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Two-disjoint-cycle-cover vertex bipancyclicity of the bipartite generalized hypercube

Ruichao Niu[a], Min Xu[a], Hong-Jian Lai[b,*]

[a] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

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ABSTRACT

Let $r_2 \geq r_1 \geq 0$ be two integers. A bipartite graph G is two-disjoint-cycle-cover vertex $[r_1, r_2]$-bipancyclic (2-DCC vertex $[r_1, r_2]$-bipancyclic in short) if for any two vertices $u, v \in V(G)$ and any even integer ℓ satisfying $r_1 \leq \ell \leq r_2$, there exist two vertex-disjoint cycles J_1 and J_2 in G with $|V(J_1)| = \ell$ and $|V(J_2)| = |V(G)| - \ell$ such that $u \in V(J_1)$ and $v \in V(J_2)$; and there also exist two vertex-disjoint cycles J_1' and J_2' in G with $|V(J_1')| = \ell$ and $|V(J_2')| = |V(G)| - \ell$ such that $v \in V(J_1')$ and $u \in V(J_2')$. We study the 2-DCC vertex bipancyclicity of the n-dimensional bipartite generalized hypercube $C(d_1, d_2, \ldots, d_n)$. As a result, we determine a family of exceptional graphs and show that for all integers $n \geq 2$, an n-dimensional bipartite generalized hypercube G is 2-DCC vertex $[4, |V(G)|/2]$-bipancyclic if and only if G is not a member in this family. Furthermore, as applications, we prove the vertex-bipancyclicity and 2-DCC bipancyclicity on n-dimensional bipartite generalized hypercube and show that the similar properties also hold for all n-dimensional bipartite k-ary n-cubes, for $n \geq 2$.

1. Introduction

Graphs are often models for interconnection networks. As presented in [11,14,15,17], among others, ring connection is one of the most common interconnection network structures. Hence embedding of rings into an interconnection network is an important issue in parallel processing. As the ring embedding problem is often modeled as finding cycles in the corresponding graph, many studies have been conducted in ring embedding problems, including investigating Hamiltonian and pancyclic properties in networks, as seen in [14–16], among others.

We follow Bondy and Murty [5] for notation and terminology not defined in this paper. A graph $G = (V, E)$ is a pair of the vertex set V and the edge set E, where V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We often use $G = (V(G), E(G))$ to emphasize the graph G. If $e = (u, v) \in E(G)$, then u and v are called the *ends* of e. We often use $P = v_1 v_2 \ldots v_k$ to denote a path in which v_1 is adjacent only to v_2, v_k is adjacent only to v_{k-1}, and v_i is exactly adjacent to v_{i-1} and v_{i+1}, for all $1 < i < k$. To emphasize the ends of a path, we also use $P[v_1, v_k]$ to denote the same path. Likewise, we often use $C = v_1 v_2 \ldots v_k v_1$ to denote a cycle which is formed from a path $P[v_1, v_k]$ by adding an edge joining v_1 and v_k. We define the *length* of a path or a cycle to be the number of its edges. A path or a cycle of length k is called a k-path or

* Corresponding author.
E-mail addresses: niurc@mail.bnu.edu.cn (R. Niu), xum@bnu.edu.cn (M. Xu), hjlai@math.wvu.edu (H.-J. Lai).

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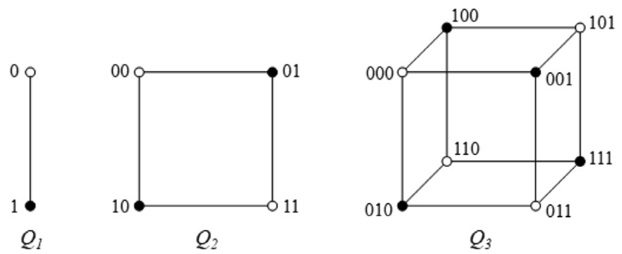

Fig. 1. Q_1, Q_2 and Q_3.

k-cycle, respectively. A *Hamilton path* (respectively, *Hamilton cycle*) of a graph G is a spanning path (respectively, spanning cycle) in G; and G is *Hamiltonian* if it has a Hamilton cycle.

In [4], Bondy defined a graph G to be *pancyclic* if it contains an ℓ-cycle for every integer ℓ with $3 \le \ell \le |V(G)|$. Based on the definition, Randerath et al. [16] defined a graph G to be *vertex-pancyclic* (respectively, *edge-pancyclic*) if every vertex (respectively, edge) of G lies on an ℓ-cycle for every ℓ with $3 \le \ell \le |V(G)|$. Define a bipartite graph G to be *bipancyclic* if it contains an ℓ-cycle for every even integer ℓ with $4 \le \ell \le |V(G)|$. The *vertex (edge)-bipancyclicity* in bipartite graphs can be defined similarly.

Expanding the notion of hamiltonicity, Kung and Chen [9] investigated the problem of embedding disjoint cycles in a graph covering every vertex exactly once. For positive integer i, j with $i \le j$, let $[i, j] = \{i, i+1, \dots, j-1, j\}$. A *two-disjoint-cycle-cover* (2-DCC for short) of a graph G is a pair of vertex-disjoint cycles J_1 and J_2 in G with $V(J_1) \cup V(J_2) = V(G)$. Furthermore, a (bipartite) graph G is *2-DCC* $[r_1, r_2]$*-(bi)pancyclic* if for any (even) integer ℓ satisfying $r_1 \le \ell \le r_2$, there exist two vertex-disjoint cycles J_1 and J_2 in G such that $|V(J_1)| = \ell$ and $|V(J_2)| = |V(G)| - \ell$. It follows from the definition that $r_2 \le \frac{|V(G)|}{2}$.

Kung et al. in [10] proposed the notion of 2-DCC vertex pancyclicity. Following [10] and motivated by the former studies, we introduce 2-DCC vertex bipancyclicity of bipartite graphs in the following.

Definition 1.1. A graph G is *2-DCC vertex* $[r_1, r_2]$*-(bi)pancyclic* if for any two distinct vertices u and v of G, there exist two vertex-disjoint cycles J_1 and J_2 in G such that both of the following two conditions hold:

(1) For any (even) integer ℓ, $r_1 \le \ell \le r_2$, J_1 contains u with length ℓ, and J_2 contains v with length $|V(G)| - \ell$.
(2) For any (even) integer ℓ, $r_1 \le \ell \le r_2$, J_1 contains u with length $|V(G)| - \ell$, and J_2 contains v with length ℓ.

By Definition 1.1, we observe that if a graph G is 2-DCC vertex $[r_1, r_2]$-(bi)pancyclic, then $r_2 \le \frac{|V(G)|}{2}$.

The purpose of the current research is to prove the 2-DCC vertex bipancyclicity of bipartite n-dimensional generalized hypercubes. We present the related preliminaries in Section 2 and justify the main result in Section 3. Section 4 is devoted to applications of our main result, including the determinations of vertex-bipancyclicity and 2-DCC bipancyclicity of the generalized hypercube and the similar properties of k-ary n-cube Q_n^k.

2. Preliminaries

Let n be a positive integer. We follow the standard notation to use C_n to denote a cycle of order n, and K_n to denote the complete graph of order n.

2.1. Cartesian product

The *Cartesian product* of two graph G and H, denoted $G \times H$, has vertex set $V(G \times H) = \{uv | u \in V(G) \text{ and } v \in V(H)\}$, where two vertices u_1v_1 and u_2v_2 are adjacent if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

Following [6], the n-dimensional hypercube, denoted Q_n, is a graph with $V(Q_n) = \{u_1u_2 \cdots u_n | u_i \in \{0, 1\}, 1 \le i \le n\}$ being the set of n-bit binary strings, where two vertices of Q_n are adjacent if and only if their binary strings differ in exactly one bit position. Thus $|V(Q_n)| = 2^n$. Let K_2 denote the complete graph on two vertices. Then utilizing Cartesian products of graph, it is routine to verify that $Q_n = K_2 \times K_2 \times \cdots \times K_2$, taking a Cartesian product of K_2 exactly n times. See Fig. 1 for examples of Q_n with $1 \le n \le 3$.

