-disjoint cycles of order at most 3 or there is a $T_{2,3,5}$, a contradiction.

Hence $l = 9$. If $|V(H_0)| \leq 9$, then $H_0 \in \mathcal{W}_0$ by Corollary 1 and one of $\{e_1, e_2\}$ is in a 2-cycle, and so $H_0(e_1, e_2)$ has a (v_{e_1}, v_{e_2})-trail, a contradiction. Then $|V(H_0)| \geq 10$ and there is at least one vertex $u \in V(H_0)\backslash V(C_0)$. If u has a neighbor outside $V(C_0)$, then there are subgraphs $T_{2,3,5}$ and P_{11}, a contradiction. Then $N_{H_0}(u) \subseteq V(C_0)$. Without loss of generality, assume that $\{ux_1, ux_3, ux_5\} \subseteq E(H_0)$, $\{ux_1, ux_3, ux_6\} \subseteq E(H_0)$ or $\{ux_2, ux_4, ux_6\} \subseteq E(H_0)$. By (4.1), $E(C_0) \cap E_0 = \emptyset$. Besides, $E(u, C_0) \cap E_0 = \emptyset$, since otherwise, there are P_{11}-minor and $T_{2,3,5}$-minor. Hence, there is an edge $e_0 \notin E(C_0) \cup E(u, C_0)$ and $e_0 \in E_0$. If $\{ux_1, ux_3, ux_5\} \subseteq E(H_0)$, then $E_0 \nsubseteq \{x_1x_3, x_1x_5, x_3x_5\}$ since $H_0 - \{x_1x_3, x_1x_5, x_3x_5\}$ is collapsible. Then at least one of $\{x_2, x_4, x_6, x_7, x_8, x_9, u\}$ has a neighbor outside $V(C_0) \cup \{u\}$ and there is a $T_{2,3,5}$-minor. In addition, there is a P_{11}-minor if one of $\{x_2, x_4, x_6, x_8, u\}$ or all of $\{x_7, x_8\}$ have neighbors outside $V(C_0) \cup \{u\}$. Then $E_0 = \{e_0\} \subseteq E(\{x_7, x_8\}, \{x_1, x_3, x_5\})$, and then $H_1 = H_0[V(C_0) \cup \{u\}] - e_0$ is a 2-edge-connected graph with order 11 and exactly one 2-vertex. By Theorem 10(1), either H_1 is collapsible, and then $H_0 - e_0$ is collapsible or $H_1 \cong P(10)(e)$ and has a P_{11}, a contradiction. By the same but easier argument, we will obtain a contradiction if either $\{ux_1, ux_3, ux_6\} \subseteq E(H_0)$ or $\{ux_2, ux_4, ux_6\} \subseteq E(H_0)$.

Case 2. $l \leq 8$.

By Theorem 13(2), H_0 is not strongly spanning trailable. Then at least one of block B_0 of H_0 is not strongly spanning trailable by Theorem 3 and $|V(B_0)| \geq 10$ by Corollary 1. By Theorem 3, $B_0 \cong W_8$. If B_0 has a cut-vertex of H_0, then at least one vertex x_0 of $V(B_0)$ belongs to a P_4 of $H_0 - V(B_0)$, and hence H_0 has P_{11} and $T_{2,3,5}$ as its subgraphs, a contradiction. Then $H_0 \cong W_8$ and $E(H_0) = E(C_0) \cup \{x_1x_5, x_2x_6, x_3x_7, x_4x_8\}$. By symmetry, assume that H_0 has no spanning (v_{f_1}, v_{f_2})-trail for $f_1 = x_1x_5, f_2 = x_3x_7$. Since H_0 and $H_0 - e_0$ are collapsible for any $e_0 \in \{f_1, f_2\}$. Then $E_0 = \{f_1, f_2\}$ by Claim 2. Besides, either $E(C_0) \subseteq E_1$ or v_2, v_4, v_6, v_8 are non-trivial. Then there is a $T_{2,3,5}$. In addition, either there is a P_{11} or each vertex of H_0 is non-trivial and $L(H) \in \mathcal{G}$. □

5 Concluding Remark

In this paper, we extend the results in [1, 12] in Theorem 2 whose proofs are quite shorter than the original ones with the help of Theorem 3. We believe Theorem 3 may be used to show that every 3-connected $\{K_{1,3}, S\}$-free graph G is Hamilton-connected for $S \in \{N_{1,1,5}, N_{1,3,3}, N_{2,2,3}\}$.

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