2.2. Generalized hypercube

The hypercube has been naturally generalizes into many other network models. Among them, the *generalized hypercube*, introduced in [6], has become a widely used topological structure of interconnection network. For an integral n-tuple $(d_1, d_2, \dots, d_n)$ with $d_i \ge 2$ for any $i \in \{1, 2, \dots, n\}$, an n-dimensional generalized hypercube, denoted as $C(d_1, d_2, \dots, d_n)$,

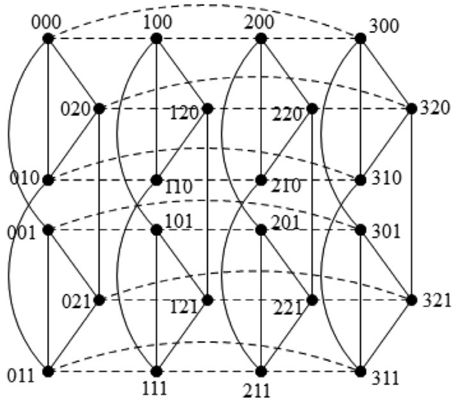

Fig. 2. The illustration of $C(4,3,2)$.

has vertex-set

$$V(C(d_1,d_2,\dots,d_n)) = \{u_1u_2\cdots u_n | u_i \in \{0,1,2,\dots,d_i-1\} \text{ and } d_i \geq 2 \text{ for every } 1 \leq i \leq n\}, \tag{1}$$

where two vertices $u_1u_2\cdots u_n$ and $v_1v_2\cdots v_n$ are adjacent in $C(d_1,d_2,\dots,d_n)$ if and only if there exists an integer $j \in \{1,2,\dots,n\}$ such that $|u_j - v_j| \equiv 1 \pmod{d_j}$ and $u_i = v_i$ for $i \in \{1,2,\dots,j-1,j+1,\dots,n\}$. It is routine to verify that

$$C(d_1,d_2,\dots,d_n) = H_1 \times H_2 \times \cdots \times H_n \tag{2}$$

is the Cartesian product of the graphs H_1, ... H_n, where, for each j with $1 \leq j \leq n$, $H_j = C_{d_j}$ if $d_j \geq 3$ and $H_j = K_2$ if $d_j = 2$. By definition, $Q_n = C(2,2,\dots,2)$, and so $C(d_1,d_2,\dots,d_n)$ generalized the notion of n-dimensional hypercube. Fig. 2 depicts a generalized hypercube $C(4,3,2)$, as an example.

The 2-dimensional generalized hypercube $C(d_1,d_2)$ has been widely used as a topological structure of interconnection network in parallel processing, such as ILLAIC IV (Illinois automatic computer), MPP (massively parallel processors), DAP (distributed array processors) and WRM (wire routing machine). In addition, the 2-dimensional generalized hypercube has also been widely used in LAN (local area network) and MAN (metropolitan are network). In some articles, the LAN and MAN are called Manhattan street network if their topological structures are same as 2-dimensional generalized hypercube [3,12].

2.3. *k*-Ary *n*-Cubes

A special case of the generalized hypercube is the k-ary n-cube. Let k and n be integers at least 2. A k-ary n-cube, denoted Q_n^k, has vertex set

$$V(Q_n^k) = \{u_1u_2\cdots u_n | u_i \in \{0,1,2,\dots,k-1\}, 1 \leq i \leq n\},$$

where two vertices $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ are adjacent in Q_n^k if and only if there exists an integer $j \in \{1,2,\dots,n\}$ such that $|u_j - v_j| \equiv 1 \pmod{k}$ and $u_i = v_i$ for $i \in \{1,2,\dots,j-1,j+1,\dots,n\}$.

The k-ary n-cubes have often been considered as a common model for multiprocessor systems due to its applications, as seen in [1,2,7,8,13], among others. By definition, $Q_n^k = C(k,k,\dots,k)$, and so $Q_n^k = C_k \times C_k \times \cdots \times C_k$ is Cartesian product of the k-cycle C_k taking n times.

In the rest of the paper, we will show the 2-DCC vertex bipancyclicity of generalized hypercube. So that we can also prove the 2-DCC vertex bipancyclicity of k-ary n-cubes.

3. Two-disjoint-cycle-cover bipancyclicity of bipartite generalized hypercube

Let $k \geq 0$ be an integer. We use $\mathbb{Z}_k = \{1,2,\dots,k\}$ to denote the cyclic group of order k and with the additive binary operation $+_k$ and with k being the additive identity in $\mathbb{Z}_k$. Let $P = v_1v_2\dots v_k$ be a path. We use $P[v_1,v_k]$ to emphasize the orientation of P is from v_1 to v_k. Thus $P[v_k,v_1]$ denotes the same path as $P[v_1,v_k]$ (as a graph) with an opposite orientation. For any $1 \leq i \leq j \leq k$, we use $P[v_i,v_j] = v_iv_{i+1}\dots v_{j-1}v_j$ to denote the subpath of P. Likewise, if $C = u_1u_2\dots u_ku_1$ is a cycle, then for any i,j with $1 \leq i < j \leq k$, $C[u_i,u_j]$ denotes the subpath $u_iu_{i+1}\dots u_{j-1}u_j$ and $C[u_j,u_i]$ denotes the subpath $u_ju_{j+1}\dots u_ku_1\dots u_{i-1}u_i$. To emphasize the orientation, we use $\overline{C} = u_ku_{k-1}u_{k-2}\dots u_1u_k$ to denote the same cycle C (as a subgraph) but with the opposite orientation. If $Q = w_1w_2\dots w_{k'}$ is a path with $v_k = w_1$ and $V(P)\cap V(Q) = \{v_k\}$, then we use PQ or $P[v_1,v_k]Q[v_k,w_{k'}]$ to denote the path $v_1v_2\dots v_kw_2\dots w_{k'}$. If $V(P)\cap V(Q) = \emptyset$ and there is a path $z_1z_2\dots z_t$ with $z_2,\dots,z_{t-1} \notin V(P)\cup V(Q)$ and with $z_1 = v_k$ and $z_t = w_1$, then we use $Pz_1\dots z_tQ$ to denote the path $v_1v_2\dots v_kz_2\dots z_tw_2\dots w_{k'}$. As cycles are considered as closed paths, we also use the same notation to denote cycles obtained by amalgamating paths.

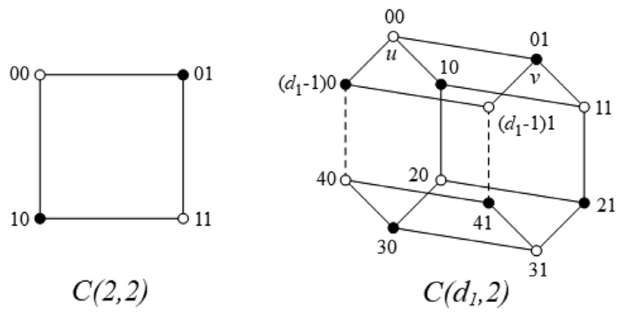

Fig. 3. Graphs in Example 3.1.

In this section, we shall investigate the 2-DCC vertex bipancyclicity of an n-dimensional generalized hypercubes $G = C(d_1, d_2, \ldots, d_n)$. We start with two examples.

Example 3.1. There exist generalized hypercubes that are not 2-DCC vertex bipancyclic, as seen in the examples below.

(i) If $G = C(2, 2)$, then $G \cong C_4$. As it has only one cycle, it cannot be 2-DCC vertex bipancyclic.

(ii) Suppose that $n = 2$, $\min\{d_1, d_2\} = 2$ and $\max\{d_1, d_2\} \geq 6$. By symmetry, we may assume that $d_1 > d_2 = 2$. For each i, j with $1 \leq i \leq 2$ and $1 \leq j \leq d_1$, let $v^i_j = u_{j-1}u_{i-1}$ denote the vertices in $V(G)$. Choose two vertices $u = v^{i_1}_{j_1}$ and $v = v^{i_2}_{j_2}$ such that $j_1 = j_2$ and $i_1 \neq i_2$. As $d_2 = 2$, every 4-cycles in G has the form $v^1_j v^1_{j+_{d_1}1} v^2_{j+_{d_1}1} v^2_j v^1_j$ as illustrated in Fig. 3. This forces that every 4-cycle containing u must also contain v. Thus such a $G = C(d_1, 2)$ with two distinguished vertices u, v as defined above cannot be 2-DCC $[4, \frac{|V(G)|}{2}]$ vertex bipancyclic. Thus when $d_1 \geq 6$, $G = C(d_1, 2)$ contain specific vertices u and v, such that G does not have disjoint cycles J_1 and J_2 with $|E(J_1)| = 4$, $|E(J_2)| = |V(G)| - 4$ such that $u \in V(J_1)$ and $v \in V(J_2)$. Such a triple (G, u, v) is called a bad triple.

Moti Motivated by Example 3.1, we define an *exceptional configuration* to be either a graph isomorphic to $C(2, 2)$, or a graph G isomorphic to a $G = C(d_1, 2)$ with distinguished vertices $u, v \in V(G)$ such that (G, u, v) is a bad triple.

Theorem 3.2. *Let $G = C(d_1, d_2, \ldots, d_n)$ be a bipartite n-dimensional generalized hypercube. Each of the following holds.*

(i) *If $n \geq 3$, then G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic.*

(ii) *Suppose that $n = 2$. Then G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic if and only if G is not an exceptional configuration.*

To prove the Theorem 3.2, we start with Lemmas 3.3, 3.4 and 3.5.

Lemma 3.3. *Let k and ℓ be integers with $k \geq 4$ and $\ell \geq 1$. Let $C = v_1 v_2 \cdots v_k v_1$ denote a cycle on k vertices and $P = w_1 w_2 \cdots w_\ell$ denote a path on ℓ vertices. For each i, j with $i \in \mathbb{Z}_k$ and $1 \leq j \leq \ell$, let v^i_j denote the vertex $v_j w_i$ in $V(C \times P)$. Each of the following holds.*

(i) *For any element $i \in \mathbb{Z}_k$, and for any integer j with $1 \leq j \leq \ell$, $C \times P$ has a spanning cycle $J = J^{(j)}$ satisfying each of the following.*
  (ia) *If $j \equiv 1$ (mod 2), then $(v^1_i, v^1_{i+_k 1})$, $(v^j_{i+_k 2}, v^j_{i+_k 3}) \in E(J)$.*
  (ib) *If $j \equiv 0$ (mod 2), then $(v^1_i, v^1_{i+_k 1})$, $(v^j_i, v^j_{i+_k 1}) \in E(J)$.*

(ii) *For any $i \in \mathbb{Z}_k$, there exists a Hamilton cycle of $C \times P$ containing the edge $(v^1_i, v^1_{i+_k 1})$. Consequently, there exists a Hamilton path of $C \times P$ joining vertices v^1_i and $v^1_{i+_k 1}$.*

Proof. For each i, j with $i \in \mathbb{Z}_k$ and $1 \leq j \leq \ell$, define cycle $C^i_k = v^i_1 v^i_2 \cdots v^i_k v^i_1$. We shall argue by induction to prove conclusion (i) of the lemma.

By inspection, if $j = 1$, then $C \times P = C$. Thus as $k \geq 4$, $J^{(1)} = C$ is a cycle of length k containing distinct edges $(v^1_i, v^1_{i+_k 1})$ and $(v^1_{i+_k 2}, v^1_{i+_k 3})$, for any $i \in \mathbb{Z}_k$.

Inductively, assume that for an odd index j with $1 \leq j \leq \ell - 1$, $C \times P[w_1, w_j]$ has a spanning cycle $J^{(j)}$ with $(v^1_i, v^1_{v_{i+_k 1}})$ and $(v^j_{i+_k 2}, v^j_{i+_k 3}) \in E(J^{(j)})$. Thus $J^{(j)}[v^j_{i+_k 3}, v^j_{i+_k 2}]$ is a spanning path of length $jk - 1$ in $C \times P[w_1, w_j]$. Let $P^{j+1} = \overline{C^{j+1}_k}[v^{j+1}_{i+_k 2}, v^{j+1}_{i+_k 3}]$. Then P^{j+1} is a spanning path in C^{j+1}_k containing edge $(v^{j+1}_i, v^{j+1}_{i+_k 1})$ in it. Define

$$J^{(j+1)} = J^{(j)}[v^j_{i+_k 3}, v^j_{i+_k 2}] v^j_{i+_k 2} v^{j+1}_{i+_k 2} P^{j+1} [v^{j+1}_{i+_k 2}, v^{j+1}_{i+_k 3}] v^{j+1}_{i+_k 3} v^j_{i+_k 3}.$$

Then it is routine to verify that $J^{(j+1)}$ is a spanning cycle of $C \times P[w_1, w_{j+1}]$ of length $(j+1) \cdot k$ with the edges $(v^1_i, v^1_{i+_k 1}) \in E(J^{(j+1)}) \cap E(C^1_k)$ and $(v^{j+1}_i, v^{j+1}_{i+_k 1}) \in E(J^{(j+1)}) \cap E(C^{j+1}_k)$.

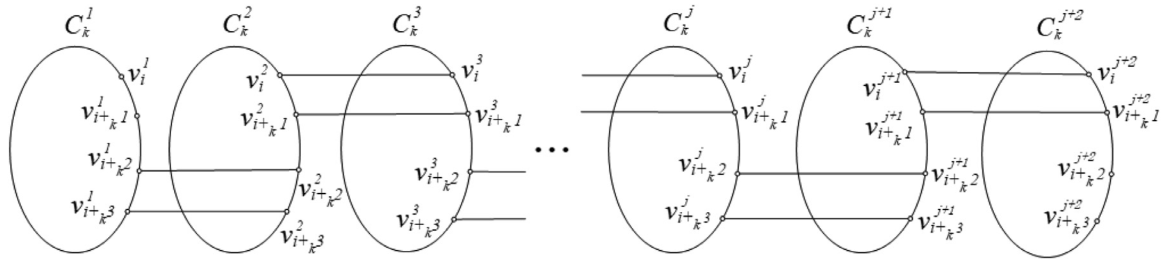

Fig. 4. The illustration of the proof of Claim 1 in Lemma 3.3.

Assume that for an even index $j+1$ with $1 \le j \le \ell - 2$, $C \times P[w_1, w_{j+1}]$ has a spanning cycle $J^{(j+1)}$ of length $(j+1) \cdot k$, and with the edges $(v_i^1, v_{i+_k1}^1) \in E(J^{(j+1)}) \cap E(C_k^1)$ and $(v_i^{j+1}, v_{i+_k1}^{j+1}) \in E(J^{(j+1)}) \cap E(C_k^{j+1})$. Thus $J^{(j+1)}[v_{i+_k1}^{j+1}, v_i^{j+1}]$ is a spanning path in $C \times P[w_1, w_{j+1}]$ of length $(j+1)k - 1$. Let $P^{j+2} = \overline{C_k^{j+2}}[v_i^{j+2}, v_{i+_k1}^{j+2}]$. Then P^{j+2} is a spanning path in C_k^{j+2} containing edge $(v_{i+_k2}^{j+2}, v_{i+_k3}^{j+2})$. We define

$$J^{(j+2)} = J^{(j+1)}[v_{i+_k1}^{j+1}, v_i^{j+1}]v_i^{j+1}v_i^{j+2}P^{j+2}[v_i^{j+2}, v_{i+_k1}^{j+2}]v_{i+_k1}^{j+2}v_{i+_k1}^{j+1}.$$

Again it is routine to verify that $J^{(j+2)}$ is a spanning cycle of $C_k \times P[w_1, w_{j+2}]$ of length $(j+2) \cdot k$ with the edges $(v_i^1, v_{i+_k1}^1) \in E(J^{(j+2)}) \cap E(C_k^1)$ and $(v_{i+_k2}^{j+2}, v_{i+_k3}^{j+2}) \in E(J^{(j+2)}) \cap E(C_k^{j+2})$. Hence Lemma 3.3(i) is proved by induction, as illustrated in Fig. 4.

By Lemma 3.3(i), $J^{(\ell)}$ is the desired spanning cycle of $C \times P[w_1, w_\ell]$ containing the edge $(v_i^1, v_{i+_k1}^1)$, and so $J^{(\ell)}[v_{i+_k1}^1, v_i^1]$ is a Hamilton path in $C \times P$, as desired. $\square$

Lemma 3.4. *If H_1 and H_2 are Hamiltonian graphs such that the Cartesian product $G = H_1 \times H_2$ is bipartite, then G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic.*

Proof. For $i = 1, 2$, let $k_i = |V(H_i)|$. Then $|V(G)| = k_1 k_2$. Since G is bipartite, by the definition of Cartesian product, both H_1 and H_2 are bipartite, and so $k_1 \equiv k_2 \equiv 0 \pmod 2$. In the rest of the proof, we by symmetry assume that $k_1 \ge k_2$ and $k_1 \ge 4$. For $i = 1, 2$, as H_i is Hamiltonian, H_1 has a Hamilton cycle $C_{k_1} = v_1 v_2 \cdots v_{k_1} v_1$ and H_2 has a Hamilton cycle $C_{k_2} = w_1 w_2 \cdots w_{k_2} w_1$.

By Definition 1.1, to prove the lemma, it suffices to prove the following.

$$\text{The Cartesian product } C_{k_1} \times C_{k_2} \text{ is 2-DCC vertex } [4, \tfrac{k_1 k_2}{2}]\text{-bipancyclic.} \tag{3}$$

Thus in the rest of the proof of this lemma, we assume that $G = C_{k_1} \times C_{k_2}$. The fact that $C_{k_1} \times C_{k_2}$ is vertex-transitive would easy some of the arguments in our proofs. Throughout the proof, we let

$$P = C_{k_2}[w_1, w_{k_2}], \text{ a Hamilton path of } H_2. \tag{4}$$

Therefore the graph $\Gamma = C_{k_1} \times P$ is an spanning subgraph of G. For each i, j with $1 \le i \le k_2$ and $1 \le j \le k_1$, let v_j^i denote the vertex $v_j w_i$ in $V(G)$, and let $C_{k_1}^i = v_1^i v_2^i \cdots v_{k_1}^i v_1^i$. Then G contains k vertex-disjoint copies of C_{k_1}, $C_{k_1}^1, C_{k_1}^2, \ldots, C_{k_1}^{k_2}$, as induced subgraphs and $V(G) = \cup_{i=1}^{k_2} V(C_{k_1}^i)$.

Let $u, v \in V(G)$ be two arbitrary vertices and let ℓ be an even integer with $4 \le \ell \le \frac{k_1 k_2}{2}$. We shall construct a 2-DCC J_1 and J_2 in G, with $|V(J_1)| = \ell$ and $|V(J_2)| = k_1 k_2 - \ell$, such that $u \in V(J_1)$ and $v \in V(J_2)$. By the vertex transitivity of $C_{k_1} \times C_{k_2}$, we always assume that $u \ne v$, $u = v_1^1 \in V(C_{k_1}^1)$, and $v = v_r^t$, where $1 \le r \le k_1$, $1 \le t \le k_2$. We will find the cycles J_1 and J_2 in each of the different cases. We prove the following claim, which will help us to prove this lemma.

Claim 1 *If length $\ell = a \cdot k_1 + b$, where a is an integer with $0 \le a < \frac{k_2}{2}$, b is an even integer with $\frac{b}{2} \ge 2$ and $k_1 - \frac{b}{2} \ge 2$, then $G = C_{k_1} \times C_{k_2}$ contains two vertex disjoint cycles J_1 of length ℓ and J_2 of length $|V(G)| - \ell$ with $u = v_1^1 \in V(J_1)$ and $v = v_r^t \in V(J_2)$.*

This claim will be verified in each one of the following three cases.

Case A: $a + 3 \le t \le k_2$ or $1 \le t \le 2$ and $v = v_r^t \ne v_1^2$.

We note that as $0 \le a < \frac{k_2}{2}$, the discussion in this case include the possibility that $a = 0$.

Case A.1: Assume first that $2 \le r \le \frac{b}{2}$. Pick a path

$$P_1 = C_{k_1}^1[v_{k_1 - \frac{b}{2} + r}^1, v_{r-1}^1] = v_{k_1 - \frac{b}{2} + r}^1 \cdots v_{k_1}^1 v_1^1 v_2^1 \cdots v_{r-2}^1 v_{r-1}^1$$

in $C_{k_1}^1$ and a path $P_2 = C_{k_1}^2[v_{k_1 - \frac{b}{2} + r}^2, v_{r-1}^2]$ in $C_{k_1}^2$. Thus each of P_1 and P_2 is of length $\frac{b}{2}$. Then

$$J_1' = P_1[v_{k_1 - \frac{b}{2} + r}^1, v_{r-1}^1]v_{r-1}^1 v_{r-1}^2 P_2[v_{r-1}^2, v_{k_1 - \frac{b}{2} + r}^2]v_{k_1 - \frac{b}{2} + r}^2 v_{k_1 - \frac{b}{2} + r}^1$$

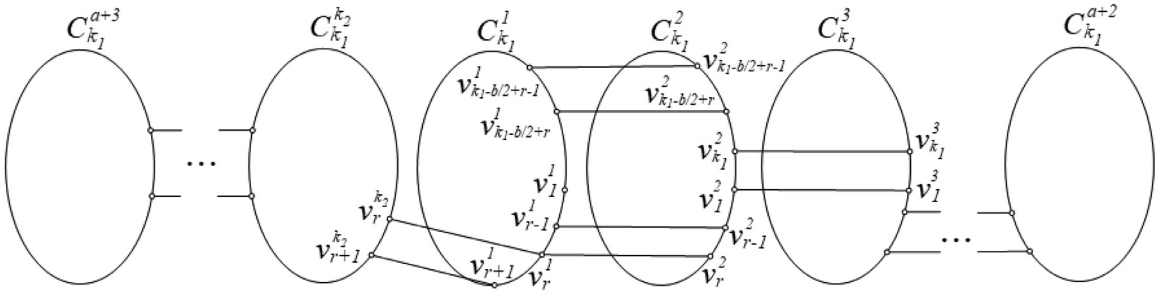

Fig. 5. The illustration of proof of Case A.1 in Claim 1.

is a cycle of length b in G with edge $(v^2_{k_1}, v^2_1) \in E(J'_1) \cap E(C^2_{k_1})$ and vertex $u = v^1_1 \in V(J'_1)$.

Let $P'_1 = C^1_{k_1}[v^1_r, v^1_{k_1-\frac{b}{2}+r-1}] = v^1_r v^1_{r+1} \cdots v^1_{k_1-\frac{b}{2}+r-1}$ and $P'_2 = C^2_{k_1}[v^2_r, v^2_{k_1-\frac{b}{2}+r-1}]$. Then P'_1 is a path in $C^1_{k_1}$ and P'_2 is a path in $C^2_{k_1} - V(J'_1)$, and each of P'_1 and P'_2 is of length $k_1 - \frac{b}{2}$. Since $k_1 - \frac{b}{2} \geq 2$, $C_{k_1} \times P[w_1, w_2] - V(J'_1)$ contains a spanning cycle

$$J'_2 = P'_1[v^1_r, v^1_{k_1-\frac{b}{2}+r-1}]v^1_{k_1-\frac{b}{2}+r-1}v^2_{k_1-\frac{b}{2}+r-1}P'_2[v^2_{k_1-\frac{b}{2}+r-1}, v^2_r]v^2_r v^1_r$$

of length $2k_1 - b$, with $(v^1_r, v^1_{r+1}) \in E(J'_2) \cap E(C^1_{k_1})$ and $\{v^1_r, v^2_r\} \subseteq V(J'_2)$.

By Lemma 3.3, $C_{k_1} \times P[w_3, w_{a+2}]$ has a spanning path $P''_1[v^3_{k_1}, v^3_1]$ and $C_{k_1} \times \overline{P}[w_{k_2}, w_{a+3}]$ has a spanning path $P''_2[v^{k_2}_r, v^{k_2}_{r+1}]$ which contains all vertices $v^{a+3}_r, v^{a+4}_r, \ldots, v^{k_2}_r$. As $V(C_{k_1} \times P[w_3, w_{a+2}]) = \cup_{i=3}^{a+2} V(C^i_{k_1})$ and $V(C_{k_1} \times \overline{P}[w_{k_2}, w_{a+3}]) = \cup_{i=a+3}^{k_2} V(C^i_{k_1})$, it follows by the definition of P in (4) that $V(P''_1[v^3_{k_1}, v^3_1]) \cap V(P''_2[v^{k_2}_r, v^{k_2}_{r+1}]) = \emptyset$. Define

$$J_1 = J'_1[v^2_1, v^2_{k_1}]v^2_{k_1}v^3_{k_1}P''_1[v^3_{k_1}, v^3_1]v^3_1 v^2_1$$
$$J_2 = J'_2[v^1_{r+1}, v^1_r]v^1_r v^{k_2}_r P''_2[v^{k_2}_r, v^{k_2}_{r+1}]v^{k_2}_{k_1}v^1_{k_1}.$$

Then J_1 is a cycle of length $a \cdot k_1 + b = \ell$ and with $u \in V(J_1)$, and J_2 is a spanning cycle in $G - V(J_1)$ of length $(2k_1 - b) + (k_2 - a - 2)k_1 = k_1k_2 - \ell$ and with $v \in V(J_2)$. By their definitions, $u \in V(J_1)$, $v \in V(J_2)$, $V(J_1) \cup V(J_2) = V(G)$ and $V(J_1) \cap V(J_2) = \emptyset$. See Fig. 5 for an illustration of this process.

Case A.2: If $\frac{b}{2} + 1 \leq r \leq k_1$ or $r = 1$ and $a + 3 \leq t \leq k_2$, similarly, pick a path $P_1 = C^1_{k_1}[v^1_1, v^1_{\frac{b}{2}}]$ in $C^1_{k_1}$ and a path $P_2 = C^2_{k_1}[v^2_1, v^2_{\frac{b}{2}}]$ in $C^2_{k_1}$ of length $\frac{b}{2}$. Then for $\frac{b}{2} \geq 2$, $C_{k_1} \times P[w_1, w_2]$ has a cycle $J'_1 = P_1[v^1_1, v^1_{\frac{b}{2}}]v^1_{\frac{b}{2}}v^2_{\frac{b}{2}}P_2[v^2_{\frac{b}{2}}, v^2_1]v^2_1 v^1_1$ of length b that contains the edge (v^2_1, v^2_2) and vertex $u = v^1_1$. Thus the path $P'_1 = C^1_{k_1}[v^1_{\frac{b}{2}+1}, v^1_{k_1}]$ is a subgraph of $C^1_{k_1} - V(J'_1)$ and the path $P'_2 = \overline{C^2_{k_1}}[v^2_{k_1}, v^2_{\frac{b}{2}+1}]$ is contained in $C^2_{k_1} - V(J'_1)$. And the length of paths P'_1, P'_2 are both $k_1 - \frac{b}{2} \geq 2$. Also we can see that P'_1 contains vertex v^1_r and P'_2 contains vertex v^2_r for $\frac{b}{2} + 1 \leq r \leq k_1$. Hence we observe that $C_{k_1} \times P[w_1, w_2] - V(J'_1)$ has a spanning cycle $J'_2 = P'_1[v^1_{\frac{b}{2}+1}, v^1_{k_1}]v^1_{k_1}v^2_{k_1}P'_2[v^2_{k_1}, v^2_{\frac{b}{2}+1}]v^2_{\frac{b}{2}+1}v^1_{\frac{b}{2}+1}$ of length $2k_1 - b$ with $(v^1_{k_1}, v^1_{k_1-1}) \in E(J'_2)$ and $\{v^1_r, v^2_r\} \subseteq V(J'_2)$ for $\frac{b}{2} + 1 \leq r \leq k_1$.

By Lemma 3.3, $C_{k_1} \times P[w_3, w_{a+2}]$ has a spanning path $P''_1[v^3_1, v^3_2]$ and $C_{k_1} \times \overline{P}[w_{k_2}, w_{a+3}]$ has a spanning path $P''_2[v^{k_2}_{k_1}, v^{k_2}_{k_1-1}]$ which contains all vertices $v^{a+3}_r, v^{a+4}_r, \ldots, v^{k_2}_r$. As $V(C_{k_1} \times P[w_3, w_{a+2}]) = \cup_{i=3}^{a+2} V(C^i_{k_1})$ and $V(C_{k_1} \times \overline{P}[w_{k_2}, w_{a+3}]) = \cup_{i=a+3}^{k_2} V(C^i_{k_1})$, it follows by the definition of P in (4) that $V(P''_1[v^3_1, v^3_2]) \cap V(P''_2[v^{k_2}_{k_1}, v^{k_2}_{k_1-1}]) = \emptyset$. Define

$$J_1 = J'_1[v^2_2, v^2_1]v^2_1 v^3_1 P''_1[v^3_1, v^3_2]v^3_2 v^2_2$$
$$J_2 = J'_2[v^1_{k_1}, v^1_{k_1-1}]v^1_{k_1-1}v^{k_2}_{k_1-1}P''_2[v^{k_2}_{k_1-1}, v^{k_2}_{k_1}]v^{k_2}_{k_1}v^1_{k_1}.$$

Then $|E(J_1)| = |E(J'_1)| + |E(P''_1)| + 1 = b + ak_1 = \ell$ and $|E(J_2)| = |E(J'_2)| + |E(P''_2)| + 1 = (2k_1 - b) + (k_2 - a - 2)k_1 = k_1k_2 - \ell$. As $u \in V(J'_1)$, $v \in V(J'_2)$ or $v \in V(P''_2)$, $V(J'_1) \cap V(J'_2) = \emptyset$ and $V(P''_1) \cap V(P''_2) = \emptyset$, we also have $u \in V(J_1)$, $v \in V(J_2)$, $V(J_1) \cap V(J_2) = \emptyset$ and $V(J_1) \cup V(J_2) = V(G)$, which is explained in Fig. 6. Thus the claim holds in this case.

Case B: $t = 2, r = 1$, for any values of a with $0 \leq a < \frac{k_2}{2}$.

Pick $P_1 = C^1_{k_1}[v^1_1, v^1_{\frac{b}{2}}]$ in $C^1_{k_1}$ and path $P_2 = C^{k_2}_{k_1}[v^{k_2}_1, v^{k_2}_{\frac{b}{2}}]$ in $C^{k_2}_{k_1}$ of length $\frac{b}{2} \geq 2$. Then $J'_1 = P_1[v^1_1, v^1_{\frac{b}{2}}]v^1_{\frac{b}{2}}v^{k_2}_{\frac{b}{2}}P_2[v^{k_2}_{\frac{b}{2}}, v^{k_2}_1]v^{k_2}_1 v^1_1$ is a cycle of length b in G containing vertex $u = v^1_1$ and with edge $(v^{k_2}_1, v^{k_2}_2)$ in it. Thus a path $P'_1 = C^1_{k_1}[v^1_{\frac{b}{2}+1}, v^1_{k_1}]$ of length $k_1 - \frac{b}{2} \geq 2$ is in $C^1_{k_1}$ and path $P'_2 = C^{k_2}_{k_1}[v^{k_2}_{\frac{b}{2}+1}, v^{k_2}_{k_1}]$ of length $k_1 - \frac{b}{2}$ is in $C^{k_2}_{k_1} - V(J'_1)$. Hence we observe that $C_{k_1} \times P[w_1, w_2] -$

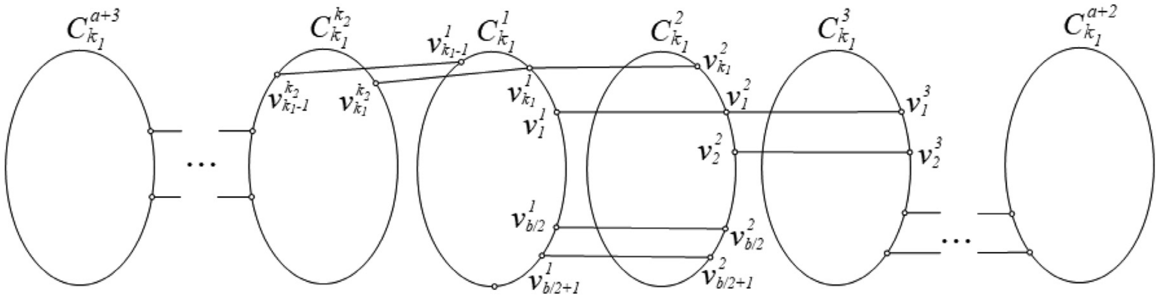

Fig. 6. The illustration of proof of Case A.2 in Claim 1.

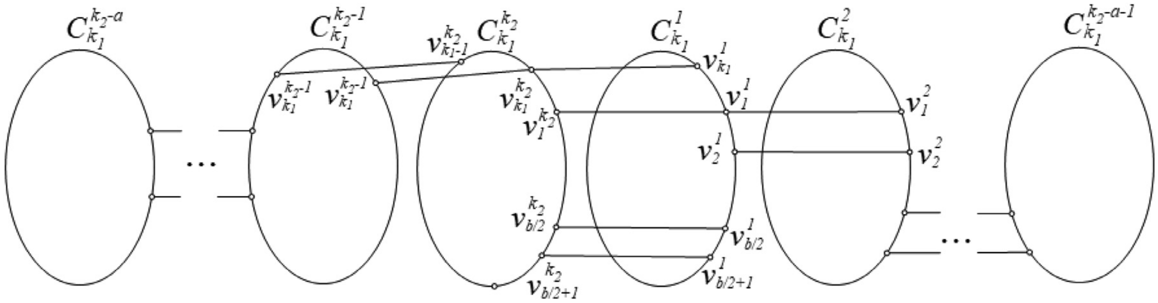

Fig. 7. The illustration of proof of Case B in Claim 1.

$V(J_1')$ has a spanning cycle $J_2' = P_1'[v^1_{\frac{b}{2}+1}, v^1_{k_1}]v^1_{k_1}v^{k_2}_{k_1}P_2'[v^{k_2}_{k_1}, v^{k_2}_{\frac{b}{2}+1}]v^{k_2}_{\frac{b}{2}+1}v^1_{\frac{b}{2}+1}$ of length $2k_1 - b$ and with the edge $(v^1_{k_1}, v^1_{k_1-1}) \in E(J_2') \cap E(C^1_{k_1})$.

By Lemma 3.3, $C_{k_1} \times P[w_{k_2-a}, w_{k_2-1}]$ has a spanning path $P_1''[v_1^{k_2-1}, v_2^{k_2-1}]$ and $C_{k_1} \times P[w_2, w_{k_2-a-1}]$ has a spanning path $P_2''[v^2_{k_1}, v^2_{k_1-1}]$ with $v = v_1^2 \in V(C^2_{k_1}) \subset V(P_2'')$. As $V(C_{k_1} \times P[w_{k_2-a}, w_{k_2-1}]) = \cup_{i=k_2-a}^{k_2-1} V(C^i_{k_1})$ and $V(C_{k_1} \times P[w_2, w_{k_2-a-1}]) = \cup_{i=2}^{k_2-a-1} V(C^i_{k_1})$, it follows by the definition of P in (4) that $V(P_1''[v_1^{k_2-1}, v_2^{k_2-1}]) \cap V(P_2''[v^2_{k_1}, v^2_{k_1-1}]) = \emptyset$. Define

$$J_1 = J_1'[v_2^{k_2}, v_1^{k_2}]v_1^{k_2}v_1^{k_2-1}P_1''[v_1^{k_2-1}, v_2^{k_2-1}]v_2^{k_2-1}v_2^{k_2}$$
$$J_2 = J_2'[v^1_{k_1}, v^1_{k_1-1}]v^1_{k_1-1}v^2_{k_1-1}P_2''[v^2_{k_1-1}, v^2_{k_1}]v^2_{k_1}v^1_{k_1}.$$

Then $|E(J_1)| = |E(J_1')| + |E(P_1'')| + 1 = b + ak_1 = \ell$ and $|E(J_2)| = |E(J_2')| + |E(P_2'')| + 1 = (2k_1 - b) + (k_2 - a - 2)k_1 = k_1k_2 - \ell$. As $u \in V(J_1')$, $v \in V(P_2'')$, $V(J_1') \cap V(J_2') = \emptyset$ and $V(P_1'') \cap V(P_2'') = \emptyset$, we also have $u \in V(J_1)$, $v \in V(J_2)$, $V(J_1) \cap V(J_2) = \emptyset$ and $V(J_1) \cup V(J_2) = V(G)$, as illustrated in Fig. 7. Thus the claim holds in this situation.

Case C: $3 \le t \le a+2$.

In this case, it is implied that $a \ge 1$. Pick a path $P_1 = C^1_{k_1}[v^1_1, v^1_{\frac{b}{2}}]$ in $C^1_{k_1}$ and a path $P_2 = C^2_{k_1}[v^2_1, v^2_{\frac{b}{2}}]$ in $C^2_{k_1}$ of length $\frac{b}{2}$. Then for $\frac{b}{2} \ge 2$, $C_{k_1} \times P[w_1, w_2]$ has a cycle $J_1' = P_1[v^1_1, v^1_{\frac{b}{2}}]v^1_{\frac{b}{2}}v^2_{\frac{b}{2}}P_2[v^2_{\frac{b}{2}}, v^2_1]v^2_1v^1_1$ of length b that contains the edge (v^1_1, v^1_2) and vertex $u = v^1_1$. Thus the path $P_1' = C^1_{k_1}[v^1_{\frac{b}{2}+1}, v^1_{k_1}]$ is a subgraph of $C^1_{k_1} - V(J_1')$ and the path $P_2' = \overline{C^2_{k_1}}[v^2_{k_1}, v^2_{\frac{b}{2}+1}]$ is contained in $C^2_{k_1} - V(J_1')$. And the length of paths P_1', P_2' are both $k_1 - \frac{b}{2} \ge 2$. Hence we observe that $C_{k_1} \times P[w_1, w_2] - V(J_1')$ has a spanning cycle $J_2' = P_1'[v^1_{\frac{b}{2}+1}, v^1_{k_1}]v^1_{k_1}v^2_{k_1}P_2'[v^2_{k_1}, v^2_{\frac{b}{2}+1}]v^2_{\frac{b}{2}+1}v^1_{\frac{b}{2}+1}$ of length $2k_1 - b$ with $(v^2_{k_1}, v^2_{k_1-1}) \in E(J_2')$.

By Lemma 3.3, $C_{k_1} \times \overline{P}[w_{k_2}, w_{k_2-a+1}]$ has a spanning path $P_1''[v_1^{k_2}, v_2^{k_2}]$. And $C_{k_1} \times P[w_3, w_{k_2-a}]$ has a spanning path $P_2''[v^3_{k_1}, v^3_{k_1-1}]$ containing vertex $v = v^t_r$ since $k_2 - a \ge \frac{k_2}{2} + 1 \ge a + 2$. As $V(C_{k_1} \times P[w_3, w_{k_2-a}]) = \cup_{i=3}^{k_2-a} V(C^i_{k_1})$ and $V(C_{k_1} \times \overline{P}[w_{k_2}, w_{k_2-a+1}]) = \cup_{i=k_2-a+1}^{k_2} V(C^i_{k_1})$, it follows by the definition of P in (4) that $V(P_1''[v_1^{k_2}, v_2^{k_2}] \cap V(P_2''[v^3_{k_1}, v^3_{k_1-1}])) = \emptyset$. Define

$$J_1 = J_1'[v^1_2, v^1_1]v^1_1v_1^{k_2}P_1''[v_1^{k_2}, v_2^{k_2}]v_2^{k_2}v^1_2$$
$$J_2 = J_2'[v^2_{k_1}, v^2_{k_1-1}]v^2_{k_1-1}v^3_{k_1-1}P_2''[v^3_{k_1-1}, v^3_{k_1}]v^3_{k_1}v^2_{k_1}.$$

Then $|E(J_1)| = |E(J_1')| + |E(P_1'')| + 1 = b + ak_1 = \ell$ and $|E(J_2)| = |E(J_2')| + |E(P_2'')| + 1 = (2k_1 - b) + (k_2 - a - 2)k_1 = k_1k_2 - \ell$. As $u \in V(J_1')$, $v \in V(P_2'')$, $V(J_1') \cap V(J_2') = \emptyset$ and $V(P_1'') \cap V(P_2'') = \emptyset$, we also have $u \in V(J_1)$, $v \in V(J_2)$, $V(J_1) \cap V(J_2) = \emptyset$ and $V(J_1) \cup V(J_2) = V(G)$, as illustrated in Fig. 8. This justifies Case C of Claim 1 as well as the claim.

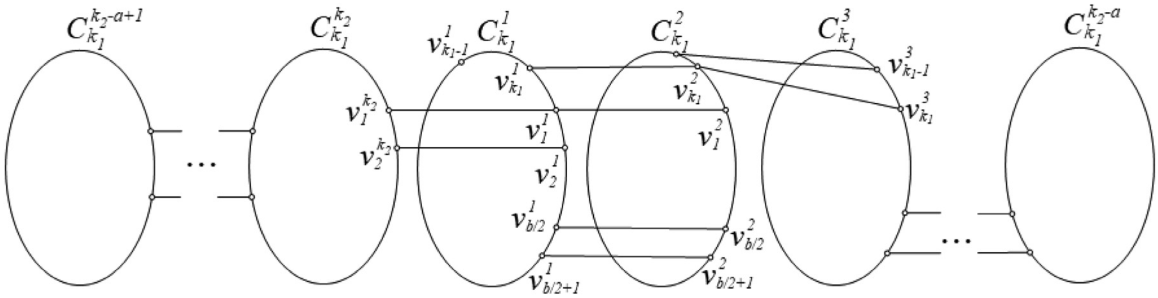

Fig. 8. The illustration of proof of Case C in Claim 1.

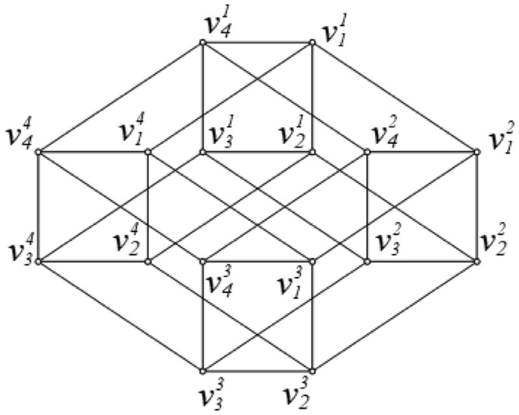

Fig. 9. The illustration of the graph $C(4,4)$.

In the following, we shall apply Claim 1 to prove Lemma 3.4 by validating Lemma 3.4 in each of the following cases.

Case 1: $4 \le \ell \le k_1$. In this case, let $a = 0, b = \ell$, then $\ell = a \cdot k_1 + b$. Since $k_1 \ge 4$, we have $\frac{b}{2} = \frac{\ell}{2} \ge 2$ and $k_1 - \frac{b}{2} \ge k_1 - \frac{k_1}{2} = \frac{k_1}{2} \ge 2$. Then according to Claim 1, the lemma holds in this case.

Case 2: $k_1 + 2 \le \ell \le \frac{k_1 \cdot k_2}{2}$.

Since ℓ and k_1 are even integers, by division algorithm, ℓ can be expressed as $\ell = a_1 \cdot k_1 + b_1$, where a_1 is an integer with $1 \le a_1 \le \frac{k_2}{2}$ and b_1 is an even integer with $0 \le b_1 \le k_1 - 1$.

Case 2.1: $b_1 \ge 4$.

In this case, $a_1 < \frac{k_1}{2}$, $\frac{b_1}{2} \ge 2$ and $k_1 - \frac{b_1}{2} > k_1 - \frac{k_1}{2} \ge 2$ since $k_1 \ge 4$. Then according to Claim 1, the lemma holds in this case.

Case 2.2: $b_1 = 2$.

If $k_1 = k_2 = 4$, it means $G = C(4,4)$ and $\ell = 6$. And Fig. 9 depicts the generalized hypercube $C(4,4)$. When $t = 1$, $r = 2$ or 3, we can construct a cycle $J_1 = v_4^1 v_1^1 v_1^2 v_1^3 v_4^3 v_4^2 v_4^1$ of length $\ell = 6$ containing vertex $u = v_1^1$ and a cycle $J_2 = v_2^1 v_3^1 v_3^2 v_2^2 v_2^3 v_3^3 v_3^4 v_4^4 v_1^4 v_2^4 v_2^1$ of length 10 containing vertex $v = v_r^1$. And we observe that $V(J_1) \cap V(J_2) = \emptyset$ and $V(J_1) \cup V(J_2) = V(G)$. When $r = 4$, according to the symmetry of the graph, this case is similar as the case $r = 2$.

When $t = 2$ or 3, we can construct a cycle $J_1 = v_1^4 v_1^1 v_2^1 v_3^1 v_4^1 v_4^4 v_1^4 \subseteq C_{k_1}^4 \cup C_{k_1}^1$ of length $\ell = 6$ with vertex $u = v_1^1$ in it but not containing vertex v since $v \in V(C_{k_1}^t)$. And we can also construct a spanning cycle J_2' in $C_{k_1} \times P[w_3, w_2]$ of length 8 containing vertex v and edge (v_4^3, v_3^3). Then we get cycle $J_2 = J_2'[v_4^3, v_3^3] v_3^3 v_3^4 v_2^4 v_2^3$ of length 12 containing vertex v. And we observe that $V(J_1) \cap V(J_2) = \emptyset$ and $V(J_1) \cup V(J_2) = V(G)$. When $t = 4$, according to the symmetry of the graph, this case is the same as the case $t = 2$.

If $\max\{k_1, k_2\} > 4$, we assume $k_1 \ge k_2$ and $k_1 > 4$. In this case $\ell = (a_1 - 1)k_1 + (b_1 + k_1)$. Let $a = a_1 - 1$ and $b = b_1 + k_1$, Thus $\ell = a \cdot k_1 + b$ with $0 \le a < \frac{k_2}{2}$, $b > k_1 > 4$, $\frac{b}{2} \ge 2$ and $k_1 - \frac{b}{2} = \frac{k_1}{2} - 1 \ge 2$ since $k_1 \ge 6$. It follows by **Claim 1** that G contains the desired disjoint cycles J_1 of length ℓ and cycle J_2 of length $|V(G| - \ell$, with $u \in V(J_1)$, $v \in V(J_2)$.

Case 2.3: $b_1 = 0$.

Then $\ell = a_1 \cdot k_1 = (a_1 - 1)k_1 + k_1$. Let $a = a_1 - 1$ and $b = k_1$. Thus $\ell = a \cdot k_1 + b$ with $0 \le a < \frac{k_2}{2}$, $b = k_1 \ge 4$, $\frac{b}{2} \ge 2$ and $k_1 - \frac{b}{2} = \frac{k_1}{2} \ge 2$. Then we turn this case into **Claim 1**. So we can directly get two disjoint cycles J_1 of length ℓ and the cycle J_2 of length $|V(G| - \ell$, with $u \in V(J_1)$, $v \in V(J_2)$.

Hence in any case, G always have a desired pair of cycles J_1 and J_2 containing vertices u and v respectively. Since the graph $C_{k_1} \times C_{k_2}$ is vertex-transitive, we can also construct cycles $\tilde{J}_1$ of length ℓ and $\tilde{J}_2$ of length $|k_1 \cdot k_2| - \ell$ containing vertices

v and u respectively with $4 \le \ell \le \frac{k_1 k_2}{2}$. And so by Definition 1.1, $C_{k_1} \times C_{k_2}$ is 2-DCC vertex $[4, \frac{k_1 k_2}{2}]$-bipancyclic. By (3), if H_1 and H_2 are Hamiltonian graphs and if $H_1 \times H_2$ is bipartite, then $H_1 \times H_2$ is also 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic. □

Lemma 3.5. *The generalized hypercube $C(d_1, d_2, \dots, d_n)$ is Hamiltonian for any integer $n \ge 2$ and $d_i \ge 2$.*

Proof. We argue by induction on n. Assume first that $n = 2$. As $C(2, 2) = C_4$ is itself a cycle, we by symmetry assume that $d_1 \ge d_2$ and $d_1 \ge 4$. By (2), $G = C(d_1, d_2) = H_1 \times H_2$, where $H_1 = C_{d_1}$ and H_2 is spanned by a path $P = w_1 w_2 \dots w_{d_2}$. It follows by Lemma 3.3 that $C(d_1, d_2)$ is Hamiltonian.

Assume that $n \ge 3$ and Lemma 3.5 holds for smaller values of n and denote $G' = C(d_1, d_2, \dots, d_{n-1})$ and $G = C(d_1, d_2, \dots, d_n)$. By induction, G' has a Hamilton cycle $C_{d'}$ of length $d' = \prod_{i=1}^{n-1} d_i$. By (2), G has a spanning graph $C(d', d_n)$. By Lemma 3.3, $C(d', d_n)$ has a spanning cycle, and so G is Hamiltonian. □

Now we are ready to present the proof for Theorem 3.2.

Proof. Denote $G = C(d_1, d_2, \dots, d_n)$. Suppose first that $n \ge 4$. Let $G'_1 = C(d_1, d_2)$ and $G'_2 = C(d_3, d_4, \dots, d_n)$. By (2), $G = G'_1 \times G'_2$. By Lemma 3.5, G'_1 and G'_2 are Hamiltonian and so G is a Cartesian product of two Hamiltonian bipartite graphs. It follows by Lemma 3.4 that G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic.

Next we assume that $n = 3$ and $\max\{d_1, d_2, d_3\} \ge 4$. Without loss of generality, we may assume that $d_1 \ge 4$. Let $G' = C(d_2, d_3)$. Then $G = C_{d_1} \times G'$. By Lemma 3.5, G' is Hamiltonian, and so as $d_1 \ge 4$, G is a Cartesian product of two Hamiltonian bipartite graphs. It follows by Lemma 3.4 that G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic.

Observe that if $n = 3$ with $\max\{d_1, d_2, d_3\} = 2$, then $G = C(2, 2, 2) = C_4 \times K_2 = C(4, 2)$, which is a special case when $n = 2$. Thus it remains to prove the theorem for the case when $n = 2$.

Assume that $n = 2$. By symmetry, we also assume that $d_1 \ge d_2$. By Example 3.1, if G is an exceptional configuration, then G is not 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic. In the following, we assume that G is not an exceptional configuration to prove that for any $u, v \in V(G)$, G has the desirable cycles as required in Definition 1.1. If $d_2 \ge 4$, then by Lemma 3.4 that G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic. Therefore, we assume that $d_2 = 2$, and so $G = C_{d_1} \times K_2$.

For each i, j with $1 \le i \le 2$ and $1 \le j \le d_1$, let $v^i_j = u_{j-1} u_{i-1}$ denote the vertex in $V(G)$, and $C^i_{d_1} = v^i_1 v^i_2 \cdots v^i_{d_1} v^i_1$. By definition of generalized hypercubes, $V(G) = V(C^1_{d_1}) \cup V(C^2_{d_1})$. By the vertex-transitivity of G, we may assume $u = v^1_1$ and $v = v^t_r$ where $r \in \{1, 2, \dots, d_1\}$, $t \in \{1, 2, \dots, d_2\}$.

Since G is not an exceptional configuration, we assume that either $v \ne v^2_1$, or $v = v^2_1$ and $G = C(4, 2)$. If $v = v^2_1$ and $G = C(4, 2)$, then the two cycles $J_1 = C^1_{d_1}$ and $J_2 = C^2_{d_1}$, each of which has length $4 = \frac{|V(G)|}{2}$ with $u \in V(J_1)$ and $v \in V(J_2)$. Hence G is 2-DCC vertex $[4, \frac{|V(G)|}{2}]$-bipancyclic. Therefore, we assume in the rest of the proof, $v \ne v^2_1$.

Suppose that $2 \le r \le \frac{\ell}{2}$ where $4 \le \ell \le d_1$. For each $i \in \{1, 2\}$, $C^i_{d_1}$ contains disjoint paths $P_i = C^i_{d_1}[v^i_{d_1 - \frac{\ell}{2} + r}, v^i_{r-1}]$ and $P'_i = C^i_{d_1}[v^i_r, v^i_{d_1 - \frac{\ell}{2} + r - 1}]$. As $r - 1 \ge 1$, $2 \le r \le \frac{\ell}{2}$ and $d_1 - \frac{\ell}{2} + r \le d_1$, we have $u = v^1_1 \in V(P_1)$ and $v^i_r \in V(P'_i)$, $i = 1, 2$. Define

$$J_1 = P_1[v^1_{d_1 - \frac{\ell}{2} + r}, v^1_{r-1}] v^1_{r-1} v^2_{r-1} P_2[v^2_{r-1}, v^2_{d_1 - \frac{\ell}{2} + r}] v^2_{d_1 - \frac{\ell}{2} + r} v^1_{d_1 - \frac{\ell}{2} + r}$$

and

$$J_2 = P'_1[v^1_r, v^1_{d_1 - \frac{l}{2} + r - 1}] v^1_{d_1 - \frac{l}{2} + r - 1} v^2_{d_1 - \frac{l}{2} + r - 1} P'_2[v^2_{d_1 - \frac{l}{2} + r - 1}, v^2_r] v^2_r v^1_r.$$

Then J_1 a cycle of length ℓ containing vertex u in G and J_2 is a cycle of length $2d_1 - \ell = |V(G)| - \ell$ containing vertex v. Thus a pair of desired cycles exist in this case.

Next we suppose that $\frac{\ell}{2} + 1 \le r \le d_1$. For each $i \in \{1, 2\}$, $C^i_{d_1}$ contains disjoint paths $P_i = C^i_{d_1}[v^i_1, v^i_{\frac{\ell}{2}}]$ and $P'_i = C^i_{d_1}[v^i_{\frac{\ell}{2}+1}, v^i_{d_1}]$. As $\frac{\ell}{2} + 1 \le r \le d_1$, $u = v^1_1 \in V(P_1)$ and for each $i = 1, 2$, $v^i_r \in V(P'_i)$. Define $J'_1 = P_1[v^1_1, v^1_{\frac{\ell}{2}}] v^1_{\frac{\ell}{2}} v^2_{\frac{\ell}{2}} P_2[v^2_{\frac{\ell}{2}}, v^2_1] v^2_1 v^1_1$ and $J'_2 = P'_1[v^1_{\frac{\ell}{2}+1}, v^1_{d_1}] v^1_{d_1} v^2_{d_1} P'_2[v^2_{d_1}, v^2_{\frac{\ell}{2}+1}] v^2_{\frac{\ell}{2}+1} v^1_{\frac{\ell}{2}+1}$. Then J'_1 is a cycle of length ℓ in G with $u \in V(J'_1)$, and J'_2 is a cycle of length $2d_1 - \ell = |V(G)| - \ell$ with $v \in V(J'_2)$. As $V(J'_1) \cap V(J'_2) = \emptyset$ and $V(J'_1) \cup V(J'_2) = V(G)$. Hence in this case, the desired cycles also exist. We conclude that if $v \ne v^2_1$, then for any ℓ with $4 \le \ell \le \frac{|V(G)|}{2}$, G always has disjoint cycles J_1 and J_2 with $|E(J_1)| = \ell$ and $|E(J_2)| = |V(G)| - \ell$ such that $u \in V(J_1)$ and $v \in V(J_2)$.

This completes the proof of the theorem. □

Remark 3.6. As shown in Example 3.1, when $n = 2$ and $d_1 \ge 6$, $d_2 = 2$, if $v = v^2_1$, G does not have a cycle of length 4 that only contains vertex u but not v. Thus it is not 2-DCC $[4, \frac{|V(G)|}{2}]$ vertex bipancyclic. However, for any vertex $v = v^t_r \ne v^2_1$, the proof of Theorem 3.2 above indicates that G always contains two disjoint cycles J_1 of length ℓ and J_2 of length $|V(G)| - \ell$ containing u and v, respectively, with required length ℓ.

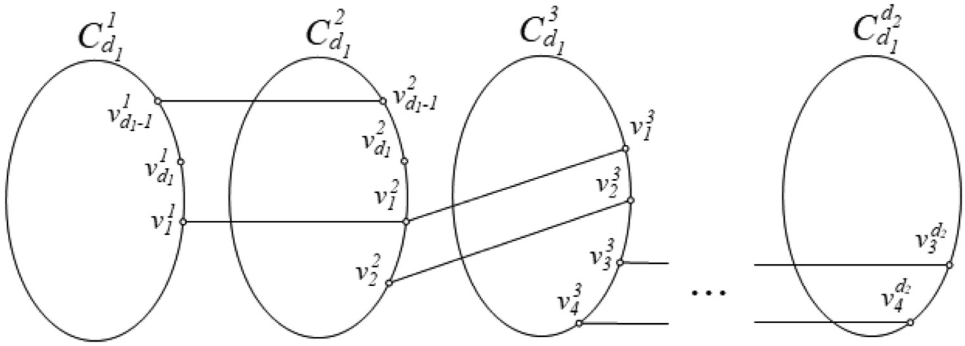

Fig. 10. Illustration of the proof of Corollary 4.2.

4. Applications

In this section, we present several corollaries of Theorem 3.2 as applications to vertex-bipancyclicity and 2-DCC bipancyclicity of bipartite generalized hypercubes as well as to the same properties of bipartite k-ary n-cubes.

Corollary 4.1. *The n-dimensional bipartite generalized hypercube is vertex-bipancyclic for* $n \geq 2$.

Proof. Let $G = C(d_1, d_2, \dots, d_n)$ be an n-dimensional bipartite generalized hypercube with (1) denoting the set of vertices in G. For any even integer ℓ with $4 \leq \ell \leq |V(G)|$, and for any vertex $v \in V(G)$, we are to find a cycle C of order ℓ such that $v \in V(C)$. If $\ell = |V(G)|$, then the existence of such C is assured by Lemma 3.5. If ℓ satisfies $4 \leq \ell \leq |V(G)| - 4$, then Theorem 3.2 warrants that G has a desirable such cycle C when $n \geq 2$ and G is not exceptional configuration.

As for the exceptional configuration, we consider the generalized hypercubes of the form $C(d_1, d_2)$ with $d_1 \geq d_2 \geq 2$. If $G = C(2, 2) = C_4$, then the corollary is obvious. Then, suppose that $d_1 \geq 6$ and $d_2 = 2$. By Remark 3.6, for any vertex $u = u_1u_2$ and $v = v_1v_2$, where $v_1 \neq u_1$, G always has two disjoint cycles J_1 of length ℓ and J_2 of length $|V(G)| - \ell$ containing u and v, respectively, with $4 \leq \ell \leq |V(G)| - 4$.

Then for any vertex $v \in V(G)$, except for $C(2, 2)$, what we should construct is a cycle of length $|V(G)| - 2$ in graph G containing vertex v.

When $n = 2$, $G = C(d_1, d_2)$. If $\max\{d_1, d_2\} \geq 4$, we can assume $d_1 \geq 4$ without loss of generality. For each i, j with $1 \leq i \leq d_2$ and $1 \leq j \leq d_1$, let $v^i_j = u_{j-1}u_{i-1}$ denote the vertex in $V(G)$, and let $C^i_{d_1} = v^i_1 v^i_2 \cdots v^i_{d_1} v^i_1$. Then G contains $C^1_{d_1}, C^2_{d_1}, \dots, C^{d_2}_{d_1}$ as induced subgraphs. Denote path $P = w_1 w_2 \cdots w_{d_2}$, then $C_{d_1} \times P$ is a spanning graph of G. And we may assume vertex $v = v^1_1$ since generalized hypercube is vertex-transitive.

We can pick two paths $P_i = C^i_{d_1}[v^i_1, v^i_{d_1-1}]$ in $C^i_{d_1}$ of length $d_1 - 2$, where $i \in \{1, 2\}$. We can see path P_1 containing vertex v. Then we can construct a cycle

$$C' = P_1[v^1_1, v^1_{d_1-1}]v^1_{d_1-1}v^2_{d_1-1}P_2[v^2_{d_1-1}, v^2_1]v^2_1v^1_1$$

of length $2d_1 - 2$ containing vertex v and with edge (v^2_1, v^2_2) in it. If $d_2 = 2$, we have got the desired the cycle. If $d_2 \geq 4$, according to Lemma 3.3, we can construct a spanning cycle C'' in graph $C_{d_1} \times P[w_3, w_{d_2}]$ of length $(d_2 - 2) \cdot d_1$ containing edge (v^3_1, v^3_2). Thus $\overline{C''}[v^3_1, v^3_2]$ is one spanning path of $C_{d_1} \times P[w_3, w_{d_2}]$. Then we have cycle $C = C'[v^2_2, v^2_1]v^2_1v^3_1\overline{C''}[v^3_1, v^3_2]$ of length $\ell = 2d_1 - 2 + (d_2 - 2) \cdot d_1 = |V(G)| - 2$ containing vertex v. So we can also get the desired cycle, as illustrated in Fig. 10.

When $n \geq 3$, $G = C(d_1, d_2, \dots, d_n)$. We denote $G' = C(d_1, d_2, \dots, d_{n-1})$, then $G = G' \times C_{d_n}$ (or $G = G' \times K_2$ if $d_n = 2$) which contains \$C(d', d_n) \$ as a subgraph According to Lemma 3.5, there exists a Hamilton cycle $C_{d'}$ in G', where $d' = \prod_{i=1}^{n-1} d_i$. Then $G'' = C_{d'} \times C_{d_n}$ (or $C_{d'} \times K_2$ if $d_n = 2$) is a spanning subgraph of G. Then we turn the case into the situation that $n = 2$. Hence we prove any vertex in G'' contained in a cycle of length $d' \cdot d_n - 2$, which means any vertex in G contained in a cycle of length $|V(G)| - 2$. $\square$

Since the 2-DCC vertex bipancyclicity can deduce the 2-DCC bipancyclicity of a graph. By Theorem 3.2 and Remark 3.6, we have the Corollary 4.2 immediately.

Corollary 4.2. *The n-dimensional bipartite generalized hypercube G is 2-DCC* $[4, \frac{|V(G)|}{2}]$*-bipancyclic except for* $C(2, 2)$ *when* $n \geq 2$.

As for k-ary n-cube $Q^k_n = C(k, k, \dots, k)$ is a special case of the generalized hypercube. We can see that $|V(Q^k_n)| = n^k$ and k is an even number if Q^k_n is bipartite. Then we can get the following corollaries.

Corollary 4.3. *The bipartite k-ary n-cube* Q^k_n *is 2-DCC vertex* $[4, \frac{k^n}{2}]$*-bipancyclic except for* Q^2_2 *when* $n \geq 2$.

Corollary 4.4. *The bipartite k-ary n-cube Q_n^k is 2-DCC $[4, \frac{k^n}{2}]$-bipancyclic except for Q_2^2 when $n \geq 2$.*

5. Concluding remark

In this paper, we show that for all integers $n \geq 2$, an n-dimensional bipartite generalized hypercube G is 2-DCC vertex $[4, |V(G)|/2]$-bipancyclic if and only if $G \neq C(2,2)$ or $G \neq C(d_1, d_2)$ with $\min\{d_1, d_2\} = 2$ and $\max\{d_1, d_2\} \geq 6$, where $d_1,\ d_2$ are even numbers. As a corollary, we prove that any n-dimensional bipartite k-ary n-cube $Q_n^k = C(k, k, \ldots, k)$ is also 2-DCC vertex $[4, \frac{k^n}{2}]$-bipancyclic except for Q_2^2 when $n \geq 2$, and show the vertex-bipancyclicity of bipartite generalized hypercubes.

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